Unit - 6 WAVEGUIDE

Page No.
Date

Topic : Wave propagation in parallel plate waveguide

→ Parallel Plate Waveguide :-

The parallel plate waveguide is formed by two conducting plates of width 'w' separated by a distance 'd' as shown below. This "waveguide" can support TEM, TE and TM modes.

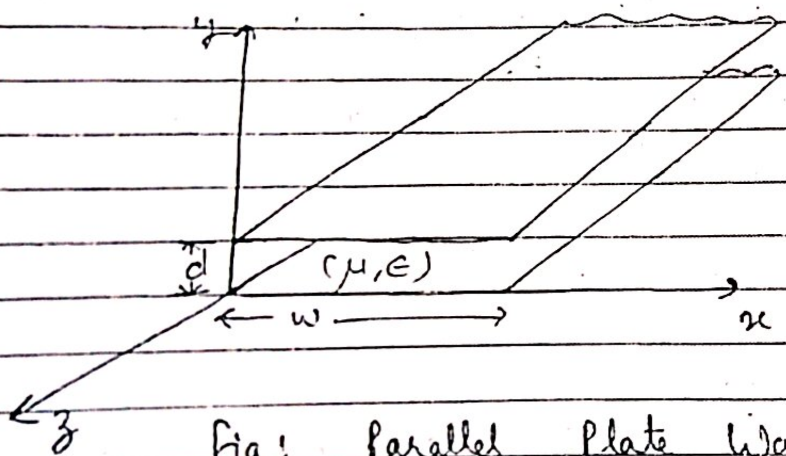


Fig: Parallel Plate Waveguide

The following assumptions are made in the determination of the various modes on the parallel plate waveguide :-

1. The waveguide is infinite in length (no reflections).

2. The waveguide conductors are PEC's and the di-electric is lossless.

3. The plate width is much larger than the plate separation ($w >> d$) so that the variation of the fields w.r.t. 'x' may be neglected.

→ Transmission lines and Waveguides :-

~~① Transmission lines~~ -

Given a particular conductor geometry for a transmission line or waveguide, only certain patterns of electric and magnetic fields (modes) can exist for propagating waves.

① Transmission line -

→ Two or more conductors (two wire, co-axial, etc.)

→ Can define a unique current and voltage and characteristic impedance along the line (use circuit equations).

② Waveguide -

→ Typically one enclosed conductor (rectangular, circular, etc).

Page No.
Date

→ Cannot define a unique voltage and current along the waveguide

∴ The propagating modes along the transmission line or waveguide may be classified a/c to which field components are present or not present in the wave.

Assuming the transmission line or waveguide is oriented with its axis along the z-axis, the modes may be classified as

i) Transverse electromagnetic modes (TEM):-

The electric and magnetic field are transverse to the dir^n of wave propagation with no longitudinal component ($E_z = H_z = 0$).

TEM modes cannot exist on single conductor guiding structures.
TEM modes are sometimes called transmission line modes since they are the dominant modes on transmission lines.

2) Transverse Electric TE :- The electric field is transverse to the dir^n of propagation while the magnetic field has both transverse and longitudinal

component [$E_z = 0$, H_z . 0].

3) Transverse Magnetic TM mode — The magnetic field is transverse to the dir^n of propagation while the electric field has both transverse and longitudinal components [$H_z = 0$, E_z . 0]

TE and TM modes are commonly referred to as waveguide modes since they are the only modes which can exist in an enclosed guiding structure. TE and TM modes are characterized by a cutoff freq^n below which they do not propagate.

TE and TM modes can exist on transmission lines but are generally undesirable.

* Analysis of waveguide general approach.

Parallel - Plate guide analysis using the wave equation.

The most direct approach in the analysis of any waveguide is through the wave equation, which we solve subject to the boundary conditions at the conducting walls.

Form of the Eq^n for the dielectric properties in the waveguide is

$$\left[\nabla^2 E_s = -K^2 E_s\right] \quad \text{--- (1)}$$

where $K = n\omega/c$

Now, we may write wave equation for TE modes, in which there will be only a y component of E. The wave equation becomes:

$$\frac{\partial^2 E_{ys}}{\partial x^2} + \frac{\partial^2 E_{ys}}{\partial y^2} + \frac{\partial^2 E_{ys}}{\partial z^2} + K^2 E_{ys} = 0 \quad \text{--- (2)}$$

We assume that the width of the guide (in the Y-dir^n) is very large compared to the plate separation d.

∴ we can assume no Y-variation in field.

So, $\frac{\partial^2 E_{ys}}{\partial y^2} = 0.$

and we know that the z- variation will be of the form $e^{-j\beta_m z}$.

Form of the field Solⁿ E_{ys} is

$$E_{ys} = E_0 f_m(x) e^{-j\beta_m z} \quad \text{--- (3)}$$

where E_0 is a constant
$f_m(x)$ is a normalised funⁿ.
We have m on β, $f(x)$, since we have several solⁿ.
Now substitude (3) into (2), we have

$$\frac{d^2 f_m(x)}{dx^2} + (K^2 - \beta_m^2) f_m(x) = 0 \quad \text{--- (4)}$$

where E_0 and $e^{-j\beta_m z}$ have divided out, and we have used the fact that

$$\frac{d^2}{dz^2} e^{-j\beta_m z} = -\beta_m^2 e^{-j\beta_m z}.$$

From Eqⁿ (4)

$$\frac{d^2 f_m(x)}{dx^2} + K_m^2 f_m(x) = 0 \quad \text{- (5)}$$

where $(K^2 - \beta_m^2) = K_m^2$

General Solⁿ of Eqⁿ (5) will be

$$f_m(x) = \cos(K_m x) + \sin(K_m x) \quad \text{--- (6)}$$

Page No.
Date

We next apply the appropriate boundary conditions in our problem to solve k_m. In Eq^n (6) only $\sin(k_m x)$ term will allow the boundary conditions to be satisfied, so, we retain it and drop the cosine term.

($x = 0$) condition is automatically satisfied by the sine function.

The $x = d$ condition is met when we choose the value of k_m such that

$$\therefore k_m = \frac{m\pi}{d}$$

To get final form of E_{ys}, put the value of $f_m(x)$ from eq^n (6) into Eq^n (3), we get

$$\boxed{E_{ys} = E_0 \sin\left(\frac{m\pi x}{d}\right) e^{-j\beta_m z}} \quad -(7)$$

'm' is the no. of half-cycles of E.F that occur over the distance d in the transverse plane.

The plane wave angle of incidence in the guide at cutoff, ~~As we learned~~ is zero, meaning that the wave simply bounces up and down between the conducting walls.

The wave must be resonant in the structure, which means that the net round trip phase shift is $2m\pi$.

With the plane waves oriented vertically $\beta_m = 0$, and so $K_m = K = 2\pi n/\lambda_{cm}$.
So at cutoff

$$\frac{m\pi}{d} = \frac{2n\pi}{\lambda_{cm}}$$

$$\boxed{d = \frac{m\lambda_{cm}}{2n}}$$

Then $E_{ys} = E_0 \sin\left(\frac{m\pi x}{d}\right)$

$$\boxed{E_{ys} = E_0 \sin\left(\frac{2n\pi x}{\lambda_{cm}}\right)}$$

We can find the magnetic field using Maxwell's equations.

$$\nabla \times E_s = -j\omega\mu H_s \quad \text{--- (8)}$$

$$\nabla \times E_s = \frac{\partial E_{ys}}{\partial x} a_z - \frac{\partial E_{ys}}{\partial z} a_x$$

$$= K_m E_0 \cos(K_m x) e^{-j\beta_m z} a_z + j\beta_m E_0 \sin(K_m x) e^{-j\beta_m z} a_x \quad \text{--- (9)}$$

We solve for H_s by dividing both sides of (8) by $-j\omega\mu$. Perform this operation on (9) we get the two magnetic field components:-

Page No.
Date

$$\boxed{H_{xs} = -\frac{\beta_m}{\omega\mu} E_0 \sin(K_m x) e^{-j\beta_m z}} \quad -(10)$$

$$\boxed{H_{zs} = j\frac{K_m}{\omega\mu} E_0 \cos(K_m x) e^{-j\beta_m z}} \quad -(11)$$

Now, taking magnitude of H_s

$$|H_s| = \sqrt{H_s \cdot H_s^*} = \sqrt{H_{xs} H_{xs}^* + H_{zs} H_{zs}^*} \quad -(12)$$

$$|H_s| = \frac{E_0}{\omega\mu} \cdot (K_m^2 + \beta_m^2)^{1/2} \left(\sin^2(K_m x) + \cos^2(K_m x)\right)^{1/2}$$

(13)

Using $K_m^2 + \beta_m^2 = K^2$ &
$\sin^2(K_m x) + \cos^2(K_m x) = 1$,

Eq^n (13) becomes

$$|H_s| = \frac{E_0 K}{\omega\mu} = \frac{\omega\sqrt{\mu\epsilon}}{\omega\mu} = \frac{E_0}{\eta} \quad -(14)$$

where $\eta = \sqrt{\mu/\epsilon}$. This result is consistent with our understanding of waveguide modes based on the superposition of plane waves, in which the relation between E_s and H_s is through the medium intrinsic impedance, η.

Date

* Attenuation in Waveguide

In Rectangular waveguide

Attenuation for propagation modes results when there are losses in the dielectric and in the imperfectly conducting guide walls. Because these losses are usually very small, we will assume, as in the case of parallel-plate waveguides, that the transverse field pattern are not appreciably affected by the losses. The attenuation constant due to losses in the dielectric can be obtained by substitute the value $\epsilon_d = \epsilon + (\sigma/j\omega)$ for ϵ in

$$\gamma = j\beta = j\sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

Result is

$$\alpha_d = \frac{\sigma\eta}{2\sqrt{1-(f_c/f)^2}} \quad \text{—} (1)$$

where σ and η are the equivalent conductivity and intrinsic impedance of the dielectric medium.

It is easy to see from Eq^n (1) that the attenuation constant of propagation waves due to losses in the dielectric decreases monotonically from an infinitely large value toward the value $\sigma\eta/2$ as the frequency increases from the cut off frequency.

To determine the attenuation constant due to wall losses, we make use of

$$\alpha_c = \frac{P_L(z)}{2P(z)} \quad \text{—} \quad (a)$$

The derivatives of α_c for the general TM and TE modes tend to be tedious. For TE mode the only nonzero field components are E_y, H_x and H_z.

Let $m=1$, $n=0$ and $h=(\pi/a)$ in

$$E_y^0(x,y) = \frac{-j\omega\mu}{h^2}\left(\frac{m\pi}{a}\right) H_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right)$$

and

$$H_x^0(x,y) = \frac{\gamma}{h^2}\left(\frac{m\pi}{a}\right) H_0 \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right)$$

and calculate time-average power flowing through a cross-section of the waveguide.

$$P(z) = \int_0^b \int_0^a -\frac{1}{2}(E_y^0)(H_x^0)^* \, dx\, dy$$

$$= \frac{1}{2}\omega\mu\beta\left(\frac{a}{\pi}\right)^2 H_0^2 \int_0^b \int_0^a \sin^2\left(\frac{\pi}{a}x\right) dx\, dy \quad \text{—} (b)$$

$$= \omega\mu\beta ab\left(\frac{aH_0}{2\pi}\right)^2$$

In order to calculate the time-average power lost in the conducting walls per unit length, we must consider all four walls.

From $J_s = a_n \times H$ &

$$H_z^0(x,y) = H_0 \cos\left(\frac{m\pi}{a}\right)x \cos\left(\frac{n\pi}{b}y\right) \;\&$$

$$H_x^0(x,y) = \frac{\gamma}{h^2}\left(\frac{m\pi}{a}\right) H_0 \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right).$$

We see that

$$J_s^0(x=0) = J_s^0(x=a) = -a_y H_z^0(x=0) = -a_y H_0 \quad (3)$$

and

$$J_s^0(y=0) = -J_s^0(y=b) = a_x H_z^0(y=0) - a_z H_x^0(y=0)$$

$$= a_x H_0 \cos\left(\frac{\pi}{a}x\right) - a_z \frac{\beta a}{\pi} H_0 \sin\left(\frac{\pi}{a}x\right) \quad (4)$$

Total power loss is then double the sum of the losses in the walls at $x=0$ and at $y=0$.

We have

$$P_L(z) = 2\left[P_L(z)\right]_{x=0} + 2\left[P_L(z)\right]_{y=0} \quad (5)$$

$$\left[P_L(z)\right]_{x=0} = \int_0^b \frac{1}{2}\left|J_s^0(x=0)\right|^2 R_s\,dy = \frac{b}{2}H_0^2 R_s \quad (6)$$

and

$$\left[P_L(z)\right]_{y=0} = \int_0^a \frac{1}{2}\left[\left|J_{sx}^0(y=0)\right|^2 + \left|J_{sz}^0(y=0)\right|^2\right]R_s\,dx$$

$$= \frac{a}{4}\left[1 + \left(\frac{\beta a}{\pi}\right)^2\right] H_0^2 R_s \quad (7)$$

Page No. | Date

Put the Value of (6) and (7) in (5)

$$P_L(z) = \left\{ b + \frac{a}{2}\left[1 + \left(\frac{\beta a}{\pi}\right)^2\right]\right\} H_0^2 R_s$$

$$= \left[b + \frac{a}{2}\left(\frac{f}{f_c}\right)^2 \right] H_0^2 R_s \quad \text{—— (8)}$$

The last expression is the result of recognizing that

$$\beta = \sqrt{\omega^2 \mu \epsilon - \left(\frac{\pi}{a}\right)^2}$$

$$= \omega\sqrt{\mu\epsilon}\sqrt{1 - \left(\frac{f_c}{f}\right)^2} \quad \text{—— (9)}$$

Put the Eq^n (2) and (8) in (9)

$$(\alpha_c)_{TE} = \frac{R_s\left[1 + (2b/a)(f_c/f)^2\right]}{\eta b\sqrt{1 - (f_c/f)^2}}$$

$$= \frac{1}{\eta b}\sqrt{\frac{\pi f \mu_\epsilon}{\sigma_c\left[1 - (f_c/f)^2\right]}}\left[1 + \frac{2b}{a}\left(\frac{f_c}{f}\right)^2\right] \quad \text{(10)}$$

Eq^n (10) reveals a rather complicated dependence of $(\alpha_c)_{TE}$ on the ratio (f_c/f). It tends to infinity when f is close to the cut-off freq^n, decreases toward a minimum as 'f' increases, and increases again steadily for further increases in f.

For a given guide width a, the attenuation decreases as b increases. However, increasing 'b' also decreases the cut-off frequency of the next higher-order mode TE_{01} (or TM), with the consequence that the available bandwidth for the dominant TE mode is reduced.

For TM

$$(\alpha_c)_{TM} = \frac{2R_s(b/a^2 + a/b^2)}{\eta ab\sqrt{1-(f_c/f)^2}\,(1/a^2 + 1/b^2)} \quad \text{(11)}$$

Attenuation Constant of the TE modes is everywhere lower than that of the TM mode. These facts have direct relevance in the choice of operating modes and frequencies.

Surface current on the waveguide walls.

For obtaining current distribution on the waveguide walls:

The modal field inside the waveguide induce surface charges and surface currents on the walls of the waveguide.
In other word we can say that the modal fields are supported by the surface charges and surface currents on the inner walls of the waveguide.
Since, the field are time varying the surface charges and surface currents also vary to time.

It should be emphasized that power carried inside the waveguide is by the fields and not by the Surface charges and surface currents.
The direction of power flow and the dirn of the current in general need not be same.

Surface current is obtained ~~through~~ as

$$J_s = \hat{n} \times H \quad \text{——— (1)}$$

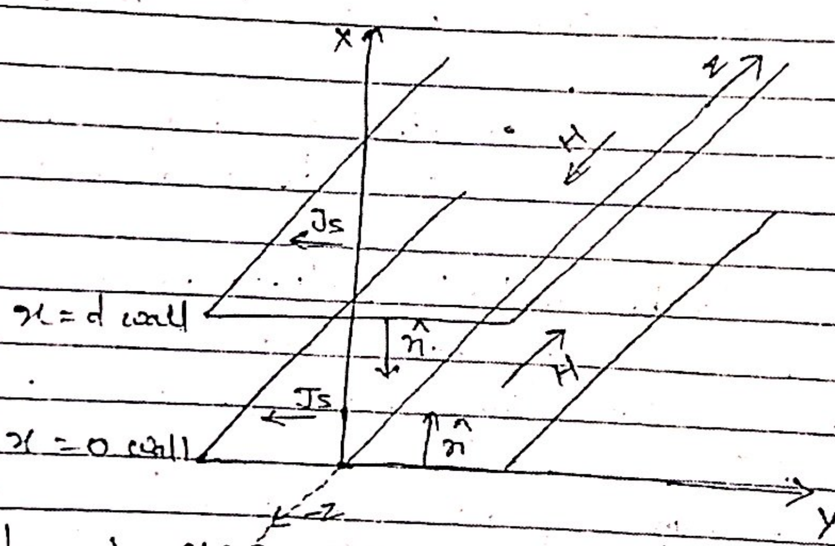
where unit vector and $\hat{n}$ is the outward normal on the repective inner surface of the waveguide.

Surface current for TE_1 mode inside para plane waveguide.

The magnetic fields for TE_1 mode are given as

$$H = \frac{2E_0}{\eta_1} \frac{\beta}{\beta_1} e^{-j\beta z} \hat{z} \quad \text{at } x = 0 \text{ wall} \quad -(2)$$

$$= \frac{-2E_0}{\eta_1} \frac{\beta}{\beta_1} e^{-j\beta z} \hat{z} \quad \text{at } x = d \text{ wall} \quad -(3)$$



Now in $x=0$ wall the normal '$\hat{n}$' is oriented in $+x$ dir^n and magnetic field is oriented in $+z$ dir^n. The current dir^n then is given by $\hat{n} \times H \rightarrow \hat{x} \times \hat{z} = -\hat{y}$ dir^n.

Similarly on $x=d$ wall the M.F is oriented in $-z$ dir^n and the normal unit '$\hat{n}$' is oriented in $-x$ dir^n. The dir^n of current is $-\hat{x} \times (-\hat{z}) = -\hat{y}$ dir^n.

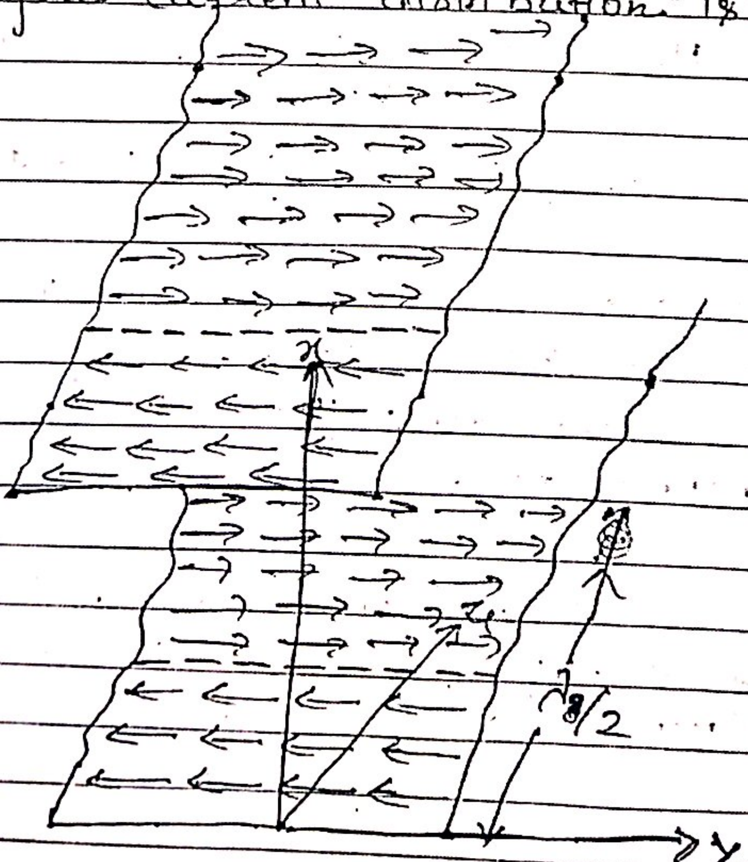
The current on both the walls do not have any variation in y-direction and have sinosoidal variation in the +z dir^n.

→ Both the walls have the current flowing in the same direction i.e. $-\hat{y}$ dir^n with equal amplitude.

→ The current always flow ⊥^{r} to the dir^n of the power flow which is in +z dir^n i.e., there is no component of the current in the direction of the power flow.

The current direction reverse for every $\lambda/2$ distance in the dir^n of the wave propagation.

The surface current distribution is in figure.



Surface current on the walls of the rectangular waveguide for TE mode.

~~Hz~~ → The magnetic field for TE mode has two component in x and z direction which are given as

$$H_x = \frac{\beta_g}{\pi} D \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi z}{\lambda_g}\right) \quad \text{— (4)}$$

$$H_z = D \cos\left(\frac{\pi x}{a}\right) \cos\left(\frac{2\pi z}{\lambda_g}\right) \quad \text{— (5)}$$

for obtaining the surface current only tangential component of the MF is needed.

For vertical walls the tangential components of the MF: are

$$H_z = D \cos\left(\frac{2\pi z}{\lambda_g}\right); x = 0 \quad \text{— (6)}$$

$$= -D \cos\left(\frac{2\pi z}{\lambda_g}\right); x = a \quad \text{— (7)}$$

The outward unit normal n is along $\hat{x}$-dir^n for $x = 0$ wall.

The current dir^n then on the wall is

$\hat{n} \times H \rightarrow \hat{x} \times \hat{z} = -\hat{y}$.

On $x = a$ wall unit normal n is in $-x$ dir^n and MF is in $-z$ dir^n & dir^n of current

$\hat{n} \times H \rightarrow \hat{x} \times \hat{z} = -\hat{y}$.

Page No.
Date

That means in both the vertical walls at this instant of time the current flows vertically downwards and since H_z does not have any variation in y-dir^n the current amplitude remain constant along the height of the wave.

On horizontal walls the surface current is due to both component x and z. On lower horizontal wall, $y = 0$, outward normal $\hat{n}$ is in y-dir^n.
Vector current distribution on the lower wall then is given as

$$J_s = \hat{n} \times H = \hat{y} \times \{ H_x \hat{x} + H_z \hat{z} = -H_x \hat{z} + H_z \hat{x}$$

$$\Rightarrow J_{sx} = H_z = D\cos\left(\frac{\pi x}{a}\right) \cos\left(\frac{2\pi z}{\lambda_g}\right) \quad (9)$$

$$J_{sz} = -H_x = -\frac{\beta a}{\pi} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi z}{\lambda_g}\right) \quad (10)$$

x and z components of the current are in space quardant in the z-direction and also in space quadration in x-direction.
when we systematically draw the vector current distribution on the horizontal waveguide wall.

→ On $y=b$ wall the magnetic field dir^n * remains same as that on the $y=0$ wall but the dir^n of normal $\hat{n}$ is reversed i.e. it is along $-\hat{y}$ dir^n.

∴ The current distribution on the upper wall then is the mirror image of the current distribution of the lower wall in fig.

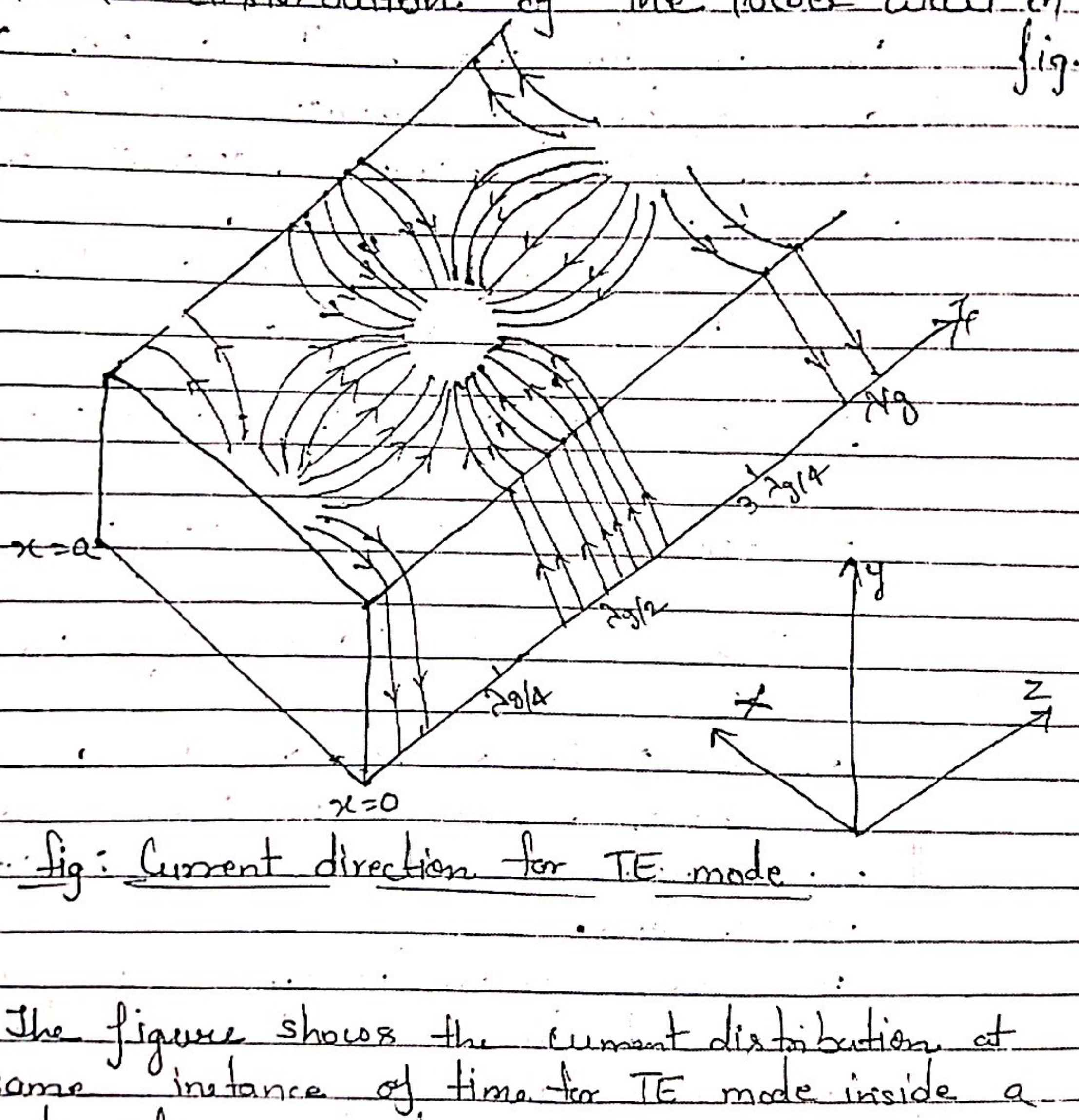


fig: Current direction for T.E mode.

→ The figure shows the current distribution at same instance of time for TE mode inside a rectangular waveguide

→ The current distribution drift inside the waveguide with a phase velocity of the mode.

Page No. 11

Field Visualization

Visualization of Modal fields inside rectangular waveguide.

In this we learn about field for TE mode in rectangular waveguide.
Magnetic field distribution for TE mode.

The visualization of the modal fields is important for identifying region from where field can be tapped efficiently by the probes.
The field probes are devices which can induce fields inside a waveguide or extract energy from the fields propagating inside the waveguide.

To visualize the three dimensional distribution of vector fields we follow a field vector untill it either closes on itself or ends upon the walls of waveguide.
We can see from the modal field expression that the fields are periodic over one guided wave length λ_g along the length of the waveguide.
So, essentially one has to develop a three dimensional picture of the fields only over a block of λ_g.

One the total field distribution is developed the distribution can be made to move with a velocity equal to the phase velocity of the mode.

- **Field for TE mode in Rectangular Waveguide.**

We visualize the fields for TE mode because the TE mode is the dominant mode of rectangular waveguide and invariably people have to deal with this mode while handling rectangular waveguide.

From general field expression for TE_{10} mode substituting $m=1$ & $n=0$, we get the field for the TE mode as follows:-

$$E_y = Re\left[-j\frac{\omega\mu a}{\pi} D \sin\left(\frac{\pi x}{a}\right) e^{-j\beta z}\right]$$

$$= -\frac{\omega\mu a}{\pi} D \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi z}{\lambda g}\right) \quad \text{--- (1)}$$

$$H_x = Re\left[\frac{j\beta a}{\pi} D \sin\left(\frac{\pi x}{a}\right) e^{-j\beta z}\right] = \frac{\beta a}{\pi} D \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi z}{\lambda g}\right) \quad (2)$$

$$H_z = Re\left[D \cos\left(\frac{\pi x}{a}\right) e^{-j\beta z}\right] = D \cos\left(\frac{\pi x}{a}\right) \cos\left(\frac{2\pi z}{\lambda g}\right) \quad (3)$$

Electric field distribution.

Since electric field has only one component the visualization of the electric field is rather simple and straight forward.

All the electric field vectors are oriented in x-dir^n. The magnitude of the electric field has sinusoidal variation in both x and z dir^n.

The electric field is zero at $x = 0$ and $x = a$ i.e. all along the vertical walls of the waveguide and maximum at $x = a/2$ i.e. half way between the two vertical walls.

So, we visualize the EF vector like an arrow with length of an arrow indicating the magnitude of the field, the arrow will be vertically oriented with largest arrow at the center of the waveguide gradually, decreasing as we move towards the vertical walls and remaining constant as we move toward horizontal walls.

Since, the EF is periodic with period λ_g in the z-dir^n the length of the field vector undergoes sinusoidal variations as we travel along the z-dir^n.

Magnetic field distribution for TE Mode

The MF has two components x and z which have different special distribution. the important thing to note is that H_x and H_z component are shifted in space by quarter cycle in both x and z dir^n. This is due to fact that H_x have a sign variation where as H_z has co-sign variation.

Physically this means that at a location x where H_x is maximum, H_z is zero and vice-versa and at location z where H_x is max^m, H_z is zero.

So, at some location say $x=0$ and $z=0$, H_z is max^m and H_x is zero.
At $x = a/2$, H_z becomes zero and H_x is max^m.
Also at $x = a$, H_z again becomes maximum with opposite direction and H_x again becomes zero.

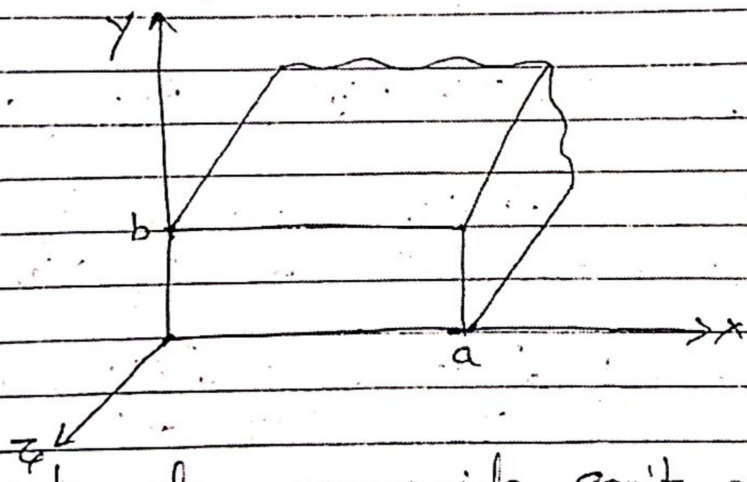
Now repeating the block of length λ_g along the length of the waveguide we can get the total distribution of electric and MF.

The whole field distribution is similar to a train with magnetic fields looking like compartments and the electric field appearing like the connecting rods.

Rectangular Waveguide

Rectangular waveguide are the one of the earliest type of the transmission lines.

A rectangular waveguide support TM & TE modes but not TEM waves because we can't define a unique voltage since there is only one conductor in a rectangular waveguide. A material with permittivity 'e' and permeability 'm' fills the inside of the conductor.



A rectangular waveguide can't propagate below some certain frequency. This freq^n is called the cut-off freq^n.

In this we discuss TM mode rectangular waveguide and TE mode rectangular waveguide.

TM Modes

Consider the shape of the rectangular waveguide above with dimensions a and b (assume $a > b$) and the parameters ϵ and μ.

For TM waves: $H_z = 0$ so E_z should be solved from equation for TM mode.

$$E_z(x, y, z) = E_z^0(x, y) e^{-\gamma z}$$

we solve the following second order partial differential Eq^n.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + h^2\right) E_z^0(x, y) = 0 \quad \text{--- (1)}$$

$$E_z^0(x, y) = X(x) Y(y) \quad \text{--- (2)}$$ By using method of separation

Now, substitute Eq^n (2) in Eq^n (1) and dividing resulting eq^n by $X(x) Y(y)$, we have

$$= -\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} + h^2 \quad \text{--- (3)}$$

In Eq^n (3) left side is a fun^n of x only and the right side is a function of y only, both sides must equal a constant in order for the Eq^n to hold for all values of x and y.

Take constant K_x^2, we obtain two separate ordinary differential Eq^n:

$$\frac{d^2 X(x)}{dx^2} + K_x^2 X(x) = 0 \quad — (4)$$

$$\frac{d^2 Y(y)}{dy^2} + K_y^2 Y(y) = 0 \quad — (5)$$

where $K_y^2 = h^2 - K_x^2$ — (6)

Now,

In x-dir^n $E_z^o(0, y) = 0$ — (7)

$E_z^o(a, y) = 0$ — (8)

In y-dir^n $E_z^o(x, 0) = 0$ — (9)

$E_z^o(x, b) = 0$ — (10)

$X(x)$ in the form of $\sin K_x x$,

$$K_x = \frac{m\pi}{a}, \quad m = 1, 2, 3, ----$$

$Y(y)$ in the form of $\sin K_y y$,

$$K_y = \frac{n\pi}{b}, \quad n = 1, 2, 3, ----$$

Sol^n for $E_z^o(x, y)$ is

$$\boxed{E_z^o(x, y) = E_0 \sin\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) \text{ v/m.}} \quad — (11)$$

Now, Eq^n (6) becomes

$$\boxed{h^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \quad — (12)$$

For TM wave,

$$H_y^0 = \frac{-j\omega\epsilon}{h^2}\frac{\partial E_z^0}{\partial x} = \frac{-j\omega\epsilon}{h^2}\left(\frac{m\pi}{a}\right)E_0 \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

$$H_x^0 = \frac{j\omega\epsilon}{h^2}\frac{\partial E_z^0}{\partial y} = \frac{j\omega\epsilon}{h^2}\left(\frac{n\pi}{b}\right)E_0 \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$$

$$E_y^0 = \frac{-\gamma}{h^2}\frac{\partial E_z^0}{\partial y} = \frac{-\gamma}{h^2}\left(\frac{n\pi}{b}\right)E_0 \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$$

$$E_x^0 = \frac{-\gamma}{h^2}\frac{\partial E_z^0}{\partial x} = \frac{-\gamma}{h^2}\left(\frac{m\pi}{a}\right)E_0 \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

where,

$$\gamma = j\beta = j\sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

Here, m and n represent possible modes and it is designated as the TM_{mn} mode.
'm' denote the number of half cycle variation of the fields in x-dir^n and
'n' denote the number of half cycle variations of the field in the y-dir^n.

TM mode the cutoff frequency is.

$$f_c = \frac{1}{2\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}\ Hz$$

TE waves in rectangular waveguide :-

For TE, $E_z = 0$, we solve

$$H_z(x, y, z) = H_z^{o}(x, y)e^{-\gamma z} \quad \text{--- (1)}$$

where, $H_z^{o}(x, z)$ satisfies the 2nd-order partial differential Eq^n

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + h^2\right)H_z^{o}(x, y) = 0 \quad \text{--- (2)}$$

The Sol^n for $H_z^{o}(x, y)$ must satisfy the following boundary conditions.

1.) In x-dir^n $\frac{\partial H_z^{o}}{\partial x} = 0$ ($E_y = 0$) at $x = 0$

$\frac{\partial H_z^{o}}{\partial x} = 0$ ($E_y = 0$) at $x = a$

2.) In y-dir^n $\frac{\partial H_z^{o}}{\partial y} = 0$ ($E_x = 0$) at $y = 0$

$\frac{\partial H_z^{o}}{\partial y} = 0$ ($E_x = 0$) at $y = b$

$$H_z^{o}(x, y) = H_0 \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) \ A/m$$

The relation b/w the eigenvalue h and $(m\pi/a)$ and $(n\pi/b)$ is same as TM mode.

$$E_x^{o}(x, y) = \frac{j\omega\mu}{h^2}\left(\frac{n\pi}{b}\right)H_0 \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

Page No
Date

$\Rightarrow E_y^0(x,y) = \frac{-j\omega\mu}{h^2}\left(\frac{m\pi}{a}\right) H_0 \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$

$\Rightarrow H_x^0(x,y) = \frac{\gamma}{h^2}\left(\frac{m\pi}{a}\right) H_0 \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi}{b}y\right)$

$\Rightarrow H_y^0(x,y) = \frac{\gamma}{h^2}\left(\frac{n\pi}{b}\right) H_0 \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right)$

where γ has same as TM modes.
For TE modes, either m or n (but not both) can be zero. If $a>b$, the cutoff freq^n is the lowest when $m=1$ and $n=0$.

$\therefore f_c = \frac{1}{2a\sqrt{\mu\epsilon}} = \frac{u}{2a}$ Hz.

Cutoff wavelength $\lambda_{TE} = 2a$ m.