JAIPUR ENGINEERING COLLEGE AND RESEARCH CENTRE

Year & Sem – I Year & II Sem
Subject –Engineering Mathematics-II
Unit -
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- **Linear Differential Equation** :
- A differential equation of the form
- $\frac{dy}{dx} + Py = Q$ (1)
- where P and Q are functions of x alone, is called a liner differential equation of first order and first degree. To solve such type of differential equation, we multiply both side with $e^{\int Pdx}$, known as integrating factor, then equation (1) becomes
- $e^{\int Pdx}\frac{dy}{dx} + Pye^{\int Pdx} = Qe^{\int Pdx}$
- Or $\frac{d}{dx}\left\{ye^{\int Pdx}\right\} = Qe^{\int Pdx}$
- On integrating both side , we get
- $ye^{\int Pdx} = Qe^{\int Pdx} + c,$ (2)
- where c is constant of integration. The equation (2) is required solution of linear differential equation (1).

- **Remark**: Sometimes the equation becomes linear if we take x as dependent variable and y as independent variable . It is of the form $\frac{dy}{dx} + P_1 y = Q_1$, where P_1 and Q_1 are function is y alone. The integration factor in this case will be $e^{\int P_1 dy}$ and solution is given by $ye^{\int P_1 dy} = Qe^{\int P dx} + c_1$, where c_1 is constant of integration.

- **Example 1:** Solve $\frac{dy}{dx} - \frac{1}{x}y = 2x^3 + 3x + 4$
- **Solution:** Here, the given equation is linear differential equation. Now comparing with standard form $\frac{dy}{dx} + Py = Q$, we have $P = \frac{1}{x}$, $Q = 2x^3 + 3x + 4$.
- Now integrating factor is
- $e^{-\int \frac{1}{x}dx} = e^{-logx} = e^{-logx^{-1}} = x^{-1} = \frac{1}{x}$
- Therefore solution is
- $y\frac{1}{x} = \int (2x^3 + 3x + 4)\frac{1}{x}dx + c$
- $\frac{y}{x} = \int \left(2x^2 + 3 + \frac{4}{x}\right)dx + c$
- $= \frac{2x^3}{3} + 3x + 4logx + c$
- Or $y = \frac{2x^4}{3} + 3x^2 + 4xlogx + cx$, which is required solution

- **Example 2:** Solve $(1+y^2)^2 dx = (\tan^{-1} y - x) dy$.
- **Solution :** the Given equation can be written as
- $\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$
- Which is linear in x
- Now , integrating factor = $e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$
- Therefore solution is
- $x e^{\tan^{-1} y} = \int e^{\tan^{-1} y} \frac{\tan^{-1} y}{1+y^2} dy + c$
- Put $e^{\tan^{-1} y} = t \Rightarrow \frac{dy}{1+y^2} = dt$
- $\therefore x e^t = \int t e^t dt + c$
- On integration , we get
- $x e^t = t e^t - e^t + c$
- Or $x = (t-1) + c e^{-t}$
- Or $x = (\tan^{-1} y - 1) + c e^{-\tan^{-1} y}$, which is required solution.

- **Equation reducible to Linear form(Bernoulli's Equation)**
- **The** equation of the type
- $\frac{dy}{dx} + Py = Qy^n$
- Where P and Q are functions of x alone (or constant) and n is a constant other than zero or unity, belongs to equation reducible form. This is also called **Bernoulli's Equation** .
- To solve such type of differential equation, dividing by y^n, we get
- $y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$
- Put $y^{1-n} = v$, on differentiating with respect to x, we get
- $(1-n)y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$
- Or $y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dv}{dx}$
- Therefore , $\frac{1}{1-n}\frac{dv}{dx} + Pv = Q$
- Or $\frac{dv}{dx} + (1-n)Pv = (1-n)Q$
- This is a linear equation whose integrating factor is $e^{(1-n)\int Pdx}$. Therefore solution is
- $ve^{(1-n)\int Pdx} = (1-n)\int Qe^{(1-n)\int Pdx}\,dx + c$, where c is constant of integration.
- Note: Do not forget to replace v by y^{-n+1} while writing the final solution.

- **Example :** Solve $\frac{dy}{dx} = \frac{1}{xy(x^2y^2+1)}$
- **Solution:** The given equation can be written as
- $\frac{dx}{dy} = x^3y^3 + xy$
- Or $\frac{dx}{dy} - xy = x^3y^3$
- On dividing by x^3, we get
- $\frac{1}{x^3}\frac{dx}{dy} - \frac{y}{x^2} = y^2$
- Put $-\frac{1}{x^2} = v$ therefore $\frac{2}{x^3}\frac{dx}{dy} = \frac{dv}{dy}$
- Then , $\frac{1}{2}\frac{dv}{dy} + vy = y^3$

- $\Rightarrow \frac{dv}{dy} + 2vy = 2y^3$, which is linear in v, therefore
- Integrating factor = $e^{\int Pdx} = e^{\int 2ydy} = e^{y^2}$
- Therefore, solution is given by
- $ve^{y^2} = \int 2y^3 e^{y^2} dy + c$
- To solve , put $y^2 = t \Rightarrow 2ydy = dt$
- $\Rightarrow ve^{y^2} = (t-1)e^t + c$
- $\Rightarrow ve^{y^2} = (y^2-1)e^{y^2} + c$
- Therefore, $-\frac{1}{x^2} = (y^2-1) + ce^{-y^2}$

- **Exact Differential Equation:**
- **An exact differential equation can always be derived directly from its general solution by differentiating without any subsequent multiplication, elimination, etc.**
- Thus ordinary differential equation of the form
- $Mdx + Ndy = 0$ (1)
- where M and N are functions of x and y , will be exact, if
- $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (2)
- Where total differential of f can be expressed as
- $df(x, y)$= $Mdx + Ndy$
- i.e. $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y} = Mdx + Ndy$
- The equation (2) is called the condition of exactness of the differential equation (1)

Method of solving:

- **Method1:**
- Compare the given equation with $Mdx + Ndy = 0$ and find M and N.
- Compute $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ and check the condition of exactness.
- Integrate $M(x, y)$ with respect to x keeping y as constant and integrate those term of $N(x, y)$ with respect to y which do not contain x.
- Write solution as $\int_{y=constant} Mdx + \int (only\ those\ tern\ of\ N, which\ not\ contain\ x)dy = c$
- **Method 2:**
- Compare the given equation with $Mdx + Ndy = 0$ and find M and N.
- Compute $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ and check the condition of exactness.
- Let $u = \int Mdx$, then find $\frac{\partial u}{\partial y}$ and $N - \frac{\partial u}{\partial y}$.
- Write solution as $u + \int \left(N - \frac{\partial u}{\partial y}\right) dy = c$
- i.e. $\int Mdx + \int \left(N - \frac{\partial u}{\partial y}\right) dy = c$

- **Example:** Solve $\left(1+e^{x/y}\right)dx+e^{x/y}(1-{}^{x}/_{y})dy$.
- **Solution:** Comparing the given equation with $Mdx+Ndy=0$, we have
- $M=1+e^{x/y}\Longrightarrow\frac{\partial M}{\partial y}=-\frac{x}{y^2}e^{x/y}$
- and $N=e^{x/y}(1-{}^{x}/_{y})\Longrightarrow\frac{\partial N}{\partial x}=e^{x/y}\left\{-\frac{1}{y}\right\}+e^{x/y}.\frac{1}{y}\{1-{}^{x}/_{y}\}=-\frac{x}{y^2}e^{x/y}$
- Here, we have $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$
- so the given equation is exact.
- Now $u=\int Mdx=\int\left(1+e^{x/y}\right)dx=x+ye^{x/y}$
- So $\frac{\partial u}{\partial y}=e^{x/y}-y.\frac{x}{y^2}e^{x/y}=e^{x/y}(1-x/y)$
- And $N-\frac{\partial u}{\partial y}=0$
- Hence , required solution is $\int Mdx+\int\left(N-\frac{\partial u}{\partial y}\right)dy=c$
- Or $x+ye^{x/y}+0=c$
- i.e. $x+ye^{x/y}=c$ where c is arbitrary constant.

- **Example:** Solve $\left(1+e^{x/y}\right)dx + e^{x/y}(1 - {x}/{y})dy$.
- **Solution:** Comparing the given equation with $Mdx + Ndy = 0$, we have
- $M = 1 + e^{x/y} \Longrightarrow \frac{\partial M}{\partial y} = -\frac{x}{y^2} e^{x/y}$
- and $N = e^{x/y}(1 - {x}/{y}) \Longrightarrow \frac{\partial N}{\partial x} = e^{x/y}\left\{-\frac{1}{y}\right\} + e^{x/y}.\frac{1}{y}\{1 - {x}/{y}\} = -\frac{x}{y^2} e^{x/y}$
- Here, we have $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
- so the given equation is exact.
- Now $u = \int Mdx = \int\left(1+e^{x/y}\right)dx = x + ye^{x/y}$
- So $\frac{\partial u}{\partial y} = e^{x/y} - y.\frac{x}{y^2}e^{x/y} = e^{x/y}(1 - x/y)$
- And $N - \frac{\partial u}{\partial y} = 0$
- Hence , required solution is $\int Mdx + \int\left(N - \frac{\partial u}{\partial y}\right)dy = c$
- Or $x + ye^{x/y} + 0 = c$
- i.e. $x + ye^{x/y} = c$ where c is arbitrary constant.

- **Example:** Solve $\{y(1+1/x)+\cos y\}dx+(x+\log x-x\sin y)dy$.
- **Solution:** Comparing the given equation with $Mdx+Ndy=0$, we have
- $M=y(1+1/x)+\cos y \Longrightarrow \frac{\partial M}{\partial y}=1+\frac{1}{x}-\sin y$
- and $N=x+\log x-x\sin y \Longrightarrow \frac{\partial N}{\partial x}=1+\frac{1}{x}-\sin y$
- Here, we have $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$
- so the given equation is exact.
- Now $u=\int(y+y/x)+\cos y\,dx=xy+y\log x+x\cos y$ (treating y as constant)
- So $\frac{\partial u}{\partial y}=x+\log x-x\sin y$
- And $N-\frac{\partial u}{\partial y}=x+\log x-x\sin y-x+\log x-x\sin y=0$
- Hence , required solution is $\int Mdx+\int\left(N-\frac{\partial u}{\partial y}\right)dy=c$
- Or $xy+y\log x+x\cos y+0=c$
- i.e. $xy+y\log x+x\cos y=c$ where c is arbitrary constant

- **Equation reducible to Exact form: the** differential equation which is not exact can be made exact by multiplying it a suitable function of x and y , known as integrating factor (I.F). we now explain the rule for finding the integrating factor.
- By inspection method: By rearranging the term of given differential equation or by dividing by a suitable function of x and y , the equation thus obtained will contain several parts integrable easily. Regarding this some list of exact differential should be useful:
- $d(xy) = x\,dy + y\,dx$ (ii) $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$
- $d\left(\frac{y}{x}\right) = \frac{xdy - xdx}{x^2}$ (iv)
- If the differential equation $Mdx + Ndy = 0$ be homogeneous equation in x an y then
- $I.F = \frac{1}{Mx+Ny}$, where $Mx + Ny \neq 0$.

- If the differential equation $Mdx + Ndy = 0$ is of the form
- $f_1(xy)y\,dx + f_2(xy)x\,dy = 0$ then
- $I.F = \frac{1}{Mx-Ny}$, where $Mx - Ny \neq 0$.
-
- If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ only then
- I.F= $e^{\int f(x)dx}$
- If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = f(y)$ only then
- I.F= $e^{\int f(y)dy}$
- If the differential equation $Mdx + Ndy = 0$ is of the form
- $x^a y^b(mydx + nxdy) + x^c y^d(pydx + qxdy) = 0$
- Where a , b ,c , d, m , n , p , q are constants,
- Then I . F. $= x^h y^k$,
- Where h , k are obtained by the condition of exactness. i.e. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

- **Example: Solve** $(\boldsymbol{xy}\sin\boldsymbol{xy}+\cos\boldsymbol{xy})\boldsymbol{ydx}+(\boldsymbol{xy}\sin\boldsymbol{xy}\ -\cos\boldsymbol{xy})\boldsymbol{x}\ \boldsymbol{dy}=\mathbf{0}$
- **Solution:** The given differential equation is
- $(xy\sin xy+\cos xy)ydx+(xy\sin xy\ -\cos xy)x\ dy=0$ (1)
- Here, we have $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$
- But (1) is a differential equation is of the form $f_1(xy)y\ dx+f_2(xy)x\ dy=0$ then
- $I.F=\frac{1}{Mx-Ny}$, where $Mx-Ny\neq 0.$
- $I.F=\frac{1}{2xy\cos xy}$, where $Mx-Ny\neq 0.$
-
-
- On multiplying (1) by $I.F=\frac{1}{2xy\cos xy}$ we have
- $tanxy(ydx+xdy)+\frac{dx}{x}-\frac{dy}{y}=0$
- which must now be exact differential equation.
- therefore $tanxy\ d(xy)+d(\log x)-d(\log y)=0$
- integrating we have , $\log\sec xy+\log x-logy=\log c$
- or $\frac{x}{y}\sec xy\ =c$ is required solution where c is a constant
- Example: Solve $(1+xy)y\ dx+(1-xy)xdy=0$

- **Example:** Solve $(x^2y-2xy^2)dx-(x^3-3x^2y)\,dy=0$
- **Solution:** The given differential equation is
- $(x^2y-2xy^2)dx-(x^3-3x^2y)\,dy=0$ (1)
- Clearly (1) is a homogeneous differential equation. Now Comparing (1) with $Mdx+Ndy=0$, we have
- $M=x^2y-2xy^2 \Longrightarrow \frac{\partial M}{\partial y}=x^2-4xy$
- and $N=-(x^3-3x^2y) \Longrightarrow \frac{\partial N}{\partial x}=-(3x^2-6xy)$
- Here, we have $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$
- so the given equation is not exact.
- Clearly (1) is a homogeneous differential equation.
- Therefore $Mx+Ny=x(x^2y-2xy^2)-y(x^3-3x^2y)=x^2y^2 \neq 0$
- So I.F of (1) $=\frac{1}{Mx+Ny}=\frac{1}{x^2y^2}$

- On multiplying (1) by $\frac{1}{x^2y^2}$, we have
- $\left(\frac{1}{y}-\frac{2}{x}\right)dx-\left(\frac{x}{y^2}-\frac{3}{y}\right)dy=0$
- which must now be exact differential equation.
- Now $u=\int Mdx=\int\left(\frac{1}{y}-\frac{2}{x}\right)dx=\frac{x}{y}-2\log x$ (treating y as constant)
- So $\frac{\partial u}{\partial y}=-\frac{x}{y^2}$
- And $N-\frac{\partial u}{\partial y}=-\left(\frac{x}{y^2}-\frac{3}{y}\right)-\frac{x}{y^2}=\frac{3}{y}$
- Hence , required solution is $\int Mdx+\int\left(N-\frac{\partial u}{\partial y}\right)dy=c$
- i.e. $\frac{x}{y}-2\log x+3\log y=c$ where c is arbitrary constant.

- Example: Solve $(y^2 + 2x^2y)dx + (2x^3 - xy)dy. = 0$
- Solution: The given equation can be written in the form
- $(y^2dx + 2x^3dy) + y(2x^2ydx - xdy) = 0$ which is of the form
- $x^a y^b(mydx + nxdy) + x^c y^d(pydx + qxdy) = 0$
- Where a , b ,c , d, m , n , p , q are constants,
- so I . F. $= x^h y^k$, now them multiplying the given equation by I.F. , we have
- $\left(x^h y^{k+2} + 2x^{h+2}y^{k+1}\right)dx + \left(2x^{h+3}y^k - x^{h+1}y^{k+1}\right)dy. = 0$
- Which can exact only when the condition of exactness $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is satisifed

- $\Longrightarrow \frac{\partial}{\partial y}((x^h y^{k+2} + 2x^{h+2}y^{k+1}) = \frac{\partial}{\partial x}(2x^{h+3}y^k - x^{h+1}y^{k+1})$
- $\Longrightarrow (k+2)x^h y^{k+1} + 2(k+1)x^{h+2}y^k = 2(h+3)x^{h+2}y^k - (h+1)x^h y^{k+1}$
-
- Now equating the coefficient of $x^h y^{k+1}$ and $x^{h+2}y^k$, we have
- $(k+2) = -(h+1)$ and $2(k+1) = 2(h+3)$
- .$\Longrightarrow k = -\frac{1}{2}$ & $h = -\frac{5}{2}$
- I.F =$x^{-\frac{5}{2}}y^{-\frac{1}{2}}$
- Now multiplying the given equation by this integrating Factor I.F =$x^{-\frac{5}{2}}y^{-\frac{1}{2}}$ and applying the method we have the solution
- $-\frac{2}{3}x^{-\frac{3}{2}}y^{-\frac{3}{2}} + 4x^{\frac{1}{2}}y^{\frac{1}{2}} = c$

- **Example:** Solve $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$
- **Solution:** The given differential equation is
- $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy =$ (1)
- Here, we have $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$
- so the given equation is not exact.
- But here If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y} = f(y)$ only then
- I.F= $e^{\int f(y)dy} = y$
- On multiplying (1) by integrating factor y, we have
- $(xy^4 + y^2)dx + 2(x^2y^3 + xy + y^5)dy = 0$
- which must now be exact differential equation.
- solution is $\int_{y=constant} Mdx + \int(only\ those\ tern\ of\ N, which\ not\ contain\ x)dy = c$
- $\int(xy^4 + y^2)dx + \int 2y^5 = c$
- On integration , we have $\frac{1}{2}x^2y^4 + xy^2 + \frac{1}{3}y^6 = c$
- which is required solution.

Refrences

1.Advanced Engineering Mathematics by Prof.ERWIN KREYSZIG (Ch.10,page no.557-580)
2. Advanced Engineering Mathematics by Prof.H.K Dass (Ch.14,page no.851-875)
3.Advanced Engineering Mathematics by B.V RAMANA (Ch.20,pageno.20.1.20.5)
4.NPTEL Lectures available on

http://www.infocobuild.com/education/audio-video-courses/mathematics/TransformTechniquesForEngineers-IIT-Madras/lecture-47.html

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Thank you!