JAIPUR ENGINEERING COLLEGE AND RESEARCH CENTRE

Year & Sem. – B. Tech II year, Sem.-IV

Subject –Discrete Mathematics Structure, Unit – 1

Topic – Relations

Presented by – Dr. Kashish Parwani

Designation - Associate Professor

Department - Mathematics

VISION OF INSTITUTE

To become a renowned centre of outcome based learning, and work towards academic, professional, cultural and social enrichment of the lives of individuals and communities.

MISSION OF INSTITUTE

❖**Focus on evaluation of learning outcomes and motivate students to inculcate research aptitude by project based learning.**

❖**Identify, based on informed perception of Indian, regional and global needs, the areas of focus and provide platform to gain knowledge and solutions.**

❖**Offer opportunities for interaction between academia and industry.**

❖**Develop human potential to its fullest extent so that intellectually capable and imaginatively gifted leaders may emerge in a range of professions.**

Engineering Mathematics: Course Outcomes

Students will be able to:

Upon successful completion of this course students will be able to:

CO1. Understand the concepts of Sets, Relations, Functions and their Operations.

CO2. Learn the concept of Propositional Logic and Finite State Machines.

CO3. Discuss and develop the Posets, Hasse Diagram, Lattices and Combinatorics.

CO4. Use and apply the concept of Algebraic Structures, Groups, Rings and Graph Theory.

Vision and Mission of the Institute

Vision of the Institute

To become a renowned centre of outcome based learning, and work towards academic, professional, cultural and social enrichment of the lives of individuals and communities.

Mission of the Institute

- Focus on evaluation of learning outcomes and motivate students to inculcate research aptitude by project based learning.
- Identify, based on informed perception of Indian, regional and global needs, the areas of focus and provide platform to gain knowledge and solutions.
- Offer opportunities for interaction between academia and industry.
- Develop human potential to its fullest extent so that intellectually capable and imaginatively gifted leaders may emerge.

RTU Scheme

4IT2-01: **Discrete Mathematics Structure**
Class: IVth Sem. B.Tech.
Branch: Information Technology
Schedule per Week- Lectures: 3
Examination Time = Three (3) Hours
Maximum Marks = 150
Evaluation: [Mid-terms (24), Assignments (06), External (120)]

4IT2-01: Discrete Mathematics Structure

Credit: 3 **Max. Marks: 150(IA:30, ETE:120)**

3L+0T+0P **End Term Exam: 3 Hours**

SN	Contents	Hours
1	**Introduction:** Objective, scope and outcome of the course.	1
2	Set Theory: Definition of sets, countable and uncountable sets, Set operations, Partition of set, Cardinality (Inclusion-Exclusion & Addition Principles) Venn Diagrams, proofs of some general identities on sets. Relation: Definition, types of relation, composition of relations, Pictorial representation of relation, Equivalence relation, Partial ordering relation, Job-Scheduling problem. Function: Definition, type of functions, one to one, into and onto function, inverse function, composition of functions, recursively defined functions, pigeonhole principle. Theorem proving Techniques: Mathematical induction, Proof by contradiction. Composition of Functions. The Pigeonhole and Generalized Pigeonhole Principles.	7
3	Propositional Logic: Proposition, First order logic, Basic logical operation, truth tables, tautologies, Contradictions, Algebra of Proposition, logical implications, logical equivalence, predicates, Normal Forms, Universal and existential quantifiers. 2 way predicate logic. Introduction to finite state machine Finite state machines as models of physical system equivalence machines, Finite state machines as language recognizers.	8
4	Posets, Hasse Diagram and Lattices: Introduction, ordered set, Hasse diagram of partially, ordered set, isomorphic ordered set, well ordered set, properties of Lattices, bounded and complemented lattices. Combinatorics: Introduction, Permutation and combination, Binomial Theorem, Multimodal Coefficients Recurrence Relation and Generating Function: Introduction to Recurrence Relation and Recursive algorithms, linear recurrence relations with constant coefficients, Homogeneous solutions, Particular solutions, Total solutions, Generating functions, Solution by method of generating functions.	8
5	Algebraic Structures: Definition, Properties, types: Semi Groups, Monoid, Groups, Abelian group, properties of groups, Subgroup, cyclic groups, Cosets, factor group, Permutation groups, Normal subgroup, Homomorphism and isomorphism of Groups, example and standard results, Rings and Fields: definition and standard results.	8
6	Graph Theory: Introduction and basic terminology of graphs, Planer graphs, Multigraphs and weighted graphs, Isomorphic graphs, Paths, Cycles and connectivity, Shortest path in weighted graph, Introduction to Eulerian paths and circuits, Hamiltonian paths and circuits, Graph coloring, chromatic number, Isomorphism and Homomorphism of graphs, matching, vertex/edge covering.	8
	Total	**40**

Dr. Kashish Parw
JECRC, Departme

COURSE OUTCOMES:

Subject – Discrete Structure Mathematics
Code – 4IT2-01
Branch – Information Technology
Semester- IVth
Upon successful completion of this course students will be able to:
By the end of this course, the students will be able to:

CO1. Understand the concepts of Sets, Relations, Functions and their Operations.

CO2. Learn the concept of Propositional Logic and Finite State Machines.

CO3. Discuss and develop the Posets, Hasse Diagram, Lattices and Combinatorics.

CO4. Use and apply the concept of Algebraic Structures, Groups, Rings and Graph Theory.

CO/PO	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
CO1	H	L	-	-	-	-	-	-	L	-	-	M
CO2	H	L	-	-	-	-	-	-	L	-	-	M
CO3	H	L	-	-	-	-	-	-	L	-	-	M
CO4	H	L	-	-	-	-	-	-	L	-	-	M

Relations

Introduction

Relationships between elements of sets are represented using the structure called a **relation**.

- Relationship between a program and its variables
- Pairs of cities linked by airline flights in a network

Relations

The most direct way to express a relationship between elements of two sets is to use ordered pairs.

For this reason, sets of ordered pairs are called binary relations.

Definition:

Let A and B be sets. A binary relation from A to B is a subset R of $A \times B = \{ (a, b) : a \in A, b \in B \}$.

Relations on a Set

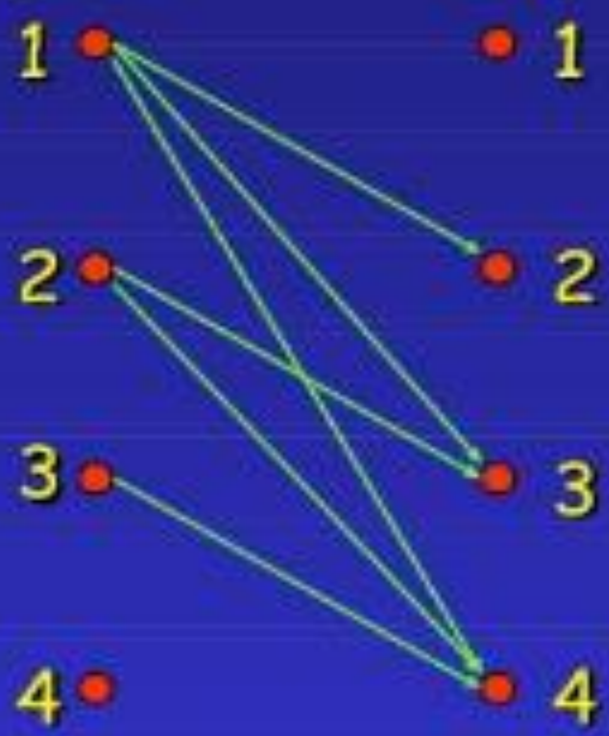
Definition: A relation on the set A is a relation from A to A.

In other words, a relation on the set A is a subset of $A \times A$.

Example: Let $A = \{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a < b\}$?

Relations on a Set

Solution: $R = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$

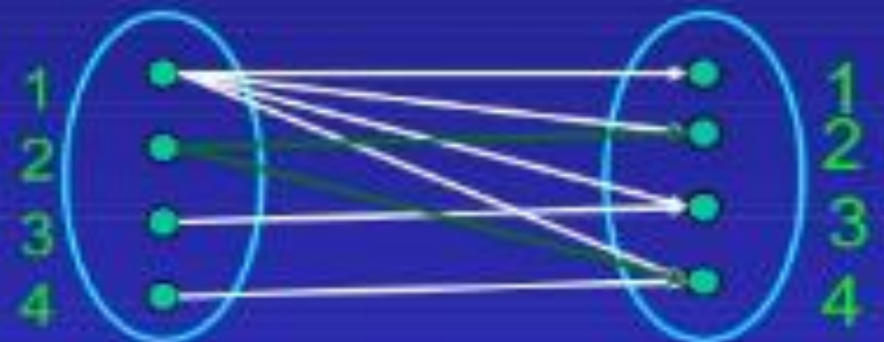


R	1	2	3	4
1		X	X	X
2			X	X
3				X
4				

Example -1.

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{ (a, b) | a \text{ divides } b \}$?

Sol : Since (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b



$$R = \{ (1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4) \}$$

Relation properties

Six properties of relations we will study:

- Reflexive
- Irreflexive
- Symmetric
- Asymmetric
- Antisymmetric
- Transitive

Dr. Kashish Parwani,
JECRC, Department of Mathematics

Reflexivity

In some relations an element is always related to itself

Let R be the relation on the set of all people consisting of pairs (x,y) where x and y have the same mother and the same father. Then $x R x$ for every person x

Definition: A relation R on a set A is called **reflexive** if $(a, a) \in R$ for every element $a \in A$.

Example

Example: Consider the following relations on {1, 2, 3, 4}

R_1 = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}
R_2 = {(1,1), (1,2), (2,1)}
R_3 = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)}
R_4 = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}
R_5 = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}
R_5 = {(3,4)}

Which of these relations are reflexive?

Directed Graph Of A Reflexive Relation

The directed graph of every reflexive relation includes an arrow from every point to the point itself (i.e., a loop).

EXAMPLE :

Let A = {1, 2, 3, 4} and define relations R1, R2, R3, and R4 on A by

- R1 = {(1, 1), (3, 3), (2, 2), (4, 4)}
- R2 = {(1, 1), (1, 4), (2, 2), (3, 3), (4, 3)}
- R3 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}
- R4 = {(1, 3), (2, 2), (2, 4), (3, 1), (4, 4)}

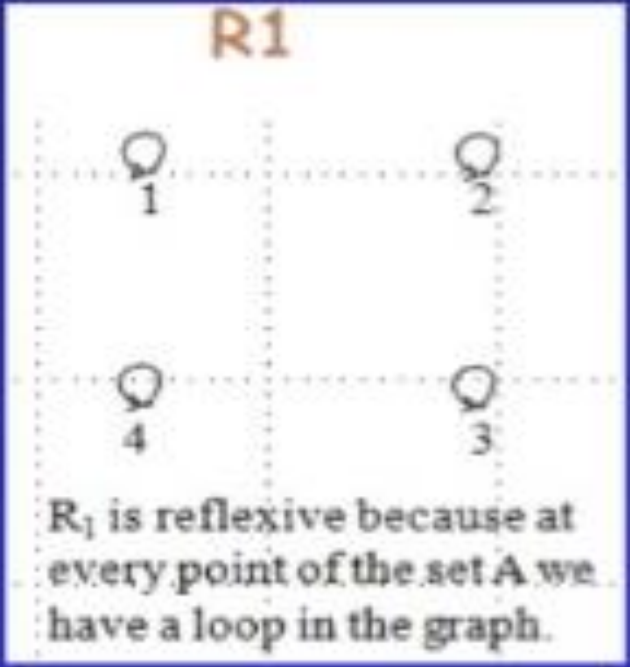
Then their directed graphs are

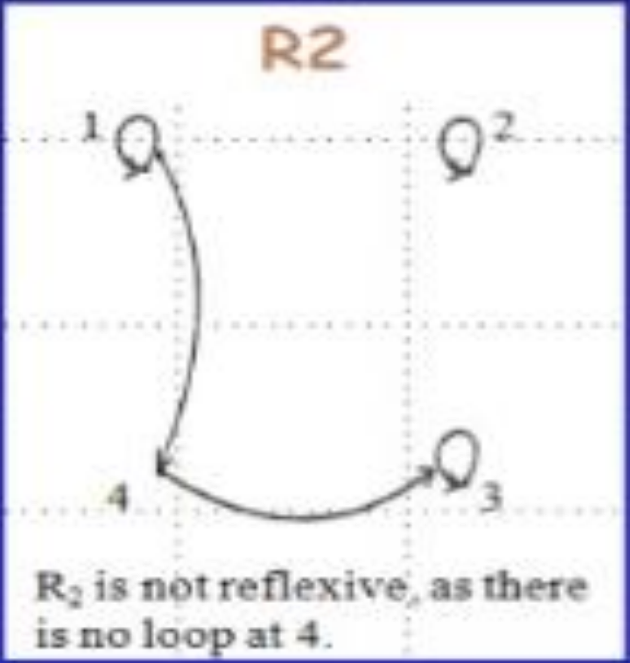
Dr. Kashish Parwani,
JECRC, Department of Mathematics

Directed Graphs Are

R1 = {(1, 1), (3, 3), (2, 2), (4, 4)}

R2 = {(1, 1), (1, 4), (2, 2), (3, 3), (4, 3)}

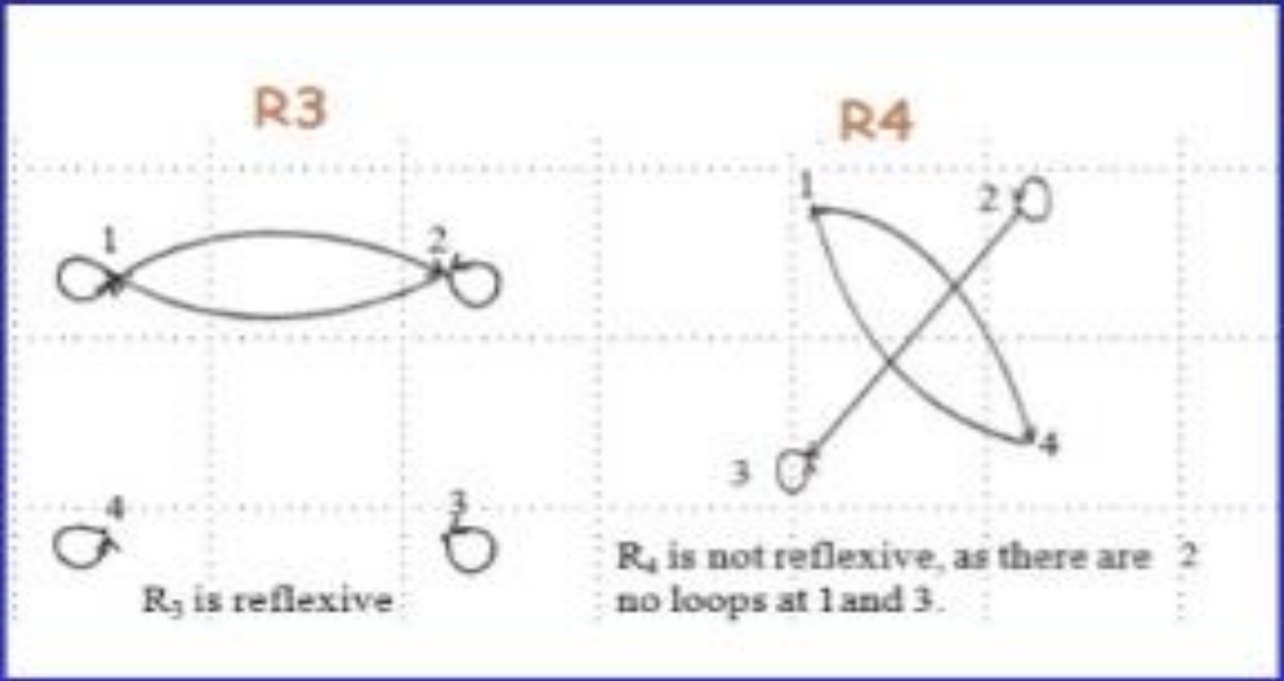




Directed Graphs Are

R3 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)}

R4 = {(1, 3), (2, 2), (2, 4), (3, 1), (4, 4)}



Matrix Representation Of A Reflexive Relation

- Let $A = \{a1, a2, \ldots, an\}$. A Relation R on A is reflexive if and only if $(ai, ai) \in R \ \forall \ i=1,2, \ldots, n$.

- Accordingly, R is **reflexive** if all the elements on the **main diagonal** of the matrix **M** representing R are equal to 1.
- The relation $R = \{(1,1), (1,3), (2,2), (3,2), (3,3)\}$ on $A = \{1,2,3\}$ represented by the following matrix M, is reflexive.

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

irreflexive

A relation is irreflexive if every element is *not* related to itself

- Or, $(a,a) \notin R$
- Irreflexivity is the opposite of reflexivity

Examples of irreflexive relations:

- $<$, $>$

Examples of relations that are not irreflexive:

- $=$, $\leq$, $\geq$

Symmetric

Definitions:

A relation R on a set A is called **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

A relation R on a set A is called **antisymmetric** if $a = b$ whenever $(a, b) \in R$ and $(b, a) \in R$.

Antisymmetric

A relation is antisymmetric if, for every $(a,b) \in R$, then $(b,a) \in R$ is true only when $a=b$

- Antisymmetry is *not* the opposite of symmetry

Examples of antisymmetric relations:

- $=$, $\leq$, $\geq$

Examples of relations that are not antisymmetric:

- $<$, $>$, isTwinOf()

Asymmetric

A relation is asymmetric if, for every $(a,b) \in R$, then $(b,a) \notin R$

- Asymmetry is the opposite of symmetry

Examples of asymmetric relations:

- $<, >$

Examples of relations that are not asymmetric:

- $=$, isTwinOf(), $\leq$, $\geq$

Example

Which are symmetric and antisymmetric

$R_1=\{(a,b)|a\leq b\}$

$R_2=\{(a,b)|a>b\}$

$R_3=\{(a,b)|a=b \text{ or } a=-b\}$

$R_4=\{(a,b)|a=b\}$

$R_5=\{(a,b)|a=b+1\}$

$R_5=\{(a,b)|a+b\leq 3\}$

Symmetric: R_3, R_4, R_6. R_3 is symmetric, if $a=b$ (or $a=-b$), then $b=a$ ($b=-a$), R_4 is symmetric as $a=b$ implies $b=a$, R_6 is symmetric as $a+b\leq 3$ implies $b+a\leq 3$

Antisymmetric: R_1, R_2, R_4, R_5. R_1 is antisymmetric as $a\leq b$ and $b\leq a$ imply $a=b$. R_2 is antisymmetric as it is impossible to have $a>b$ and $b>a$, R_4 is antisymmteric as two elements are related w.r.t. R_4 if and only if they are equal. R_5 is antisymmetric as it is impossible

Symmetric

Let A = {1, 2, 3, 4} and define relations R1, R2,

R3, and R4on A as follows.

- R1 = {(1, 1), (1, 3), (2, 4), (3, 1), (4,2)}
- R2 = {(1, 1), (2, 2), (3, 3), (4, 4)}
- R3 = {(2, 2), (2, 3), (3, 4)}
- R4 = {(1, 1), (2, 2), (3, 3), (4, 3), (4, 4)}
  - Then R1 is symmetric because for every order pair (a,b) in R1, we have (b,a) in R1, for example we have (1,3) in R1 then we have (3,1) in R1. similarly all other ordered pairs can be checked.
  - R2 is also symmetric.
  - R3 is not symmetric, because (2,3) $\in$ R3 but (3,2) $\notin$ R3.
  - R4 is not symmetric because (4,3) $\in$ R4 but (3,4) $\notin$ R4

Directed Graph Of A Symmetric Relation

For a symmetric directed graph whenever there is an arrow going from one point of the graph to a second, there is an arrow going from the second point back to the first.

EXAMPLE

- Let A = {1, 2, 3, 4} and define relations R1, R2, R3, and R4 on A by the directed graphs:
  - R1 = {(1, 1), (1, 3), (2, 4), (3, 1), (4,2)}
  - R2 = {(1, 1), (2, 2), (3, 3), (4, 4)}
  - R3 = {(2, 2), (2, 3), (3, 4)}
  - R4= {(1, 1), (2, 2), (3, 3), (4, 3), (4, 4)}

Directed Graph Of A Symmetric Relation

Both of above relations are Symmetric

Matrix Representation Of A Symmetric Relation

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

Your Task

For each of these relations on the set{1,2,3,4}, decide Whether it is Symmetric, Antisymmetric.

a) {(2,2), (2,3), (2,4), (3,2), (3,3), (3,4)}
b){(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)}
c) {(2,4), (4,2)}
d){(1,2), (2,3), (3,4)}
e) {(1,1), (2,2), (3,3),(4,4)}
f) {(1,3), (1,4), (2,3), (2,4), (3,1), (3,4)

Symmetric: b , c ,e

Antisymmetric: d , e

Transitive

Definition: A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$.

If $a < b$ and $b < c$, then $a < c$

- Thus, $<$ is transitive

If $a = b$ and $b = c$, then $a = c$

- Thus, $=$ is transitive

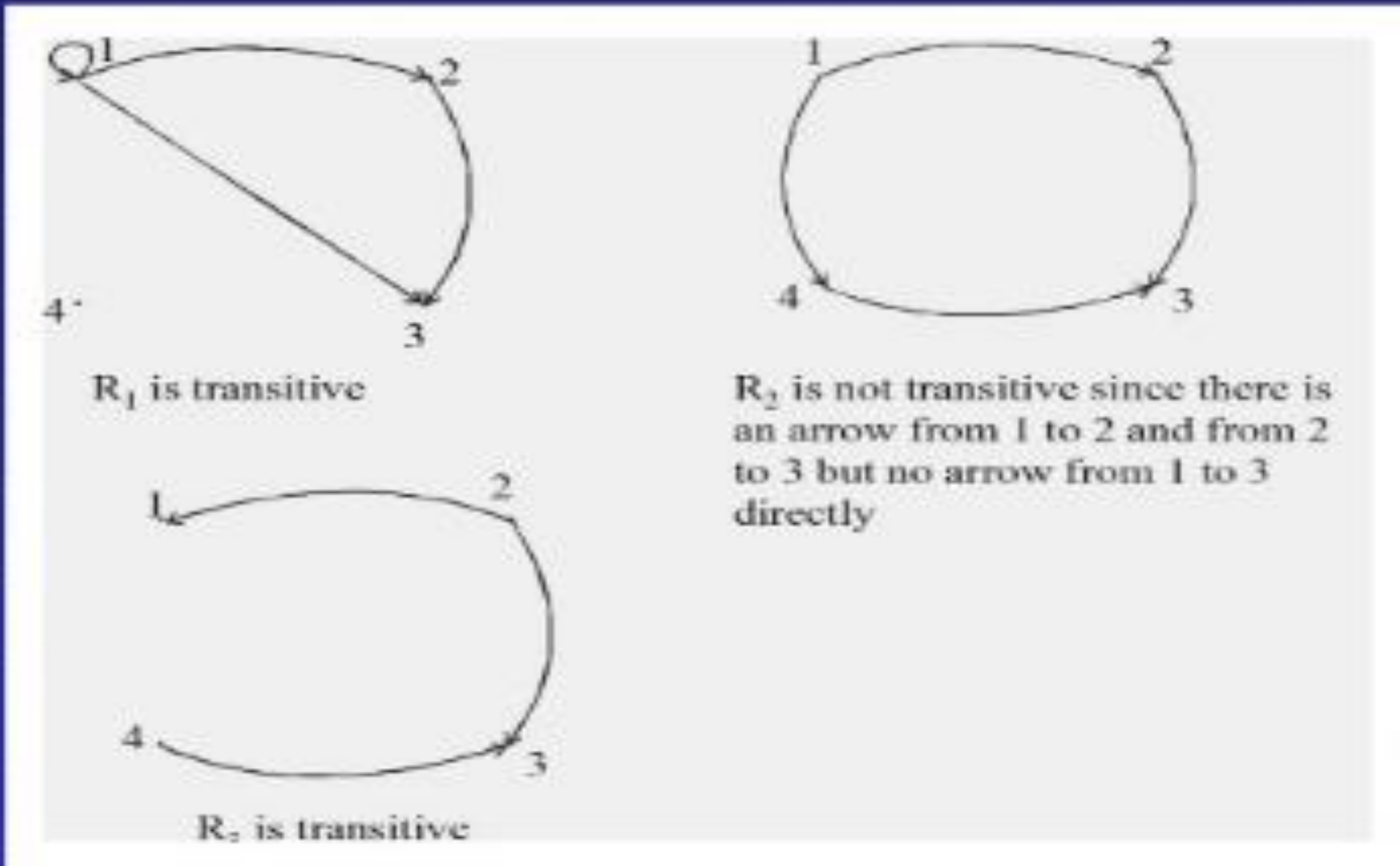
Directed Graph Of A Transitive Relation

For a transitive directed graph, whenever there is an arrow going from one point to the second, and from the second to the third, there is an arrow going directly from the first to the third.

EXAMPLE

- Let A = {1, 2, 3, 4} and define relations R1, R2 and R3 on A by the directed graphs:
- R1 = {(1, 1), (1, 2), (1, 3), (2, 3)}
- R2 = {(1, 2), (1, 4), (2, 3), (3, 4)}
- R3 = {(2, 1), (2, 4), (2, 3), (3,4)}

Directed Graph Of A Transitive Relation



Dr. Kashish Parwani,
JECRC, Department of Mathematics

Representing Relations Using Digraphs

Definition: A directed graph, or digraph, consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs).

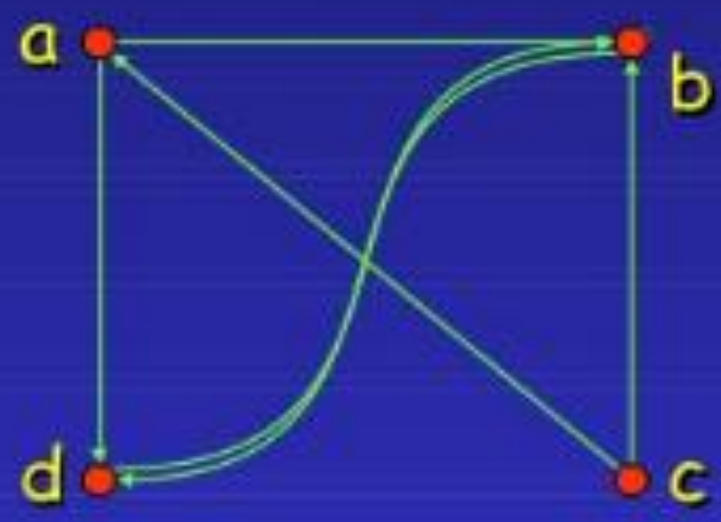
The vertex a is called the initial vertex of the edge (a, b), and the vertex b is called the terminal vertex of this edge.

We can use arrows to display graphs.

Dr. Kashish Parwani,
JECRC, Department of Mathematics

Representing Relations Using Digraphs

Example: Display the digraph with V = {a, b, c, d}, E = {(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)}.



An edge of the form (b, b) is called a **loop**.

Representing Relations

The Boolean operations **join** and **meet** (you remember?) can be used to determine the matrices representing the **union** and the **intersection** of two relations, respectively.

To obtain the **join** of two zero-one matrices, we apply the Boolean "or" function to all corresponding elements in the matrices.

To obtain the **meet** of two zero-one matrices, we apply the Boolean "and" function to all corresponding elements in the matrices.

Representing Relations

Example: Let the relations R and S be represented by the matrices

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing R□S and R□S?

Solution: These matrices are given by

$$M_{R \cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad M_{R \cap S} = M_R \wedge M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

85

Representing Relations Using Matrices

Example: Find the matrix representing R^2, where the matrix representing R is given by

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Solution: The matrix for R^2 is given by

$$M_{R^2} = M_R^{[2]} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

91

Equivalence Relations

Equivalence relations are used to relate objects that are similar in some way.

Definition: A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

Two elements that are related by an equivalence relation R are called **equivalent**.

Equivalence Relations

Since R is symmetric, a is equivalent to b whenever b is equivalent to a.

Since R is reflexive, every element is equivalent to itself.

Since R is transitive, if a and b are equivalent and b and c are equivalent, then a and c are equivalent.

Obviously, these three properties are necessary for a reasonable definition of equivalence.

96

Equivalence Relations

Example: Suppose that R is the relation on the set of strings that consist of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

Solution:

- R is reflexive, because $l(a) = l(a)$ and therefore aRa for any string a.
- R is symmetric, because if $l(a) = l(b)$ then $l(b) = l(a)$, so if aRb then bRa.
- R is transitive, because if $l(a) = l(b)$ and $l(b) = l(c)$, then $l(a) = l(c)$, so aRb and bRc implies aRc.

R is an equivalence relation.

Practice Question:

1. Prove that the composite of two bijections is a bijection.

2. Consider a set A = {a, b, c, d, e, f,} and a relation R defined on A given by R{(a,a),(a,b),(b,a),(b,b),(c,c),(d,d),(d,e),(d,f),(e,d)(e,e),(e,f),(f,d),(f,e),(f,f)}.Write the matrix representation M_R of the relation and hence prove that it is an equivalence relation.

3. Given A={1,2,3,4}and R=(1,1),(1,3),(1,4),(3,2),(4,2),(4,4) represent relation by using diagram.

4. Enlist properties of equivalence relations.

5.6. Given A={1,2,3,4}and R=(1,1),(1,3),(1,4),(3,2),(4,2),(4,4),consider relation by diagram.

Suggested links from NPTEL & other Platforms:

1. https://nptel.ac.in/courses/111/106/111106086/
2. https://nptel.ac.in/courses/111/107/111107058/
3. https://nptel.ac.in/courses/106/106/106106183/#

JECRC Foundation

JAIPUR ENGINEERING COLLEGE AND RESEARCH CENTRE

Thank you!

Dr. Kashish Parwani,
JECRC, Department of Mathematics