Syllabus

Mechanics of solids

UNIT-4

1	Introduction to slope and deflection of beams.
2	Relation between deflection, bending moment, shear force and load.
3	Transverse deflection of beams and shafts under static loading using area moment method
4	Transverse deflection of beams and shafts under static loading using double integration method
5	**Thin-walled Pressure Vessels:** Stresses in cylindrical and spherical vessels
6	**BC**-Thick Walled Pressure Vessels

UNIT-5

Deflections of beams

Introduction

We showed that the loading actions at any section of a simply-supported beam or cantilever can be resolved into a bending moment and a shearing force. Subsequently, in unit 3 and 4, we discussed ways of estimating the stresses due to these bending moments and shearing forces. There is, however, another aspect of the problem of bending which remains to be treated, namely, the calculation of the ***stiffness*** of a beam. In most practical cases, it is necessary that a beam should be not only strong enough for its purpose, but also that it should have the requisite stiffness, that is, it should not deflect from its original position by more than a certain amount. Again, there are certain types of beams, such as those clamped by more than two supports and beams with their ends held in such a way that they must keep their original directions, for which we cannot calculate bending moments and shearing forces without studying the deformations of the axis of the beam; these problems are statically indeterminate, in fact.

In this chapter we consider methods of finding the deflected form of a beam under a given system of external loads and having known conditions of support.

Elastic bending of straight beams

It was shown that a straight beam of uniform cross-section, when subjected to end couples ***M*** applied about a principal axis, bends into a circular arc of radius ***R,***

$$\frac{1}{R} = \frac{M}{EI}$$

given by

where *EI*, which is the product of Young's modulus *E* and the second moment of area *I* about the relevant principal axis, is the flexural stiffness of the beam; equation holds only for *elastic* bending.

Where a beam is subjected to shearing forces, as well as bending moments, the axis of the beam is no longer bent to a circular arc. To deal with this type of problem, we assume that equation. still defines the radius of curvature at any point of the beam where the bending moment is ***M.*** This implies that where the bending moment varies from one section of the beam to another, the radius of curvature also vanes from section to section, in accordance with equation .

In the unstrained condition of the beam, *Cz* is the longitudinal centroidal axis, Figure, and ***Cx,*** *Cy* are the principal axes in the cross-section. The co-ordinate axes ***Cx,*** *Cy* are so arranged that the y-axis is vertically downwards. This is convenient as most practical loading conditions give rise to vertically downwards deflections. Suppose bending moments are applied about axes parallel to ***Cx, so*** that bending is restricted to the yz-plane, because C_x and C_y are principal axes.

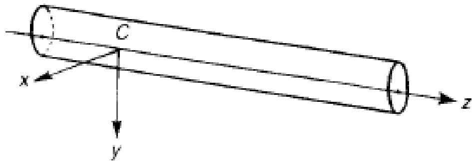


Figure Longitudinal and principal centroidal axis for a straight beam.

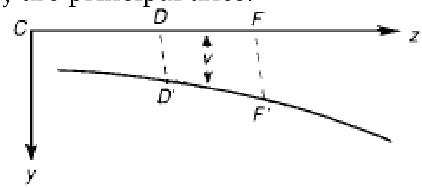


Figure Displacements **of** the longitudinal axis ***of*** the beam.

Consider a short length of the unstrained beam, corresponding with D F on the axis ***Cz,*** Figure 5.2. In the strained condition D and F are dsplaced to D' and F', respectively, which lies in the yz-plane. Any point such as D on the axis ***Cz*** is displaced by **an** amount ***v*** parallel to *Cy;* it is also hsplaced a small, but negligible, amount parallel to *Cz.*

The radius of curvature R at any section of the beam is then given by

$$\frac{1}{R} = \frac{\dfrac{d^2v}{dz^2}}{\pm\left[1+\left(\dfrac{dv}{dz}\right)^2\right]^{3/2}}$$

-

We are concerned generally with only small deflections, in which ***v*** is small; this implies that ***(dv/dz)*** is small, and that $(dv/dz)^2$ eligible compared with **unity.** Then, with sufficient accuracy, we may write

$$\frac{1}{R} = \pm\frac{d^2v}{dz^2}$$

The equations give

$$\pm\, EI\frac{d^2v}{dz^2} = M$$

We must now consider whether the positive or negative sign is relevant in this equation; we have already adopted the convention in Section 4.4 that sagging bending moments are positive. When a length of the beam is subjected to sagging bending moments, as in Figure, the value of ***(dv/dz)***along the length diminishes as z increases; hence a sagging moment implies that the curvature is negative. Then

$$EI \frac{d^2v}{dz^2} = -M$$

where *M* is the ***sagging***bending moment.

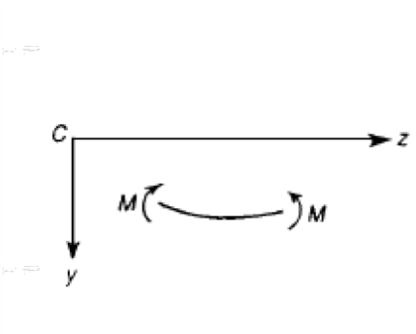


Figure: Curvature induced by sagging bending moment.

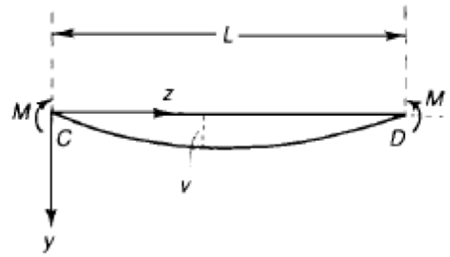


Figure: Deflected form of a beam in pure bending.

Where the beam is loaded on its axis of shear centres, so that no twisting occurs, M may be written in terms of shearing force F and intensity ***w*** of vertical loading at any section. From equation we have

$$\frac{d^2M}{dz^2} = \frac{dF}{dz} = -w$$

On substituting for M from equation , we have

$$\frac{d^2}{dz^2}\left[-EI\frac{d^2v}{dz^2}\right] = \frac{dF}{dz} = -w$$

Thls relation is true if *EI* vanes from one section of a beam to another. Where *El* is constant along the length of a beam,

$$-EI\frac{d^4v}{dz^4} = \frac{dF}{dz} = -w$$

As an example of the use of equation , consider the case of a uniform beam carrying couples

D at its ends, Figure. The bending moment at any section is ***M* ,** so the beam is under a constant **bendmg** moment. Equation gives

$$EI \frac{d^2 v}{dz^2} = -M$$

On integrating once, we have

$$EI \frac{dv}{dz} = -Mz + A$$

where ***A*** is a constant. On integrating once more

$$EIv = -\frac{1}{2}Mz^2 + Az + B$$

where *B* is another constant. If we measure v relative to a line *CD* joining the ends of the beam, vis zero at each end. Then v = 0, for *z* = 0 and *z* = *L*.

On substituting these two conditions into equation, we have

$$B = 0 \quad \text{and} \quad A = \frac{1}{2}ML$$

The equation may be written

$$EIv = \frac{1}{2}Mz(L - z)$$

At the mid-length, $z = \frac{1}{2}L$, and

$$v = \frac{ML^2}{8EI}$$

which is the greatest deflection. At the ends $z = 0$ and $z = L/2$,

$$\frac{dv}{dz} = \frac{ML}{2EI} \text{ at } C; \quad \frac{dv}{dz} = -\frac{ML}{2EI} \text{ at } D$$

It is important to appreciate that equation, expressing the radius of curvature ***R*** in terms of ***v,*** is only true if the displacement v is small.

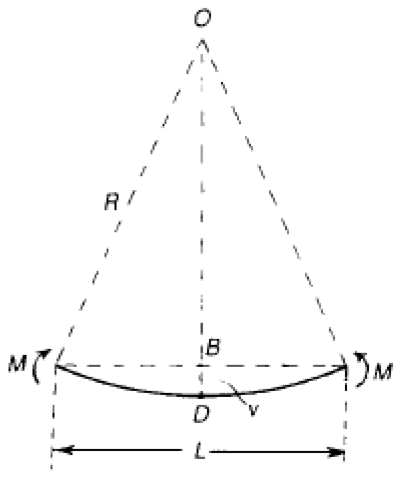


Figure Distortion **of a beam** in **pure bending.**

We can study more accurately the pure bending of a beam by considering it to be deformed into the arc of a circle, Figure; as the bending moment *M* is constant at all sections of the beam, the radius of curvature ***R*** is the same for all sections. If ***L*** is the length between the ends, Figure, and ***D***is the mid-point,

$$OB = \sqrt{R^2 - (L^2/4)}$$

Thus the central deflection ***v*,** is

Then

$$v = R\left[1 - \sqrt{1 - \frac{L^2}{4R^2}}\right]$$

Suppose *L/R* is considerably less than unity; then

$$v = R\left[\frac{1}{2}\left(\frac{L^2}{4R^2}\right) + \frac{1}{8}\left(\frac{L^2}{4R^2}\right)^2 + \ldots\right]$$

which can be written

$$v = \frac{L^2}{8R}\left[1 + \frac{L^2}{4R^2} + \ldots\right]$$

But

$$\frac{1}{R} = \frac{M}{EI}$$

and so

$$v = \frac{ML^2}{8EI}\left[1 + \frac{M^2L^2}{4(EI)^2} + \ldots\right]$$

Clearly, if $(L^2/4R^2)$is negligible compared with **unity**we have, approximately,

$$v = \frac{ML^2}{8EI}$$

which agrees with equation . The more accurate equation shows that, when $\boldsymbol{(L^2/4R^2)}$

is not negligible, the relationship between ***v*** and M is non-linear; for all practical purposes this refinement is unimportant, and we find simple linear relationships of the type of equation are sufficiently accurate for engineering purposes.

Simply-supported beam carrying a uniformly distributed load

A beam of uniform flexural stiffness EI and span ***L*** is simply-supported at its ends, Figure 13.6; it carries a uniformly distributed lateral load of ***w*** per unit length, which induces bending in the ***yz*** plane only. Then the reactions at the ends are each equal to ***1/2wL;*** if z is measured from the end C, the bending moment at a distance z from C is

$$M = \frac{1}{2}wLz - \frac{1}{2}wz^2$$

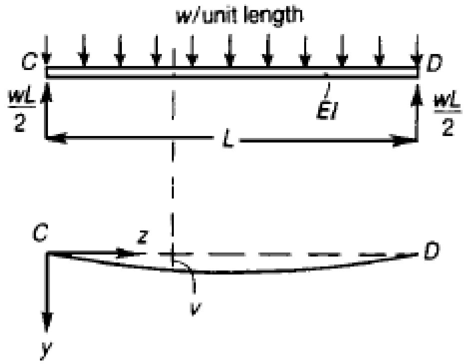


Figure Simply-supported beam carrying a uniformly supported load.

Then from equation,

$$EI\frac{d^2v}{dz^2} = -M = -\frac{1}{2}wLz + \frac{1}{2}wz^2$$

On integrating,

$$EI\frac{dv}{dz} = -\frac{wLz^2}{4} + \frac{wz^3}{6} + A$$

and

$$EIv = -\frac{wLz^3}{12} + \frac{wz^4}{24} + Az + B$$

Suppose $v = 0$ at the ends $z = 0$ and $z = L$; then

$$B = 0, \quad \text{and} \quad A = wL^3/24$$

Cantilever with a concentrated load

Then equation becomes

$$EIv = \frac{wz}{24}\left[L^3 - 2Lz^2 + z^3\right]$$

The deflection at the mid-length, $z = ½L$, is

$$v = \frac{5wL^4}{384EI}$$

Cantilever with a concentrated load

B uniform cantilever of flexural stiffness Eland length ***L*** carries a vertical concentrated load Wat the free end, Figure 13.7. The bending moment a distance ***z*** from the built-in end is

$$M = -W(L - z)$$

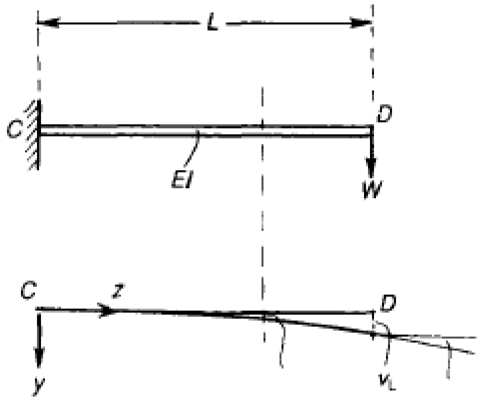


Figure Cantilever carrying a vertical load at the remote end.

Hence equation gives

$$EI\frac{d^2v}{dz^2} = W(L - z)$$

Then

$$EI\frac{dv}{dz} = W\left(Lz - \frac{1}{2}z^2\right) + A$$

and

$$EIv = W\left(\frac{1}{2}Lz^2 - \frac{1}{6}z^3\right) + Az + B$$

At the end $z = 0$, there is zero slope in the deflected form, ***so*** that dv/dz = 0; then equation gives
A = 0. Furthermore, at ***z***= 0 there is also no deflection, ***so*** that ***B*** = 0. Then

$$EIv = \frac{Wz^2}{6}(3L - z)$$

At the free end, $z = L$,

$$v_L = \frac{WL^3}{3EI}$$

The slope of the beam at the free end is

$$\theta_L = \left(\frac{dv}{dz}\right)_{z = L} = \frac{WL^2}{2EI}$$

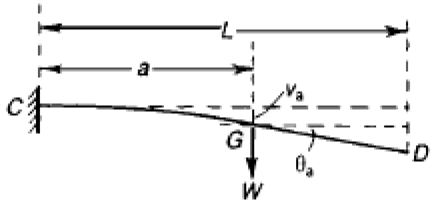
When the cantilever is loaded at some point between the ends, at a distance ***a,*** say, from the built-in support, Figure, the beam between ***G*** and ***D*** carries no bending moments and therefore remains straight. The deflection at G can be

$$v_a = \frac{Wa^3}{3EI}$$

and the slope at $z = a$ is

$$\theta_a = \frac{Wa^2}{2EI}$$

Then the deflection at the free end D of the cantilever is



deduced from equation ; for $z = \boldsymbol{a}$,

Figure Cantilever with a load applied between the ends.

$$v_L = \frac{Wa^3}{3EI} + (L - a)\frac{Wa^2}{2EI}$$

$$= \frac{Wa^2}{6EI}(3L - a)$$

Cantilever with a uniformly distributed load

A uniform cantilever, Figure, carries a uniformly distributed load of w per unit length over the whole of its length. The bending moment at **a** distance *z* from

$$M = -\frac{1}{2}w(L - z)^2$$

C is

Then, from equation *(13.5)*,

$$EI\frac{d^2v}{dz^2} = \frac{1}{2}w(L - z)^2 = \frac{1}{2}w(L^2 - 2Lz + z^2)$$

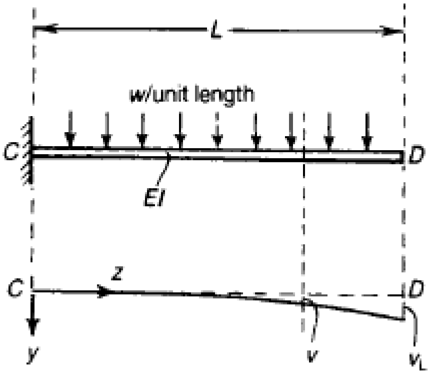


Figure Cantilever carrying **a** uniformly distributed load.

Thus

$$EI\frac{dv}{dz} = \frac{1}{2}w\left(L^2z - Lz^2 + \frac{1}{3}z^3\right) + A$$

and

$$EIv = \frac{1}{2}w\left(\frac{1}{2}L^2z^2 - \frac{1}{3}Lz^3 + \frac{1}{12}z^4\right) + Az + B$$

At the built end, $z = 0$, and we have

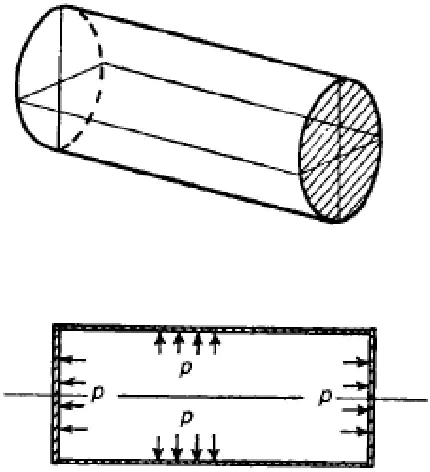
Thus $A = B = 0$. Then

$$EIv = \frac{1}{24} w \left(6L^2 z^2 \quad 4Lz^3 + z^4\right)$$

Thin shells under internal pressure

Thin cylindrical shell of circular cross-section

A problem in which combined stresses are present is that of a cylindrical shell under internal pressure. Suppose a long circular shell is subjected to an internal pressure p, which may be due to a fluid or gas enclosed w i b the cylinder, Figure. The internal pressure acting on the long sides of the cylinder gives rise **to** a circumferential stress in the wall of the cylinder; if the ends of the cylinder are closed, the pressure acting on these ends is transmitted to the walls of the cylinder, thus producing a longitudinal stress in the walls.



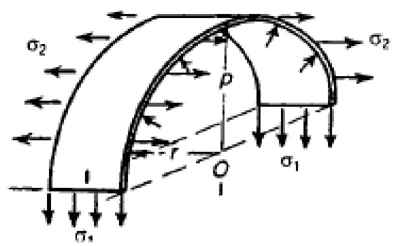


Figure Long thin cylindrical shell with closed ends under internal pressure.

Figure Circumferential and longitudinal stresses in a thin cylinder with **closed** ends under internal pressure.

Suppose ***r*** is the mean radius of the cylinder, and that its thickness ***t*** is small compared with ***r.*** Consider a unit length of the cylinder remote from the closed ends, Figure; suppose we cut **t h ~ s** unit length with a diametral plane, as in

Figure. The tensile stresses acting on the cut sections are **o,,** acting circumferentially, and ***02,***acting longitudinally. There is an internal pressure p on the inside of the half-shell. Consider equilibrium of the half-shell in a plane perpendcular to the axis of the cylinder, **as** in Figure ; the total force due to the internal pressure ***p*** in the direction OA is

$$p \times (2r \times 1)$$

because we are dealing with a unit length of the cylinder. **This** force is opposed by the stresses a,; for equilibrium we must have

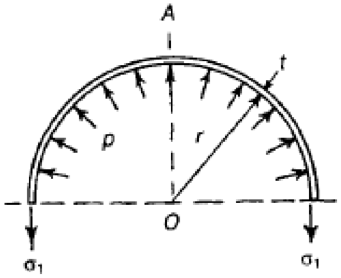
$$p \times (2r \times 1) = \sigma_1 \times 2(t \times 1)$$

Then

$$\sigma_1 = \frac{pr}{t}$$

σ_1

We shall call this the ***circumferential***(or ***hoop) stress.***



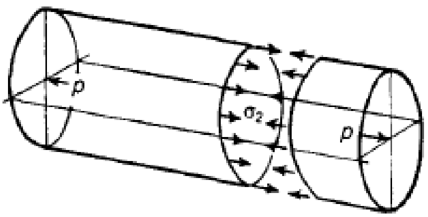


Figure Derivation of circumferential stress. **Figure** Derivation **of** longitudinal stress.

Now consider any transverse cross-section of the cylinder remote from the ends, Figure **6.4;**
the total longitudinal force on each closed end due to internal pressure is

$$p \times \pi r^2$$

At any section this is resisted by the internal stresses **a2,**Figure 6.4. For equilibrium we must have

$$p \times \pi r^2 = \sigma_2 \times 2\pi r t$$

which gives

$$\sigma_2 = \frac{pr}{2t}$$

We shall call this the *longitudinal stress.* Thus the longitudinal stress, σ_2, is only half the circumferential stress, σ_1.

The stresses acting on an element of the wall of the cylinder consist of a circumferential stress σ_1, a longitudinal stress σ_2, and a radial stress p on the internal face of the element, Figure 6.5. As (r/t) is very much greater than unity, p is small compared with σ_1 and σ_2. The state of stress in the wall of the cylinder approximates then to a simple two-dimensional system with principal stresses σ_1 and σ_2.

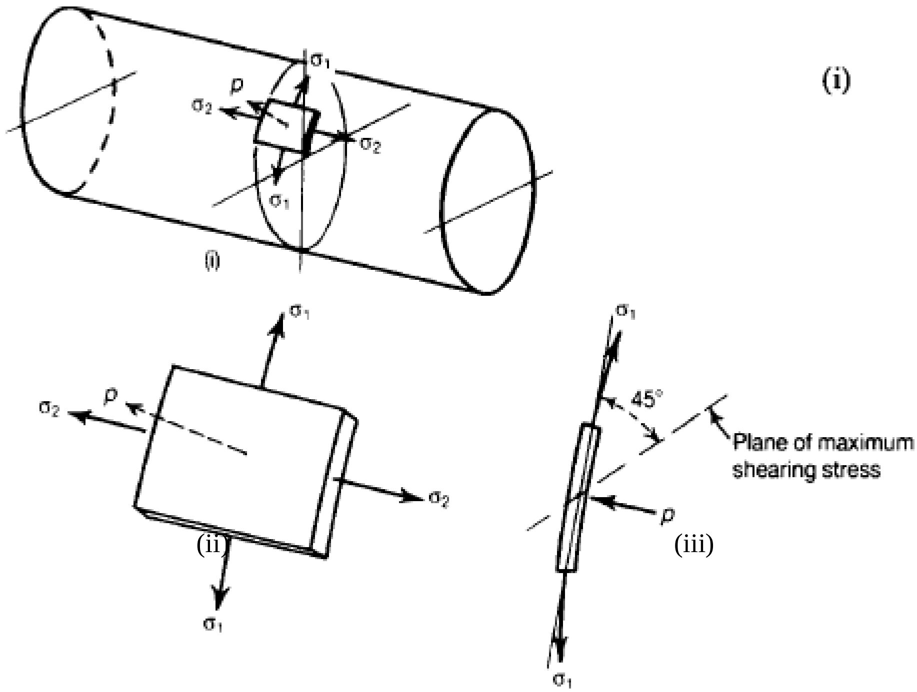


Figure Stresses acting on an element of the wall of a circular cylindrical shell with closed ends under internal pressure.

The maximum shearing stress in the plane of σ_1 and σ_2 is therefore

$$\tau_{max} = \frac{1}{2}\left(\sigma_1 - \sigma_2\right) = \frac{pr}{4t}$$

This is not, however, the maximum shearing stress in the wall of the cylinder, for, in the plane of σ_1 and p, the maximum shearing stress is

$$\tau_{max} = \frac{1}{2}(\tau_{max} = \frac{1}{2}(\sigma_1) = \frac{pr}{2t}$$

Since p is negligible compared with ***G,;***again, in the plane of Sigma and p, the maximum shearing stress is

$$\tau_{max} = \frac{1}{2}(\sigma_2) = \frac{pr}{4t}$$

The greatest of these maximum shearing stresses is given by equation (6.3); it occurs on a plane at 45" to the tangent and parallel to the longitudinal axis of the cylinder, Figure 6.5(iii).

The circumferential and longitudinal stresses are accompanied by direct strains. If the material of the cylinder is elastic, the corresponding strains are given by

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2) = \frac{pr}{Et}\left(1 - \frac{1}{2}\nu\right)$$

$$\varepsilon_1 = \frac{1}{E}(\sigma_2 - \nu\sigma_1) = \frac{pr}{Et}\left(\frac{1}{2} - \nu\right)$$

The circumference of the cylinder increases therefore by a small amount ***2πre,;*** the increase in mean radius is therefore 'E, The increase in length of a unit length of the cylinder is ***E,, so*** the change in internal volume of a unit length of the cylinder is

$$\delta V = \pi (r + r\varepsilon_1)^2 (1 + \varepsilon_2) - \pi r^2$$

The volumetric strain is therefore

$$\frac{\delta V}{\pi r^2} = (1 + \varepsilon_1)^2 (1 + \varepsilon_2) - 1$$

But ε_1 and ε_2 are small quantities, so the volumetric strain is

$$(1 + \varepsilon_1)^2 (1 + \varepsilon_2) - 1 \doteq (1 + 2\varepsilon_1)(1 + \varepsilon_2) - 1$$

$$\doteq 2\varepsilon_1 + \varepsilon_2$$

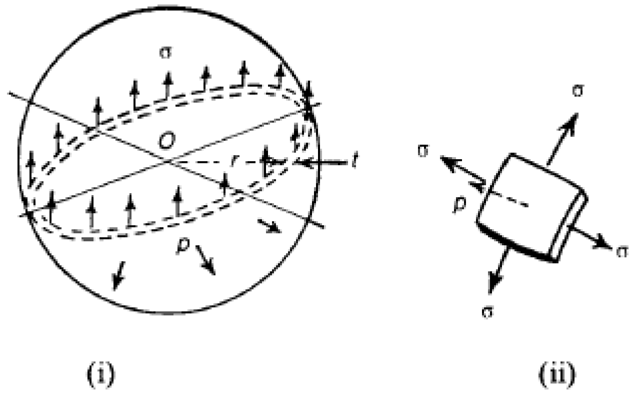
In terms of σ_1 and σ_2 this becomes

$$2\varepsilon_1 + \varepsilon_2 = \frac{pr}{Et}\left[2\left(1 - \frac{1}{2}\nu\right) + \left(\frac{1}{2} - \nu\right)\right] = \frac{pr}{Et}\left(\frac{5}{2} - 2\nu\right)$$

Thin spherical shell

We consider next a thin spherical shell of means radius ***r,*** and thickness ***t,*** which is subjected to **an** internal pressure ***p* .** Consider any diameter plane through the shell, Figure ; the total force normal to this plane due t o p acting on a hemisphere

$$p \times \pi r^2$$



is

Figure Membrane stresses in a thin spherical shell under internal pressure.

This is opposed by a tensile stress (I in the walls of the shell. By symmetry (I is the same at all points of the shell; for equilibrium of the hemisphere we must have

$$p \times \pi r^2 = \sigma \times 2\pi r t$$

This gives

$$\sigma = \frac{pr}{2t}$$

At any point of the shell the direct stress (I has the same magnitude in all directions in the plane **of** the surface of the shell; the state of stress is shown in Figure (ii). ***A s p*** is small compared with (I, the maximum shearing stress occurs on planes at **45'** to the tangent plane at any point.

I f the shell remains elastic, the circumference of the sphere in any diametral plane is strained an amount

$$\varepsilon = \frac{1}{E}(\sigma - \nu\sigma) = (1 - \nu)\frac{\sigma}{E}$$

The volumetric strain of the enclosed volume of the sphere is therefore

$$3\varepsilon = 3(1 - \nu)\frac{\sigma}{E} = 3(1 - \nu)\frac{pr}{2Et}$$

Equation is intended for determining membrane stresses in a perfect thin-walled spherical shell. If, however, the spherical shell is fabricated, so that its joint is weaker than the remainder of the shell, then equation takes on the following modified form:

$$\sigma = \text{stress} = \frac{pr}{2\eta t}$$

where

$$\eta = \text{joint efficiency} \leq 1$$

Cylindrical shell with hemispherical ends

Some pressure vessels are fabricated with hemispherical ends; this has the advantage of reducing the bending stresses in the cylinder when the ends are flat. Suppose the thicknesses ***t,*** and ***t2*** of the cylindrical section and the hemispherical end, respectively , are proportioned so that the radial expansion is the same for both cylinder and hemisphere; in this way we eliminate bending stresses at the junction of the **two**parts.

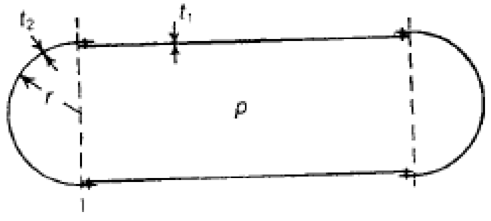


Figure Cylindrical shell with hemispherical ends, so Designed **as**to minimize the effects of bending stresses.

From equations**,** the circumferential strain in the cylinder is

$$\frac{pr}{Et_1}\left(1 - \frac{1}{2}v\right)$$

and from equation the circumferential strain in the hemisphere is

$$(1 - v)\frac{pr}{2Et_2}$$

If these strains are equal, then

$$\frac{pr}{Et_1}\left(1 - \frac{1}{2}v\right) = \frac{pr}{2Et_2}$$