Syllabus

Mechanics of solids[3ME4-07]

UNIT-3

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10	**BC**-Combined Direct and Bending Stresses

UNIT-3

Analysis of stress and strain

Introduction

Up to the present we have confined our attention to considerations of simple direct and shearing stresses. **But** in most practical problems we have to deal with combinations of these stresses.

The strengths and elastic properties of materials are determined usually by simple tensile and compressive tests. How are we to make use of the results of such tests when we know that stress in a given practical problem is compounded from a tensile stress in one direction, a compressive stress in some other direction, and a shearing stress in a third direction? Clearly we cannot make tests of a material under all possible combinations of stress to determine its strength. It is essential, in fact, to study stresses and strains in more general terms; the analysis which follows should be regarded as having a direct and important bearing on practical strength problems, and is not merely a display of mathematical ingenuity.

Shearing stresses in a tensile test specimen

A long uniform bar, Figure**,** has a rectangular cross-section of area ***A* .** The edges of the bar are parallel to perpendicular axes **Ox,** QY, Oz. The bar is uniformly stressed in tension in the **x-** direction, the tensile stress on a cross-section of the bar parallel to **Ox** being ***ox.*** Consider the stresses acting on an inclined cross-section of the bar; an inclined plane is taken at an angle **0** to the yz-plane. The resultant force at the end cross-section **of** the bar is acting parallel to **Ox.**

$$P = A\sigma_x$$

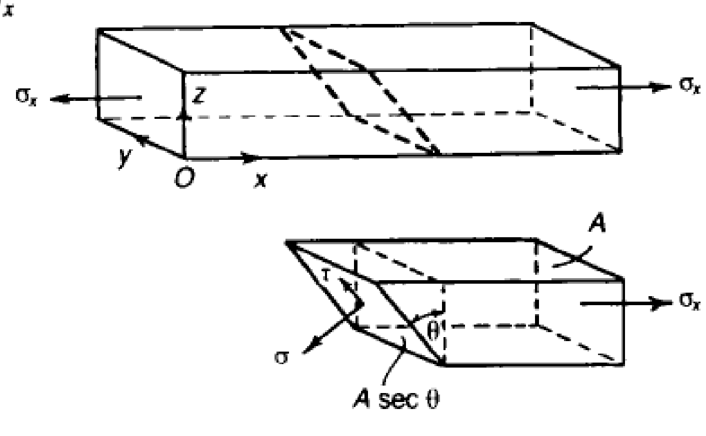


Figure :Stresses on an inclined plane through a tensile test piece.

For equilibrium the resultant force parallel to **Ox** on an inclined cross-section is also $P = A_o$. At the inclined cross-section in Figure, resolve the force A_o, into two components-one perpendicular, and the other tangential, to the inclined cross-section, the latter component acting parallel to the xz-plane. These two components have values, respectively, of

$$A\sigma_x \cos\theta \text{ and } A\sigma_x \sin\theta$$

The area of the inclined cross-section is

$$A \sec\theta$$

so that the normal and tangential stresses acting on the inclined cross-section are

$$\sigma = \frac{A\sigma_x \cos\theta}{A \sec\theta} = \sigma_x \cos^2\theta$$

$$\tau = \frac{A\sigma_x \sin\theta}{A \sec\theta} = \sigma_x \cos\theta \sin\theta$$

is the ***direct stress*** and ***T*** the ***shearing stress*** on the inclined plane. It should be noted that the stresses on an inclined plane are not simply the resolutions of ***ox*** perpendicular and tangential to that plane; the important point in Figure is that the area of an inclined cross-section of the bar is different from that of a normal cross-section. The shearing stress **T** may be written in the form

$$\tau = \sigma_x \cos\theta \sin\theta = \tfrac{1}{2}\sigma_x \sin 2\theta$$

At **0** = 0" the cross-section is perpendicular to the axis of the bar, and ***T*** = 0; **T** increases as **0** increases until it attains a maximum of !4 ***ox*** at 6 = **45** " ; ***T*** then diminishes as 0 increases further until it is again zero at **0** = 90". Thus on any inclined cross-section of a tensile test- piece, shearing stresses are always present; the shearing stresses are greatest on planes at **45**
to the longitudinal axis of the bar.

Failure of materials in compression

Shearing stresses are also developed in a bar under uniform compression. The failure of some materials in compression is due **to** the development of critical shearing stresses on planes inclined to the direction of compression. Figure 5.3 shows **two** failures of compressed timbers; failure is due primarily to breakdown in shear on planes inclined to the direction of compression.

General two-dimensional stress system

two-dimensional stress system is one in which the stresses at any point in a body act in the same plane. Consider a thin rectangular block of material, ***abcd,*** two faces of which are parallel to the xy-plane, Figure**.** A two-dimensional state of stress exists if the stresses on the remaining four faces are parallel to the xy-plane. In general, suppose the forces acting on the faces are *P*, Q, ***R,*** **S,** parallel to the xy- plane, Figure**.** Each of these forces can be resolved into components *P,* etc., of ***forces*** in Figure is now replaced by its equivalent system of ***stresses;*** the rectangular block of Figure is in uniform state of two-dimensional stress; over the two faces parallel to ***Ox*** are direct and shearing stresses ***Oy*** and *T,~*, respectively. The thickness is assumed to be 1 unit of length, for convenience, the other sides having lengths ***a*** and ***b.*** Equilibrium of the block in the ***x***- andy-directions is already ensured; for rotational equilibrium of the block in the *xy*-plane we must have

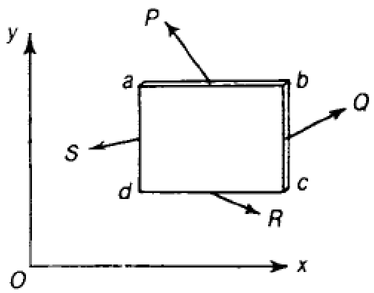


Figure General two-dimensional inclined state of stress. stress system.

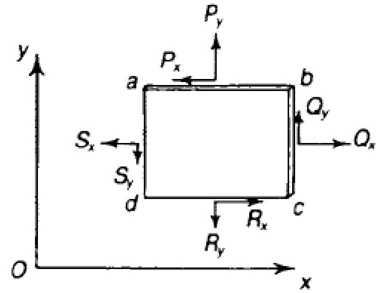
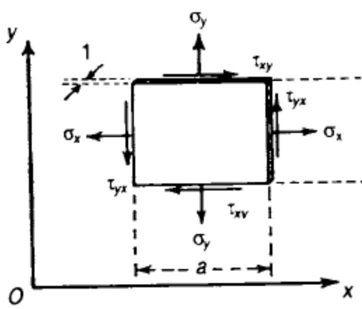


Figure Stresses on an plane in **a** two-dimensional

$$(ab)\,\tau_{xy} = (ab)\,\tau_{yx}$$

$$\tau_{xy} = \tau_{yx}$$

Then the shearing stresses on perpendicular planes are equal and ***complementary*** **as** we found in the simpler case of pure shear


y
1
σ_y
τ_{xy}
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b
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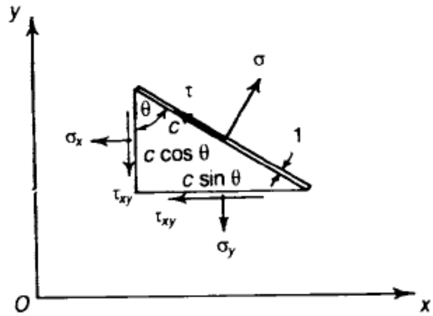
y
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c
1
σ_x
$c \cos \theta$
$c \sin \theta$
τ_{xy}
τ_{xy}
σ_y
O
x

Stresses on an inclined plane

Consider the stresses acting on an inclined plane of the uniformly stressed rectangular block of Figure ***5.6;*** the inclined plane makes **an** angle **ө** with ***O,,*** and cuts off a ‘triangular’ block, Figure . The length of the hypotenuse is ***c,*** and the thickness of the block is taken again as one unit **of** length, for convenience. The values of direct stress, ***0,*** and shearing stress, ***T,*** on the inclined plane are found by considering equilibrium of the triangular block. The direct stress acts along the normal to the inclined plane. Resolve the forces on the three sides of the block parallel *to* the **a**

$$\sigma\,(c.1) = \sigma_x\,(c\cos\theta\cos\theta) + \sigma_y\,(c\sin\theta\sin\theta) + \tau_{xy}\,(c\cos\theta\sin\theta) + \tau_{xy}\,(c\sin\theta\cos\theta)$$

This gives

$$\sigma = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

Now resolve forces in a direction parallel to the inclined plane:

$$\tau.(c\ 1) = -\sigma_x\,(c\cos\theta\sin\theta) + \sigma_y\,(c\sin\theta\cos\theta) + \tau_{xy}\,(c\cos\theta\cos\theta) - \tau_{xy}\,(c\sin\theta\sin\theta)$$

This gives

$$\tau = -\sigma_x\cos\theta\sin\theta + \sigma_y\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

The expressions for a and **T** are written more conveniently in the forms:

$$\sigma = ½(\sigma_x + \sigma_y) + ½(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau = -½(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta$$

The shearing stress T vanishes when

$$½(\sigma_x - \sigma_y)\sin 2\theta = \tau_{xy}\cos 2\theta$$

that is, when

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

or when

$$2\theta = \tan^{-1}\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad \text{or} \quad \tan^{-1}\frac{2\tau_{xy}}{\sigma_x - \sigma_y} + 180^\circ$$

These may be written

$$\theta = \frac{1}{2}\tan^{-1}\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad \text{or} \quad \frac{1}{2}\tan^{-1}\frac{2\tau_{xy}}{\sigma_x - \sigma_y} + 90^\circ$$

In a two-dimensional stress system there are thus two planes, separated by go", **on** which the shearing stress is zero. These planes are called the principal planes, and the corresponding values of **o** are called the principal stresses. The direct stress **t**$_s$ is a maximum when

$$\frac{d\sigma}{d\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

that is, when

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

which is identical with equation ***(5.8),*** defining the directions of the principal stresses; the principal ***stresses*** are also the maximum and ***minimum***direct stresses in the material.

Values *of* the principal stresses

The directions of the principal planes are given by equation . For any two-dimensional stress system, in which the values of **ox,** ***cry*** and ***T~*** are **known,** tan28 is calculable; two values of 8, separated by go", can then be found. The principal stresses are then calculated by substituting these vales of 8 into equation.

Alternatively, the principal stresses can be calculated more directly without finding the principal planes. Earlier we defined a principal plane as one **on** which there is **no** shearing stress; in Figure it is assumed that no shearing stress acts on a plane.

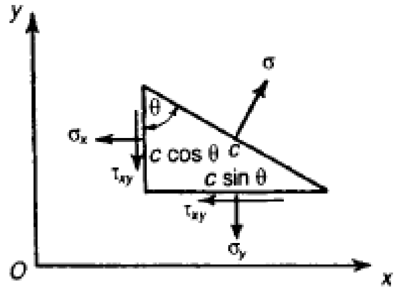


Figure: A principal stress acting on an inclined plane; there is no shearing stress ***T*** associated with a principal stress ***o.***

For equilibrium of the triangular block in the x-direction,

$$\sigma(c\cos\theta) - \sigma_x(c\cos\theta) = \tau_{xy}(c\sin\theta)$$

and so

$$\sigma - \sigma_x = \tau_{xy}\tan\theta$$

For equilibrium of the block in the y-direction

$$\sigma(c\sin\theta) - \sigma_y(c\sin\theta) = \tau_{xy}(c\cos\theta)$$

and **thus**

$$\sigma - \sigma_y = \tau_{xy}\cot\theta$$

On eliminating 8 between equations and ; by multiplying these equations together, we get

$$(\sigma - \sigma_x)(\sigma - \sigma_y) = \tau^2_{xy}$$

This equation is quadratic in o; the solutions are

$$\sigma_1 = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + 4\tau_{xy}^2} \quad = \text{maximum principal stress}$$

$$\sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + 4\tau_{xy}^2} \quad = \text{minimum principal stress}$$

which are the values of the principal stresses; these stresses occur on mutually perpendicular planes.

Maximum shearing stress

The principal planes define directions of zero shearing stress; on some intermediate plane the shearing stress attains a maximum value. The shearing stress is given by equation (2.7); **T** attains a maximum value with respect to **0** when

$$\frac{d\tau}{d\theta} = -(\sigma_x - \sigma_y)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0$$

i.e., when

$$\cot 2\theta = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

The planes of maximum shearing stress are inclined then at **45"** to the principal planes. On substituting this value of cot **28** into equation ,the maximum numerical value of **T is**

$$\tau_{max} = \sqrt{\left[\frac{1}{2}\left(\sigma_x - \sigma_y\right)\right]^2 + \left[\tau_{xy}\right]^2}$$

But from equations ,

$$\sqrt{\left[\frac{1}{2}\left(\sigma_x - \sigma_y\right)\right]^2 + \left[\tau_{xy}\right]^2} = \sigma_1 - \frac{1}{2}\left(\sigma_x + \sigma_y\right) = \frac{1}{2}\left(\sigma_x + \sigma_y\right) - \sigma_2$$

where **o,**and ***o2***are the principal stresses of the stress system. Then by adding together the two equations on the right hand side, we get

$$2\sqrt{\left[\frac{1}{2}\left(\sigma_x - \sigma_y\right)\right]^2 + \left[\tau_{xy}\right]^2} = \sigma_1 - \sigma_2$$

and equation becomes

$$\tau_{max} = \frac{1}{2}\left(\sigma_1 - \sigma_2\right)$$

The maximum shearing stress is therefore half the difference between the principal stresses of the system.

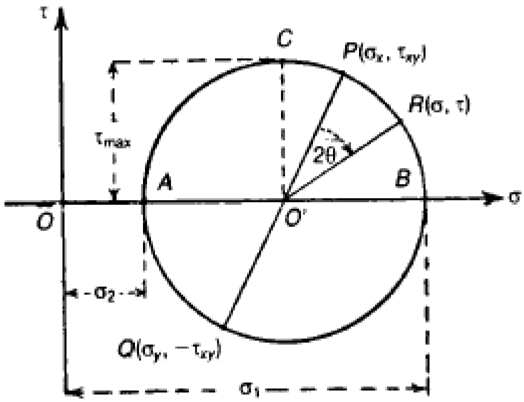
Mohr's circle of stress

geometrical interpretation of equations and leads to a simple method of stress analysis. Now, we have found already that

$$\sigma = \frac{1}{2}\left(\sigma_x + \sigma_y\right) + \frac{1}{2}\left(\sigma_x - \sigma_y\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau = -\frac{1}{2}\left(\sigma_x - \sigma_y\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

Take two perpendicular axes ***00*** & Figure 2.9; on h s co-ordinate system set off the point having co-ordinates ***(ox,*** TJ and ***(o,TJ,*** corresponding to the known stresses in the ***x-*** andy-directions. The line *PQ* joining these two points is bisected by the **Oa**axis at a point 0'. With a centre at 0', construct a circle passing through *P* and *Q*. The stresses ***o*** and **T** on a plane at an angle 8 to Oy are found by setting off a radius of the circle at an angle 28 to *PQ,* Figure 5.9; 28 is measured in a clockwise direction from 0' *P*.



Mohr's circle of stress. The points ***P***and Q correspond to the stress state(σ_x, τ_{xy}) and $(\sigma_y, -\tau_{xy})$ respectively, and are diametrically opposite; the state of stress (σ, τ) on a plane inclined at an angle **0** to ***9,***is given by the point R.

The co-ordinates of the point ***R(o,***T) give the direct and shearing stresses on the plane. We may write the above equations in the forms

$$\sigma - \frac{1}{2}(\sigma_x + \sigma_y) = \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$-\tau = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

Square each equation and add; then we have

$$\left[\sigma - \frac{1}{2}(\sigma_x + \sigma_y)\right]^2 + \tau^2 = \left[\frac{1}{2}(\sigma_x - \sigma_y)\right]^2 + \left[\tau_{xy}\right]^2$$

Thus all corresponding values of **o**and ***T*** lie on a circle of radius

$$\sqrt{\left[\frac{1}{2}(\sigma_x - \sigma_y)\right]^2 + \tau_{xy}^2}$$

with its centre at the point $(\frac{1}{2}[\sigma_x + \sigma_y], 0)$ Figure**.**

This circle defining all possible states of stress is known as Mohr's Circle ***of stress;*** the principal stresses are defined by the points A and ***E*** **,** at which T = 0. The maximum shearing stress, which is given by the point C, is clearly the radius of the circle.

Principal stresses and strains

We have seen that in a two-dimensional system of stresses there are always two mutually perpendicular directions in which there are no shearing stresses; the direct stresses on these planes were referred to as principal stresses, IS, and **IS*.** As there are no shearing stresses in these two mutually perpendicular directions, there are also no shearing strains; for the principal directions the corresponding direct strains are given by

$$E\varepsilon_1 = \sigma_1 - \nu\sigma_2$$

$$E\varepsilon_2 = \sigma_2 - \nu\sigma_1$$

The direct strains, **E,, E,,** are the principal strains already discussed in Mohr‘s circle of strain. It follows that the principal strains occur in directions parallel to the principal stresses.