JAIPUR ENGINEERING COLLEGE AND RESEARCH CENTRE

Year & Sem – I Year & II Sem
Subject –Engineering Mathematics
Unit –III (Ordinary differential equations of higher orders)
Presented by – (Dr. Tripati Gupta, Associate Professor)

VISION AND MISSION OF INSTITUTE

VISION OF INSTITUTE

To became a renowned centre of outcome based learning and work towards academic professional, cultural and social enrichment of the lives of individuals and communities .

MISSION OF INSTITUTE

- Focus on evaluation of learning, outcomes and motivate students to research aptitude by project based learning.
- Identify based on informed perception of Indian, regional and global needs, the area of focus and provide platform to gain knowledge and solutions.
- Offer opportunities for interaction between academic and industry .
- Develop human potential to its fullest extent so that intellectually capable and imaginatively gifted leaders may emerge.

Engineering Mathematics: Course Outcomes

Students will be able to:

CO-1	To understand the concept of rank of matrix, inverse, Eigen values & vectors along with solution of linear simultaneous equation determine inverse of a matrix using Cayley Hamilton Theorem
CO-2	To solve Ordinary D.E of first order, first degree and first order higher degree using various methods
CO-3	To find the complete solution of D.E of higher order with constant coefficient & variable coefficients & their methods of solution.
CO-4	To solve partial differential equations with its applications in Laplace equation, Heat & Wave equation

CONTENTS (TO BE COVERED)

Ordinary differential equations of higher orders

Linear Differential Equations with Constant Coefficients

The differential equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = Q$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants and Q is a function of x, is said to be nth order ordinary differential equation in y with constant coefficients.

Equations with $Q=0$

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n)y = 0$$

or $\quad f(D)\, y = 0 \quad \text{——} \quad (1)$

If $y = e^{mx}$ is a solution, (1) gives

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_n = 0 \quad \text{——} \quad (2)$$

If $f_1(x)$ is a solution of (2), $c_1 f_1(x)$ is also a solution of (2), where c_1 is an arbitrary constant.

If $f_1(x), f_2(x), \ldots, f_n(x)$ are solutions, then $c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)$ is also a solution, where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

If $f_1(x), f_2(x), \ldots, f_n(x)$ are solutions of (2) and $\phi(x)$ is a particular solution of (1), then

$$y = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) + \phi(x)$$

is the most general solution of (1) as it contains the requisite number of arbitrary constants.

The first part of the solution i.e

$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)$,

which involve n arbitrary constants, n being the order of the differential equation, is called Complimentary function (C.F.). The second part of the solution $\phi(x)$, which does not contain any arbitrary constants is called particular Integral (P.I.).

This equation, the L.H.S of which is obtained by replacing D in f(D) by m, is called auxiliary equation (A.E.). If the differential equation is of nth order, the auxilliary equation is of nth degree and hence has n roots.

Depending upon the nature of roots, we

have the following cases: (2)

(a) Roots of A.E. different:

(i) Real, non-surds: If the roots are $m_1, m_2, \ldots, m_n$, the solution in this case is given by

$$C.F. = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

(ii) Roots of A.E. Imaginary: such roots always occur in pairs. If two roots are $m_1 \pm i m_2$, the corresponding part of solution is

$$C.F. = e^{m_1 x}(A_1 \cos m_1 x + A_2 \sin m_2 x)$$

(iii) Roots of A.E are real and in surds: Such roots also occur in pairs. If the roots be $m_1 \pm \sqrt{m_2}$, the corresponding part of C.F. is

$$C.F. = e^{m_1 x}\left[A_1 \cosh\sqrt{m_2}\,x + A_2 \sinh\sqrt{m_2}\,x\right]$$

(b) Roots of auxilliary equation equal:

(i) For two equal roots, each equal to m_1, of auxiliary equation, the corresponding C.F. is given by

$$C.F. = (C_1 + C_2 x)e^{m_1 x}$$

(ii) If the roots of A.E. are $m_1 \pm i m_2, m_1 \pm i m_2$

roots sol^n will be

$$C.F. = e^{m_1 x}\left[(c_1+c_2x)\cos m_2 x + (c_3+c_4x)\sin m_2 x\right]$$

(iii) If the roots of A.E. are $m_1 \pm \sqrt{m_2}$, $m_1 \pm \sqrt{m_2}$, we have

$$C.F. = e^{m_1 x}\left[(c_1+c_2x)\cosh\sqrt{m_2}\,x + (c_3+c_4x)\sinh\sqrt{m_2}\,x\right]$$

2. Solve $(D^3 - 13D^2 + 12)y = 0$ $\quad D \equiv \dfrac{d}{dx}$

Solution The auxiliary equation is

$$m^3 - 13m^2 + 12 = 0$$

$$(m-1)\left[m^2 + m - 12\right] = 0$$

$$\Rightarrow (m-1)(m+4)(m-3) = 0$$

$$\Rightarrow m = 1, 3, -4$$

$\therefore$ The solution is

$$y = c_1 e^{x} + c_2 e^{3x} + c_3 e^{-4x}$$

Q. Solve $(D^4 + 8D^2 + 16)y = 0$

Solution The auxiliary equation is

$$m^4 + 8m^2 + 16 = 0$$

$$(m^2 + 4)^2 = 0$$

$$\Rightarrow m^2 = -4, -4 \Rightarrow m = \pm 2i, \pm 2i$$

$\therefore$ The solution is

$$y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$$

The Particular Integral

For the differential equation

$$f(D)y = Q$$

$$P.I. = \frac{1}{f(D)}Q$$

If $f(D) \equiv (D-\alpha)$, let $v = \frac{1}{D-\alpha}Q$

$$\Rightarrow (D-\alpha)v = Q$$

$$\Rightarrow \frac{dv}{dx} - \alpha v = Q$$

This is linear differential equation in v and its solution is

$$v e^{-\alpha x} = \int e^{-\alpha x} Q\, dx + c$$

$$\Rightarrow v = c e^{\alpha x} + e^{\alpha x}\int e^{-\alpha x} Q\, dx$$

Thus if $\frac{1}{f(D)} = \frac{A_1}{(D-\alpha_1)} + \frac{A_2}{(D-\alpha_2)} + \dots + \frac{A_n}{(D-\alpha_n)}$

we have

$$\frac{1}{f(D)}Q = \left[\frac{A_1}{(D-\alpha_1)} + \frac{A_2}{(D-\alpha_2)} + \dots + \frac{A_n}{(D-\alpha_n)}\right]Q$$

Special

Special methods for particular Integral

If in differential equation $f(D)y = Q$, then Q may be in the following forms

(i) e^{ax}

(ii) $\sin ax$ or $\cos ax$

(iii) x^n, when n is positive integer

(iv) $e^{ax} V$, where V is any function of x,

(v) xV

(i) <u>To find $\frac{1}{f(D)} e^{ax}$</u> (5)

$$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \text{, provided } f(a) \neq 0$$

Also

$$\frac{1}{f(D)} e^{ax} = \frac{1}{\phi(a)} \frac{x^r}{r!} \text{, provided } \phi(a) \neq 0$$

Here $f(D) = (D-a)^r \phi(D)$

(ii) To find $\frac{1}{f(D)} \sin ax$ and $\frac{1}{f(D)} \cos ax$

$$\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax \text{, Provided } f(-a^2) \neq$$

Similarly

$$\frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax \text{, Provided } f(-a^2) \neq 0$$

Also

$$\frac{1}{(D^2+a^2)} \sin ax = -\frac{x}{2a} \cos ax$$

$$\frac{1}{(D^2+a^2)} \cos ax = \frac{x}{2a} \sin ax$$

(iii) To find $\frac{1}{f(D)} x^m$, where m is a positive integer.

Consider $\frac{1}{(D-a)} x^m = \frac{1}{-a\left(1-\frac{D}{a}\right)} x^m$

$$= -\frac{1}{a}\left(1-\frac{D}{a}\right)^{-1} x^m$$

(6)

$$= -\frac{1}{a}\left(1+\frac{D}{a}+\frac{D^2}{a^2}+\frac{D^3}{a^3}+\cdots\right)x^m$$

$$= -\frac{1}{a}\left[x^m+\frac{m x^{m-1}}{a}+\frac{m(m-1)x^{m-2}}{a^2}\right.$$

$$\left.+\cdots+\frac{m(m-1)\cdots 2\cdot 1}{a^m}\right] \quad -(2)$$

(iv) $\frac{1}{f(D)} e^{ax} U$, where U is any function of x.

$$\frac{1}{f(D)}\left[e^{ax}U\right] = e^{ax}\frac{1}{f(D+a)}U$$

(v) To evaluate $\frac{1}{f(D)} xU$,

$$\frac{1}{f(D)}[xU] = x\frac{1}{f(D)}U + \frac{d}{dD}\left[\frac{1}{f(D)}\right]U.$$

Q. Solve $(D^2-1)y = \cosh x \cos x$

Solution: The auxiliary equation is

$$m^2 - 1 = 0 \Rightarrow m = \pm 1$$

$$\therefore \text{C.F.} = c_1 e^{x} + c_2 e^{-x}$$

$$\text{P.I.} = \frac{1}{D^2-1} \cosh x \cos x$$

$$= \frac{1}{2} \frac{1}{(D^2-1)} (e^{x} + e^{-x}) \cos x \qquad \because \cosh x = \frac{1}{2}(e^{x}+e^{-x})$$

$$= \frac{1}{2} \frac{1}{(D^2-1)} e^{x} \cos x + \frac{1}{2} \frac{1}{D^2-1} e^{-x} \cos x$$

$$= \frac{1}{2} e^{x} \frac{1}{[(D+1)^2-1]} \cos x + \frac{1}{2} e^{-x} \frac{1}{[(D-1)^2-1]} \cos x$$

(7)

$$= \frac{1}{2} e^{x} \frac{1}{D^2+2D} \cos x + \frac{1}{2} e^{-x} \frac{1}{D^2-2D} \cos x$$

$$= \frac{1}{2} e^{x} \frac{1}{2D-1} \cos x + \frac{1}{2} e^{-x} \frac{1}{-2D-1} \cos x$$

$$= \frac{1}{2} e^{x} \frac{(2D+1)}{4D^2-1} \cos x - \frac{1}{2} e^{-x} \frac{(2D-1)}{4D^2-1} \cos x$$

$$= \frac{1}{2} e^{x} \frac{(2D+1)}{-5} \cos x - \frac{1}{2} e^{-x} \frac{(2D-1)}{-5} \cos x$$

$$= -\frac{1}{10} e^{x} (2D+1) \cos x + \frac{1}{10} e^{-x} (2D-1) \cos x$$

$$= -\frac{1}{10} e^{x} [-2\sin x + \cos x] + \frac{1}{10} e^{-x} (-2\sin x - \cos x)$$

$$= \frac{1}{5} \sin x (e^{x} - e^{-x}) - \frac{1}{10} \cos x (e^{x} + e^{-x})$$

$$= \frac{2}{5} \sin x \sinh x - \frac{1}{5} \cos x \cosh x$$

$\therefore$ The general solution is

$$y = c_1 e^{-x} + c_2 e^{x} + \frac{2}{5} \sin x \sinh x - \frac{1}{5} \cos x \cosh x$$

Q. Solve :

$$(D^2-2D-4)y = e^x \cos x - \sin^2 x$$

Solⁿ: The A.E. is

$$m^2-2m-4=0$$

$$m = \frac{2 \pm \sqrt{20}}{2} = (1 \pm \sqrt{5})$$

$$C.F. = c_1 e^{(1+\sqrt{5})x} + c_2 e^{(1-\sqrt{5})x}$$

$$P.I. = \frac{1}{(D^2-2D-4)}\left[e^x \cos x - \sin^2 x\right]$$

$$= \frac{1}{(D^2-2D-4)}\left[e^x \cos x - \frac{1-\cos 2x}{2}\right]$$

$$= \frac{1}{D^2-2D-4}\left[e^x \cos x\right] - \frac{1}{2}\frac{1}{(D^2-2D-4)}(1) + \frac{1}{2}\frac{1}{(D^2-2D-4)}\cos 2x$$

$$= -\frac{1}{6}e^{x}\cos x + \frac{1}{8} - \frac{1}{4(D+4)}\cos 2x$$

$$= -\frac{1}{6}e^{x}\cos x + \frac{1}{8} - \frac{1}{4}\frac{(D-4)}{D^2-16}\cos 2x$$

$$= -\frac{1}{6}e^{x}\cos x + \frac{1}{8} - \frac{1}{4}\frac{(D-4)}{-20}\cos 2x$$

$$= -\frac{1}{6}e^{x}\cos x + \frac{1}{8} + \frac{1}{80}(-2\sin 2x - 4\cos 2x)$$

$$= \frac{1}{8} - \frac{1}{6}e^{x}\cos x - \frac{1}{40}\sin 2x - \frac{1}{20}\cos 2x$$

$\therefore$ Required solution is

$y = C.F. + P.I.$

Q. $(D^2-4D+4)y = 8x^2 e^{2x}\sin 2x$

Sol: A.E. $m^2-4m+4=0$

$\Rightarrow (m-2)^2=0 \Rightarrow m=2,2$

C.F. $y=(c_1+c_2x)e^{2x}$

P.I. $= y = \frac{1}{(D-2)^2}$ [Img. part of $\{8\frac{1}{(D-2)^2}x^2 e^{2x} e^{2ix}$

$= 8\ \mathrm{Im}\frac{1}{(D-2)^2}\left[x^2 e^{(2+2i)x}\right]$

$= 8\ \mathrm{Im}\left[e^{(2+2i)x}\frac{1}{(D-2+2+2i)^2}\left[x^2\right]\right]$

$= 8\ \mathrm{Im}\left[e^{2(1+i)x}\frac{1}{(D+2i)^2}x^2\right]$

$= 8\ \mathrm{Im}\left[e^{2(1+i)x}\frac{1}{(2i)^2}\left[1+\frac{D}{2i}\right]^{-2}x^2\right]$

$$= 8\,\mathrm{Im}\left[e^{2(1+i)x}\frac{1}{(2i)^2}\left[1+\frac{D}{2i}\right]^{-2}x^2\right]$$

$$= 8\,\mathrm{Im}\left\{e^{2(1+i)x}\frac{1}{(-4)}\left[1-2\frac{D}{2i}+3\left(\frac{D}{2i}\right)^2+\cdots\right]x^2\right\}$$

$$= -2e^{2x}\,\mathrm{Im}\left\{e^{2ix}\left[x^2+2ix-\frac{3}{4}(2)\right]\right\}$$

$$= -2e^{2x}\left[2x\cos 2x+\left(x^2-\frac{3}{2}\right)\sin 2x\right]$$

Thus

$$y = C.F. + P.I.$$

$$= (c_1+c_2x)e^{2x}$$

$$-2e^{2x}\left[2x\cos 2x+\left(x^2-\frac{3}{2}\right)\sin 2x\right]$$

References

1. **Thomas' calculus; Maurice D. Weir, Joel Hass; Person Publications.**
2. **Higher Engineering Mathematics; B V Ramana; Tata Mc Graw Hill.**
3. **Engineering Mathematics; Bali, Iyengar; Laxmi Publications.**

4. NPTEL Lectures available on

- https://www.youtube.com/watch?v=tHqx1qxA8q4
- https://www.youtube.com/watch?v=btOCUmJkrrg

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Thank you!