B-Spline Curves:

⇒ These are most widely used class of approximating splines.

⇒ Advantages over Bezier Curves

1) The degree of a B-spline polynomial can be set independently of the number of control points. (with certain limitations)

2) B-spline allow local control over the shape of a spline curve or surface.

⇒ B-Spline are more complex than Beziers

⇒ B-spline curves:

$$P(u) = \sum_{k=0}^{n} P_k B_{k,d}(u) \quad u_{min} \le u \le u_{max} \quad ①$$

$$2 \le d \le n+1$$

P_k ← input set of $(n+1)$ control points

$B_{k,d}$ → are polynomial of degree $(d-1)$

$$B_{k,1}(u) = \begin{cases} 1, & \text{is } u_k \le u < u_{k+1} \\ 0, & \text{otherwise} \end{cases}$$

$$B_{k,d}(u) = \frac{u - u_k}{u_{k+d-1} - u_k} B_{k,d-1}(u) + \frac{u_{k+d} - u}{u_{k+d} - u_{k+1}} B_{k+1,d-1}(u)$$

⇒ where each blending function is defined over d subintervals of the ~~total~~ range of u.

⇒ The selected set of subinterval endpoints u_j is referred to as knot vector.

⇒ We can choose any values ~~for u_{min} &~~ ~~u_{max}~~ for the subintervals endpoints satisfying relation $u_j \leq u_{j+1}$

⇒ values for u_{min} & u_{max} then depend on the number of control points we ~~select~~ select.

⇒ Properties of B-spline curves:

1) The Polynomial curve has degree $(d-1)$ & C^{d-2} continuity over the range of u.

2) for $(n+1)$ control points, the curve is described with $(n+1)$ blending functions.

3) Each blending function $B_{k,d}$ is defined over d subintervals of the total range of u, starting at knot value u_k.

4) The range of parameter u is divided into $(n+d)$ subintervals by the $(n+d+1)$ values specified in the knot vector.

5] With knot values labeled as $[u_0, u_1, \ldots u_{n+d}]$ the resulting B-spline curve is defined only in the interval from knot value u_{d-1} upto knot value u_{n+1}.

6) Each section of the spline curve (b/w two successive knot values) is influenced by d control points.

7) Any one control point can affect the shape of at most d - curve sections.

8) $$\sum_{k=0}^{n} B_{k,d}(u) = 1$$

$$u_{d-1} \le u \le u_{n+1}$$

⇒ 1) Uniform, Periodic B-splines:

Knot vector: $[-1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5, 2.00]$

uniform → $\begin{cases} [0.0, 0.2, 0.4, 0.6, 0.8, 1.0] \\ [0, 1, 2, 3, 4, 5, 6, 7] \end{cases}$

$$B_{k,d}(u) = B_{k+1,d}(u+\Delta u)$$
$$= B_{\ldots, d}(u+2\Delta u)$$

3). Calculation of B-spline blending functions for uniform, integer knot vector

Sol: ~~delet~~ $d = n = 3$

Knot values $= n+d+1 = 3+3+1 = 7$

Knot vector $= [0, 1, 2, 3, 4, 5, 6]$

$u \rightarrow$ 0 to 6

with $n+d \rightarrow$ subintervals

$3+3 = 6$ subintervals

=> Each of the four blending functions spans $d = 3$ subintervals of the total range of u.

eqn (2)
$$B_{0,3}(u) = \begin{cases} \frac{1}{2}u^2 & 0 \le u \le 1 \\ \frac{1}{2}(u)(2-u) + \frac{1}{2}(u-1)(3-u) & 1 < u \le 2 \\ \frac{1}{2}(3-u)^2 & 2 < u \le 3 \end{cases}$$

using eqn (2).

$$B_{0,3}(u) = \frac{u-u_0}{u_2-u_0} B_{0,2}(u) + \frac{u_3-u}{u_3-u_1} B_{1,2}(u)$$

$$B_{0,2}(u) = \frac{u-u_0}{u_1-u_0} \underset{\uparrow\, 1}{B_{0,1}(u)} + \underset{\uparrow\, \text{zero}}{\frac{u_2-u}{u_2-u_1}} B_{1,1}(u)$$

$$B_{1,2}(u) = \frac{u-u_1}{u_2-u_1} B_{1,1}(u) + \frac{u_3-u}{u_3-u_2} B_{2,1}(u)$$

$$B_{0,3}(u) = \frac{u-u_0}{u_2-u_0}\cdot\frac{(u-u_0)}{(u_1-u_0)} + \frac{u_3-u}{u_3-u_1}\underbrace{B_{1,2}(u)}_{\text{zero}}$$

$$= \frac{u-0}{2-0}\cdot\frac{u-0}{1-0}$$

$$= \frac{u^2}{2} = \tfrac{1}{2}u^2$$

$$B_{1,3} = \begin{cases} \frac{1}{2}(u-1)^2 & 1 \le u < 2 \\ \frac{1}{2}(u-1)(3-u) + \frac{1}{2}(u-2)(4-u) & 2 \le u < 3 \\ \frac{1}{2}(4-u)^2 & 3 \le u < 4 \end{cases}$$

$$B_{2,3}(u) = \begin{cases} \frac{1}{2}(u-2)^2 & 2 \le u < 3 \\ \frac{1}{2}(u-2)(4-u) + \frac{1}{2}(u-3)(5-u) & 3 \le u < 4 \\ \frac{1}{2}(5-u)^2 & 4 \le u < 5 \end{cases}$$

$$B_{3,3}(u) = \begin{cases} \frac{1}{2}(u-3)^2 & 3 \le u < 4 \\ \frac{1}{2}(u-3)(5-u) + \frac{1}{2}(u-4)(6-u) & 4 \le u < 5 \\ \frac{1}{2}(6-u)^2 & 5 \le u < 6 \end{cases}$$

1) Uniform
2) Open Uniform
3) Non-Uniform

2) <u>Open Uniform</u>: $\{0, 0, 1, 2, 3, 3\}$, $d=2$, $n=3$

$\{0, 0, 0, 0, 1, 2, 2, 2, 2\}$ $d=4, n=4$

Open uniform Knot vector.

$$u_j \begin{cases} 0 & \text{for } 0 \le j \le d \\ j-d+1 & d < j \le n \\ n-d+2 & \text{for } j > n \end{cases}$$

Ex.. $d=3$, $n=4$

$\{u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$
$= \{0, 0, 0, 1, 2, 3, 3, 3\}$

$B_{0,3}(u) \Rightarrow K=0, d=3$ $\quad u_0 \le u \le u_3$
$0 \le u \le 1$ $\quad (1-u)^2$

$B_{1,3}(u) \Rightarrow K=1, d=3$, $\quad$ ~~$u_1 \le u \le u_4$~~

$$\begin{cases} \frac{1}{2} u (4-3u) & 0 \le u < 1 \\ \frac{1}{2}(2-u)^2 & 1 \le u < 2 \end{cases}$$

$$B_{2,3}(u) = \begin{cases} \frac{1}{2}u^2 & 0 \le u < 1 \\ \frac{1}{2}u(2-u) + \frac{1}{2}(u-1)(3-u) & 1 \le u < 2 \\ \frac{1}{2}(3-u)^2 & 2 \le u < 3 \end{cases}$$

$$B_{3,3}(u) = \begin{cases} \frac{1}{2}(u-1)^2 & 1 \le u < 2 \\ \frac{1}{2}(3-u)(3u-5) & 2 \le u < 3 \end{cases}$$

$$B_{4,3}(u) = (u-2)^2 \qquad 2 \le u < 3$$

$\Rightarrow$ <u>Non-Uniform:</u>

$\{0, 1, 2, 3, 3, 4\}$

<u>Ex:</u> $\{0, 0, 0, 1, 1, 3, 3, 3\}$

$\{0, 2, 2, 3, 3, 6\}$

...t Periodic

$n+1=4$

$P(u) = p_0\left(\frac{1}{2}u^2\right) + p_1$ $\quad \{0,1,2,3,4,5,6,7\}$

$u_0 \quad u_1 \quad u_2 \quad u_3 \quad u_4$

$d-1=3$

$d=4 \quad n+1 =$ control points

$$B_{0,4} = \frac{u-u_0}{u_3-u_0} B_{0,3}(u) + \frac{u_4-u}{u_4-u_1} B_{1,3}(u)$$

$$B_{0,3}(u) = \frac{u-u_0}{u_2-u_0} B_{0,2}(u) + \frac{u_3-u}{u_3-u_1} B_{1,2}(u)$$

$$B_{0,2}(u) = \frac{u-u_0}{u_1-u_0} B_{0,1}(u) + \frac{u_2-u}{u_2-u_1} B_{1,1}(u)$$

$$B_{1,3}(u) = \frac{u-u_1}{u_3-u_1} B_{1,2}(u) + \frac{u_4-u}{u_4-u_2} B_{2,2}(u)$$

$$B_{1,2}(u) = \frac{u-u_1}{u_2-u_1} B_{1,1}(u) + \frac{u_3-u}{u_3-u_2} B_{2,1}(u)$$

$$= \frac{u-u_0}{u_3-u_0}\left[\frac{u-u_0}{u_2-u_0}\left[\frac{u-u_0}{u_1-u_0}\right]\right]$$

$$= \frac{u-0}{3-0}\left[\frac{u-0}{2-0}\left[u-0\right]\right]$$

$$= \frac{1}{6}u^3$$

$\Rightarrow$ Open-Uniform Quadratic B-Splines:

(2)

Ex: $d=3$, $n=4$

$$\{u_0, u_1, u_2, \dots u_7\} = \{0,0,0,1,2,3,3,3\}$$

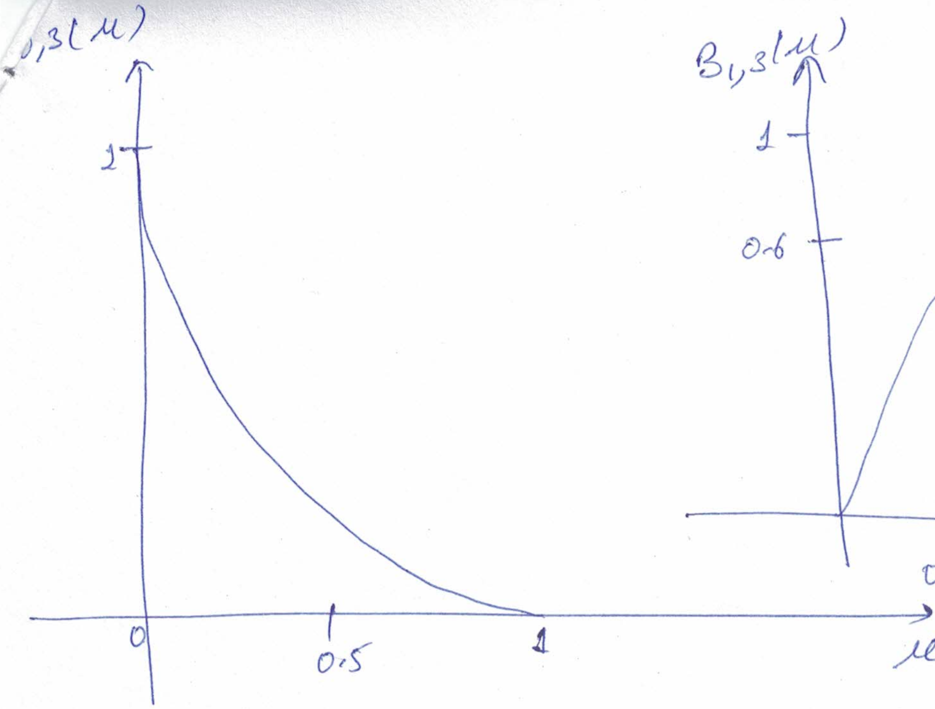
1) $B_{0,3} = (1-u)^2 \qquad 0 \le u \le 1$

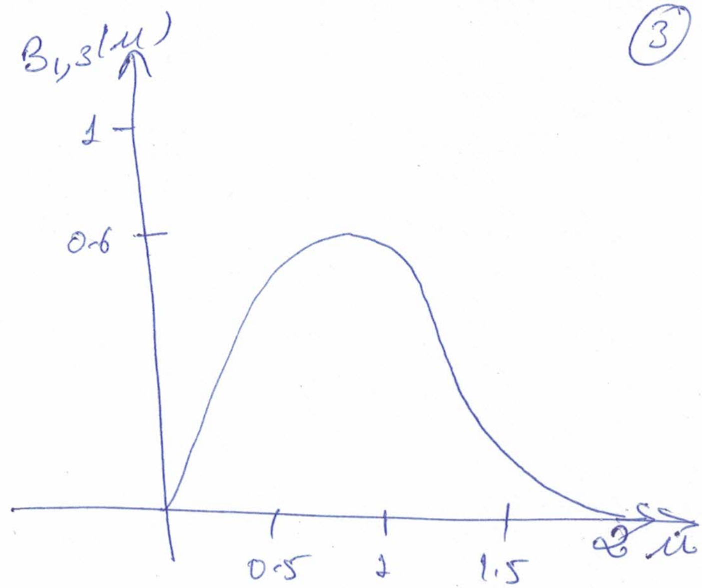
2) $$B_{1,3}(u) = \begin{cases} \frac{1}{2}u(4-3u) & 0 \le u \le 1 \\ \frac{1}{2}(2-u)^2 & 1 \le u \le 2 \end{cases}$$

3) $$B_{2,3}(u) \begin{cases} \frac{1}{2}u^2 & 0 \le u \le 1 \\ \frac{1}{2}u(2-u) + \frac{1}{2}(u-1)(3-u) & 1 \le u \le 2 \\ \frac{1}{2}(3-u)^2 & 2 \le u < 3 \end{cases}$$

4) $$B_{3,3}(u) \begin{cases} \frac{1}{2}(u-1)^2 & 1 \le u \le 2 \\ \frac{1}{2}(3-u)(3u-5) & 2 \le u < 3 \end{cases}$$

5) $B_{4,3}(u) = (u-2)^2 \qquad 2 \le u < 3$


0,3(u)
1
0
0.5
1
u


B1,3(u)
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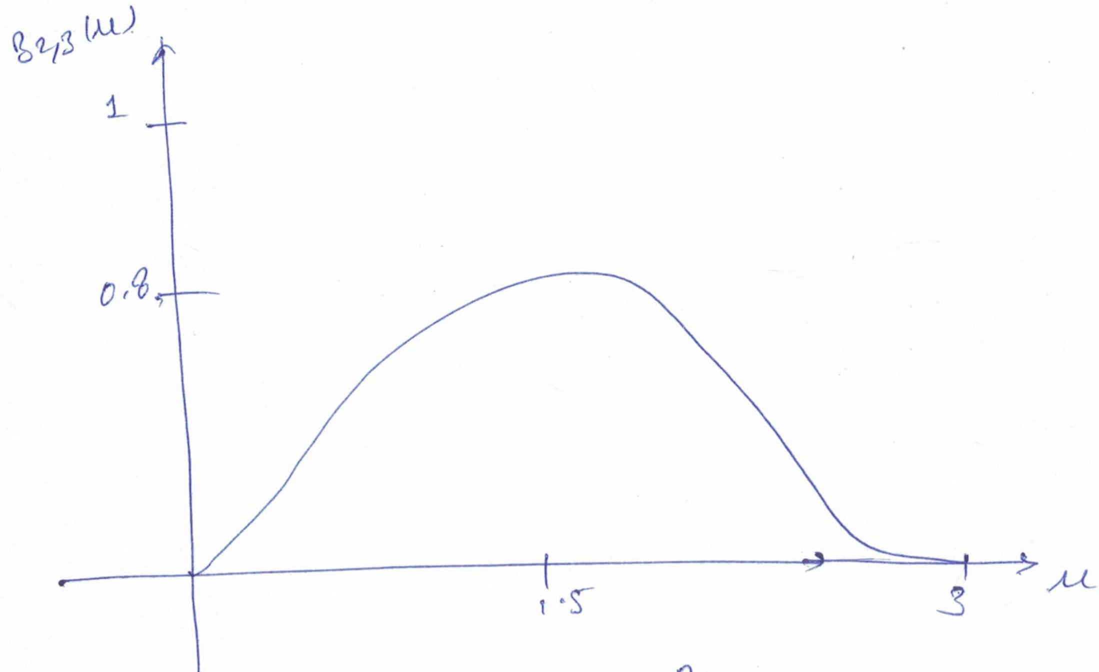
0.6
0.5
1
1.5
2
u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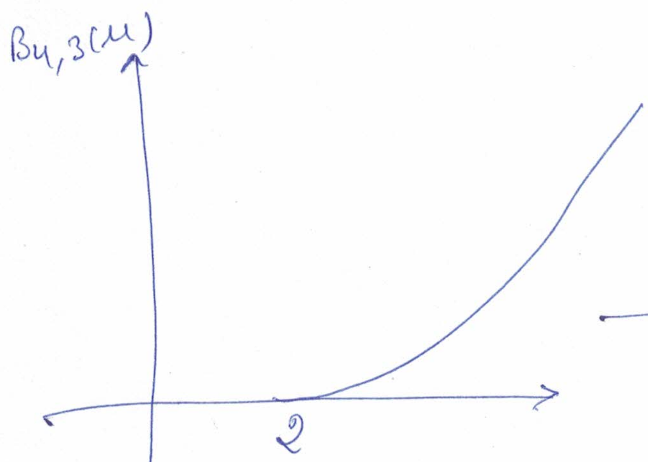
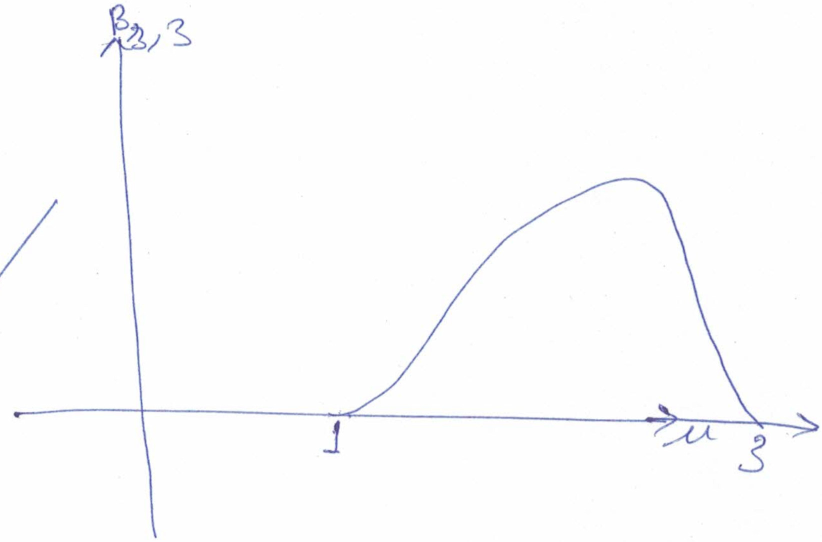
1.5
3
u

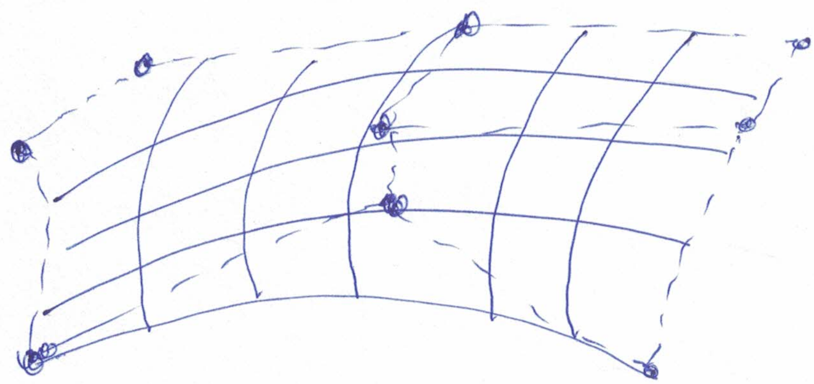

B4,3(u)
2


B3,3
1
u
3

1. Bezier Surfaces:

$$P(u,v) = \sum_{j=0}^{m} \sum_{k=0}^{n} P_{j,k} \, BEZ_{j,m}(v) \, BEZ_{k,n}(u)$$

⇒ $(m+1)$ by $(n+1)$ control points



2. B-spline Surfaces:

$$P(u,v) = \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} P_{k_1,k_2} \, B_{k_1,d_1}(u) \, B_{k_2,d_2}(v)$$

⇒ (n_1+1) by (n_2+1) ⇒ control points

⇒ d_1 & d_2 ⇒ degree of polynomial

$$2 \le d_1 \le n_1$$
$$2 \le d_2 \le n_2$$

$$k_1 \le u \le k_1+1$$
$$k_2 \le v \le k_2+1$$