Method of Dimensional analysis

1. Rayleigh's method
2. Buckingham's π-theorem

(1) **Rayleigh's method** :→ Used for where three, or four variables only.

$$X = f(X_1, X_2, X_3)$$

(N) $$X = k\, X_1^a \cdot X_2^b \cdot X_3^c$$

$$F_D = f(D, V, \rho, \mu).$$

$$F = \underbrace{K}_{\text{Non-dimensional factor/constant}} D^a \cdot V^b \cdot \rho^c \cdot \mu^d$$

$$MLT^{-2} = K L^a \cdot (LT^{-1})^b \cdot (ML^{-3})^c \cdot (ML^{-1}T^{-1})^d$$

Power of M, $1 = c + d$

Power of L, $1 = a + b - 3c - d$

Power of T, $-2 = -b - d$

→ There are four unknown (a, b, c, d) but eq^n are three

→ not possible to find the values of a, b, c, d, But three of them can be expressed in terms of fourth variable.

→ a, b, c are expressed in terms of d.

$$c = 1 - d \quad \text{——— (i)}$$

$$b = 2 - d \quad \text{——— (ii)}$$

$$a = 1 - b + 3c + d$$

$$a = 1 - (2 - d) + 3(1 - d) + d.$$

$$= 1 - 2 + d + 3 - 3d + d$$

$$a = 2 - d \quad \text{——— (iii)}$$

$$F = K D^{(2-d)} \cdot V^{(2-d)} \cdot \rho^{(1-d)} \cdot \mu^d.$$

$$\boxed{F = K D^2 V^2 \rho \left(\frac{\mu}{DV\rho}\right)^d}$$ Ans

$$F = K D^2 V^2 \rho\, \phi\left(\frac{\mu}{\rho V D}\right)$$

(N)

$$\eta = f(\rho, \mu, \omega, D, Q)$$

$$\eta = K \rho^a . \mu^b . \omega^c . D^d . Q^e$$

$$M^0L^0T^0 = K(ML^{-3})^a . (ML^{-1}T^{-1})^b . (T^{-1})^c . (L)^d . (L^3T^{-1})^e.$$

Power of M, $0 = a + b$

power of L, $0 = -3a - b + d + 3e$

Power of T, $0 = -b - c - e$

There are five unknowns but equations are three. So express three unknowns in terms of the other two unknowns. viscosity (μ) & discharge (Q) is important in this problem & hence a, c, d is expressed in terms of b, e.

$$a = -b.$$

$$c = -b - e$$

$$d = 3a + b - 3e.$$

$$= 3(-b) + b - 3e = -2b - 3e.$$

$$\eta = K \rho^{-b} . \mu^b . \omega^{-b-e} . D^{-2b-3e} . Q^e.$$

$$\boxed{\eta = \cancel{K\rho} K\left(\frac{\mu}{\rho \omega D^2}\right)^b . \left(\frac{Q}{\omega D^3}\right)^e}$$

$$\boxed{\eta = K\left(\frac{\mu}{\rho \omega D^2}\right) . \left(\frac{Q}{\omega D^3}\right)}$$

Imp: Buckingham's π-Theorem :→

"If there are n variables (independent & dependent variables) in a physical phenomenon and if these variables contain m fundamental dimensions (M, L, T), then the variables are arranged into $(n-m)$ dimensionless terms. Each term is called π-term".

→ Let x_1, x_2, x_3 —— x_n are variables in a physical problem.

→ Let x_1 be the dependent variable & x_2, x_3 — x_n are the independent variables on which x_1 depends.

$$x_1 = f(x_2, x_3 \text{ —— } x_n) \rightarrow \text{also written as.} \quad (1)$$

$$f(x_1, x_2, x_3 \text{ —— } x_n) = 0. \quad (2)$$

eqn (2) contains n variable
if there are m fundamental variable then there are $(n-m)$ dimensionless terms.

no. of π terms $= (n-m)$

$$f(\pi_1, \pi_2, \pi_3 \text{ —— } \pi_{n-m}) = 0. \quad (3)$$

→ Now each π-term have $(m+1)$ variables.

$m \rightarrow$ fundamental dimension (M, L, T)
↳ also called repeating variables.
$1 \rightarrow$ non-repeating variables.

$$\left.\begin{aligned} \pi_1 &= x_2^{a_1} \cdot x_3^{b_1} \cdot x_4^{c_1} \cdot x_1 \\ \pi_2 &= x_2^{a_2} \cdot x_3^{b_2} \cdot x_4^{c_2} \cdot x_5 \\ &\vdots \\ \pi_{n-m} &= x_2^{a_{n-m}} \cdot x_3^{b_{n-m}} \cdot x_4^{c_m} \cdot x_n \end{aligned}\right\} \quad (4)$$

the values of a, b, c, ... etc. are ascertained & substituted in (4) & values of π-terms are obtained.
The final eqn by expressing any one of π-terms as a function of others as.

$$\pi_1 = \phi[\pi_2, \pi_3 \text{ — } \pi_{n-m}]$$
$$\pi_2 = \phi_1[\pi_1, \pi_3 \text{ —— } \pi_{n-m}]$$

(N) $\eta = f(\rho, \mu, \omega, D, Q)$

$f(\eta, \rho, \mu, \omega, D, Q) = 0.$

$n = 6$

$m = 3$

$(n-m) = 6-3 = 3$ Π-terms

$f(\Pi_1, \Pi_2, \Pi_3) = 0.$

$\omega \Rightarrow rad/sec = \frac{1}{T}.$

First Π-term $\Pi_1 = D^{a_1} \cdot \omega^{b_1} \rho^{c_1} \cdot \eta$

$$\left[\begin{array}{c} \rho\ V\ D \\ \rho\ V\ L \\ \mu, V, L \\ \mu\ V\ D \end{array}\right]$$

$M^0L^0T^0 = [L]^{a_1} [T^{-1}]^{b_1} \cdot [ML^{-3}]^{c_1} [M^0L^0T^0]$

Power of M, $0 = c_1 + 0 \Rightarrow c_1 = 0$

Power of L, $0 = a_1 - 3c_1 + 0 \Rightarrow a_1 = 0.$

Power of T, $0 = -b_1 + 0 \Rightarrow b_1 = 0.$

$\Pi_1 = D^0 \omega^0 \rho^0 \cdot \eta.$

$\boxed{\Pi_1 = \eta}$

If a variable is dimensionless, it itself is a Π-term.

Second Π-term (Π_2):-

$\Pi_2 = D_2^{a_2} \cdot \omega^{b_2} \cdot \rho^{c_2} \cdot \mu.$

$M^0L^0T^0 = [L]^{a_2} \cdot [T^{-1}]^{b_2} \cdot [ML^{-3}]^{c_2} \cdot [ML^{-1}T^{-1}]$

Power of M, $0 = c_2 + 1 \Rightarrow \boxed{c_2 = -1}$

Power of L, $0 = a_2 - 3c_2 - 1 \Rightarrow a_2 = 3c_2 + 1 \Rightarrow \boxed{a_2 = -2}$

Power of T, $0 = -b_2 - 1 \Rightarrow \boxed{b_2 = -1}$

$\Pi_2 = D_2^{-2} \omega^{-1} \cdot \rho^{-1} \cdot \mu.$

$\Pi_2 = \dfrac{\mu}{D_2^2 \cdot \rho\omega}$ $\longrightarrow$ second Π-term.

Third Π-term :-

$\Pi_3 = D_2^{a_3} \cdot \omega^{b_3} \cdot \rho^{c_3} \cdot Q$

$$\left[\begin{array}{l} Q = A \cdot V \\ \quad = \dfrac{Vol}{t} = \dfrac{L^3}{T} \\ \quad = L^3T^{-1} \end{array}\right]$$

$M^0L^0T^0 = [L]^{a_3} \cdot [T^{-1}]^{b_2} \cdot [ML^{-3}]^{c_3} \cdot (L^3T^{-1}]$

Power of M, $\boxed{0 = c_3}$

Power of L, $0 = a_3 - 3c_3 + 3 \Rightarrow \boxed{a_3 = -3}$

Power of T, $0 = -b_2 - 1 \Rightarrow \boxed{b_3 = -1}$

$$\pi_3 = D_2^{-3} \cdot \omega^{-1} \cdot \rho^{0} \cdot Q.$$

$$\boxed{\pi_3 = \frac{Q}{D_2^{3} \cdot \omega}}$$

$$\boxed{f_1\left(\eta, \frac{\mu}{D^2 \omega \rho}, \frac{Q}{D^2 \omega}\right) = 0}$$

$$\boxed{\eta = \phi\left(\frac{\mu}{D^2 \omega \rho}, \frac{Q}{D^2 \omega}\right)}$$ Ans

Similitude :→ similarity b/w model & its prototype in every aspect.

- Three types of similarity must exist b/w Model & Prototype

1. Geometric similarity → similarity of linear dimension
2. Kinematic similarity → " of motion
3. Dynamic similarity → " " forces

1. Geometric similarity :→

The ratio of linear dimension of prototype & Model are equal.

L_m, b_m, D_m, A_m, V_m → for Model

L_p, b_p, D_p, A_p, V_p → for prototype

$$\boxed{\frac{L_p}{L_m} = \frac{b_p}{b_m} = \frac{D_p}{D_m} = L_r} \text{ — Scale ratio.}$$

Area ratio :- $\frac{A_p}{A_m} = \frac{L_p \times b_p}{L_m \times b_m} = L_r \times L_r = L_r^2$

Volume ratio :→ $\frac{V_p}{V_m} = \left(\frac{L_p}{L_m}\right)^3 = \left(\frac{b_p}{b_m}\right)^3 = \left(\frac{D_p}{D_m}\right)^3$

2. Kinematic similarity :→ Similarity of Motion b/w model & prototype.

→ velocity & acceleration at the corresponding points in the Model and at the corresponding points in the prototype are same.

→ velocity & acceleration are vector quantities, hence not only ratio of magnitude of velocity & acc. at the corresponding points in model & prototype should be same, but dir of vel. & acc. at corresponding points in the model & prototype should be parallel.

V_{p_1} = vel. of fluid at point 1 in prototype

V_{p_2} = vel. of fluid at point 2 in prototype.

a_{p_1} = Acceleration of fluid at 1 in prototype
a_{p_2} = Acceleration of fluid at 2 in prototype

$$\boxed{\frac{V_{p_1}}{V_{m_1}} = \frac{V_{p_2}}{V_{m_2}} = V_r}$$ — velocity ratio.

$$\boxed{\frac{a_{p_1}}{a_{m_1}} = \frac{a_{p_2}}{a_{m_2}} = a_r}$$ — acceleration ratio.

3. **Dynamic similarity** → Similarity of forces b/w Model & prototype.

$$\boxed{\frac{(F_i)_p}{(F_i)_m} = \frac{(F_v)_p}{(F_v)_m} = \frac{(F_g)_p}{(F_g)_m} = F_r}$$ → force ratio

Explain Dimensionless Numbers with expression.

These are the numbers which are obtained by dividing inertia force by viscous force or gravity force or pressure force or surface tension force or elastic force. There are 5 dimensionless numbers.

1) Reynold's Number (Re) - Ratio of Inertia force of flowing fluid and viscous force of fluid. Inertia Force (F_i) = Mass x Acceleration of flowing fluid

Viscous force (F_v) = Shear stress x Area $= \tau \times A = \left(\mu \frac{du}{dy}\right) \times A = \mu \frac{V}{L} \times A$

Reynold's Number $Re = \frac{F_i}{F_v} = \frac{\rho A V^2}{\mu \frac{V}{L} \times A} = \frac{\rho V L}{\mu} = \frac{V \times L}{(\mu/\rho)} = \frac{V \times L}{\nu}$

2) Froude's Number (Fe) – Square root of inertia force of flowing fluid to gravity force. $Fe = \sqrt{\frac{F_i}{F_g}}$

F_g = gravity force = Mass x Acceleration due to gravity
$= \rho \times \text{Volume} \times g = \rho \times L^3 \times g = \rho \times L^2 \times L \times g = \rho \times A \times L \times g$

$\therefore Fe = \sqrt{\frac{F_i}{F_g}} = \sqrt{\frac{\rho A V^2}{\rho A L g}} = \sqrt{\frac{V^2}{Lg}} = \frac{V}{\sqrt{Lg}}$

3) Euler's Number (Eu) – Square root of ratio of inertia force of flowing fluid to pressure force. $\therefore Eu = \sqrt{\frac{F_i}{F_p}} \Rightarrow$

where F_p = Intensity of pressure x Area = $P \times A$
$F_i = \rho A V^2$

$\therefore Eu = \sqrt{\frac{\rho A V^2}{P \times A}} = \frac{V}{\sqrt{P/\rho}}$

4) Weber's Number (We) – Square root of ratio of inertia force of flowing fluid to surface tension force. $\therefore We = \sqrt{\frac{F_i}{F_s}}$

where F_i = Inertia force = $\rho A V^2$; F_s = Surface tension force

F_s = Surface tension per unit length x length = $\sigma \times L$

$We = \sqrt{\frac{\rho A V^2}{\sigma \times L}} = \sqrt{\frac{\rho \times L^2 \times V^2}{\sigma \times L}} = \sqrt{\frac{\rho L \times V^2}{\sigma}} = \sqrt{\frac{V^2}{\sigma/\rho L}} = \frac{V}{\sqrt{\sigma/\rho L}}$

5) Mach's Number (M) – Square root of ratio of inertia force of flowing fluid to elastic force. $M = \sqrt{\frac{F_i}{F_e}}$ $F_i = \rho A V^2$; F_e = elastic force = Elastic stress x Area
$= K \times A = K \times L^2$

$\therefore M = \sqrt{\frac{\rho A V^2}{K \times L^2}} = \sqrt{\frac{\rho \times L^2 \times V^2}{K \times L^2}} = \sqrt{\frac{V^2}{K/\rho}} = \frac{V}{\sqrt{K/\rho}}$

1) Reynold's Model law - It is law in which models are based on Reynold's Number.
(i) Pipe Flow (ii) Resistance experienced by sub-marine, airplanes etc.

V_m = Velocity of fluid in model ρ_m = Density of fluid in model
L_m = length of model μ_m = Viscosity of fluid in model

$$(Re)_m = (Re)_p \longrightarrow \frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p}$$

2) Weber's Model law - $(We)_{model} = (We)_{Prototype}$.
V_m = Velocity of fluid in model ρ_m = Density of fluid in model
σ_m = Surface tension force in model L_m = length of model.
$V_p, \sigma_p, \rho_p, L_p$ = Corresponding values in prototype.

$$\frac{V_m}{\sqrt{\sigma_m / \rho_m L_m}} = \frac{V_p}{\sqrt{\sigma_p / \rho_p L_p}}$$

Applications - (i) Capillary waves in channel
(ii) capillary rise in narrow passage.

3) Euler's Model law $(Eu)_{model} = (Eu)_{Prototype}$

$$\frac{V_m}{\sqrt{\frac{P_m}{\rho_m}}} = \frac{V_p}{\sqrt{\frac{P_p}{\rho_p}}}$$ If fluid is same - $$\frac{V_m}{\sqrt{P_m}} = \frac{V_p}{\sqrt{P_p}}$$

Applications - This law is used where phenomena of cavitation takes place.

Q4 η depends on density ρ, dynamic viscosity μ, angular velocity ω, diameter D and discharge Q. Express η

ans4 We may write - $\eta = f(\rho, \mu, \omega, D, Q)$ or $f'(\eta, \rho, \mu, \omega, D, Q) = 0$
Thus total variables, $n = 6$
η = Dimensionless, $\rho = ML^{-3}$, $\mu = ML^{-1}T^{-1}$, $\omega = T^{-1}$, $D = L$, $Q = L^3T^{-1}$
$\therefore m = 3$

Number of π terms $= n - m = 6 - 3 = 3$
Equation is now written as $f_1(\pi_1, \pi_2, \pi_3) = 0$