Maxwell’s Equations

3	Maxwell's Equations-Basics of Vectors, Vector calculus, Basic laws of Electromagnetics,Maxwell's Equations, Boundary conditions at Media Interface.	**03**

Outline

- Scalars and vectors
- Vector addition and subtraction
- Vector Product
- Faraday's law
- Transformer and motional electromotive forces
- Displacement current
- Maxwell's equations in final forms
- Time-varying potentials

$$A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$$

$$\mathbf{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{A}}{A}$$

$$\mathbf{a}_A = \frac{A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

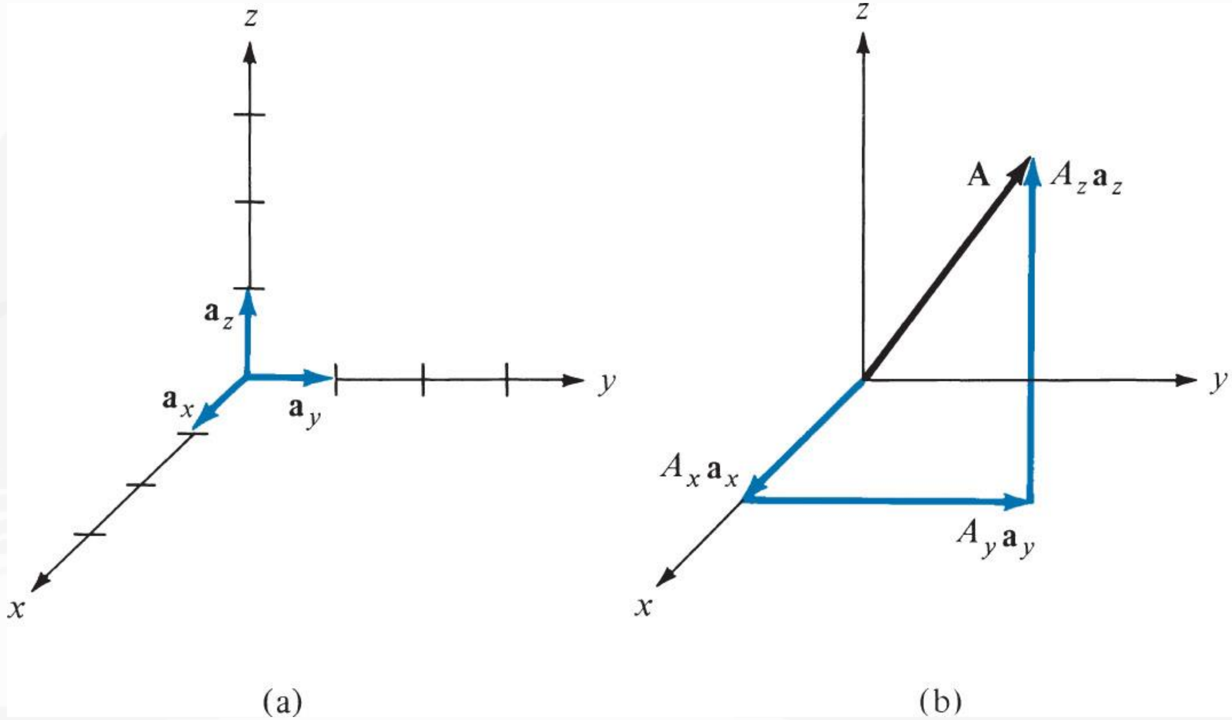


FIGURE 1.1 (**a**) Unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$, (**b**) components of **A** along $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$.

Practice Problem- # 1

Given vectors $\mathbf{A} = \mathbf{a}_x + 3\mathbf{a}_z$ and $\mathbf{B} = 5\mathbf{a}_x + 2\mathbf{a}_y - 6\mathbf{a}_z$, determine

(a) $|\mathbf{A} + \mathbf{B}|$

(b) $5\mathbf{A} - \mathbf{B}$

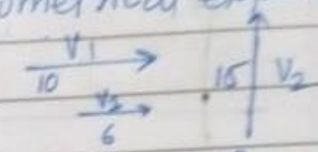
(c) The component of **A** along $\mathbf{a}_y$

(d) A unit vector parallel to $3\mathbf{A} + \mathbf{B}$

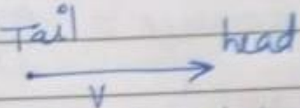
Vector :- Expression

Algebric expression | geometrical expression

$\vec{a}$

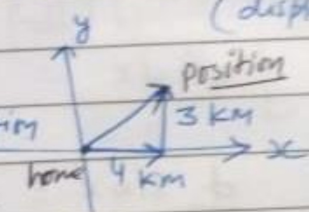


Contents – ① magnitude — $|\vec{a}| = a$ (sometimes)
② direction — $\hat{a}$ (unit vector)

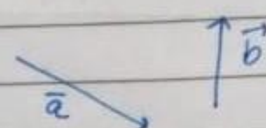


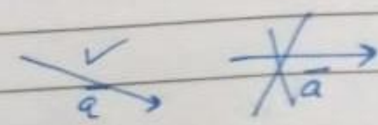
Addition : condition – two quanity should be vectors and have same nature. (velocity – veloc) (displam – disp)

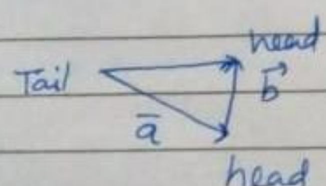
Resultant Addition =
① magnitude – Geometrical operation
② direction – Second Geometrical operation.



Triangle method of Addition :- One particle under effect of two vector simultaneously





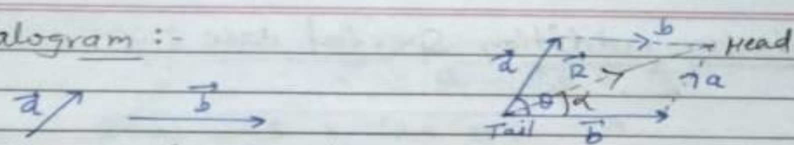


$\vec{c} = \vec{a} + \vec{b}$

↑ Resultant — components

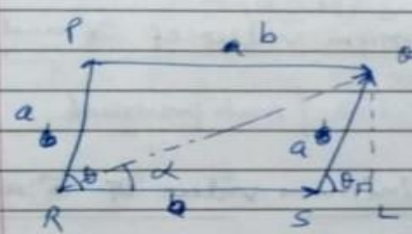
Entropy, electric potential, population → Scalar

Parallalogram :-



Parallalogram law :- if two adajacent sides of a parallalogram are represented by two vectors, then resultant is given by diagonal passing through intersection.

Resultant → Tail to head.



$$\frac{SL}{a} = \cos\theta$$

$$\begin{cases} SL = a\cos\theta \\ QL = a\sin\theta \end{cases}$$

In right angle triangal $\Delta\, RQL$

$$RQ^2 = QL^2 + RL^2$$

$$= (a\sin\theta)^2 + (b + a\cos\theta)^2$$

$$= a^2\sin^2\theta + b^2 + a^2\cos^2\theta + 2ab\cos\theta$$

$$RQ^2 = a^2 + b^2 + 2ab\cos\theta$$

Resultant $RQ = \sqrt{a^2 + b^2 + 2ab\cos\theta}$ → magnitude of resultant

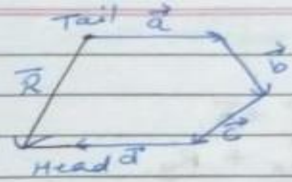
For direction

$$\tan\alpha = \frac{QL}{RL} = \frac{a\sin\theta}{b + a\cos\theta}$$

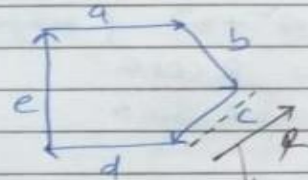
$$\alpha = \tan^{-1}(\quad '' \quad)$$

$$\alpha = \tan^{-1}\left(\frac{a\sin\theta}{b + a\cos\theta}\right)$$

Resultant ⇒ $RQ = \sqrt{a^2 + b^2 + 2ab\cos\theta}$



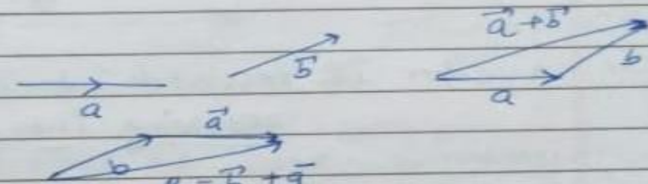
$$\vec{R} = \vec{a} + \vec{b} + \vec{c} + \vec{d}$$



$\vec{R} = 0$ (If all the vectors make closed polygone, then resultant is zero.)

Note If many vectors on addition make a closed polygon then negative of any one vector is resultant of rest of vectors.

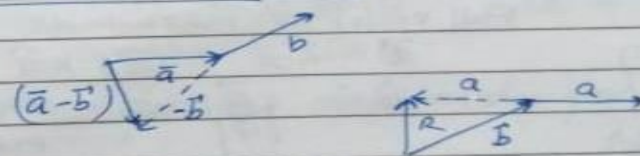
Note ① $\vec{a} + \vec{b}$



$\vec{b} + \vec{a}$

$$\boxed{\vec{a} + \vec{b} = \vec{b} + \vec{a}}$$ Commutative

② $\vec{a} - \vec{b}$



$\vec{b} - \vec{a}$

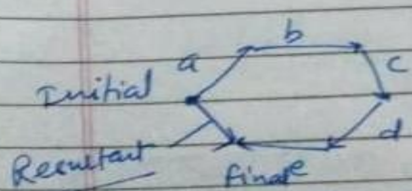
Substraction is non-commulative

Associative property: $\vec{a} + \vec{b} + \vec{c} \rightarrow (\vec{a}+\vec{b}) + \vec{c}$

$\vec{a} + (\vec{b} + \vec{c})$

$(\vec{a} + \vec{c}) + \vec{b}$

only addition



$R = \vec{a} + \vec{b} + \vec{c} + \vec{d} + \vec{e}$

1 2 3 4 5 order of draw

2 4 1 5 3

Any serial in polygone addition gives same result (direction & magnitude

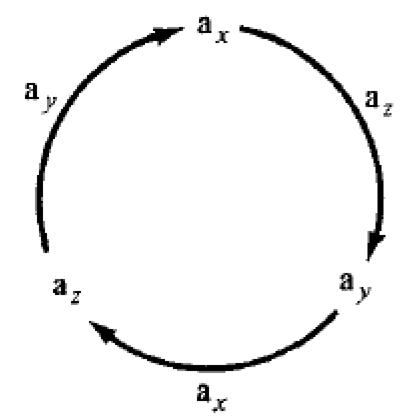
VECTOR MULTIPLICATION

1. Scalar (or dot) product: $\mathbf{A} \cdot \mathbf{B}$
2. Vector (or cross) product: $\mathbf{A} \times \mathbf{B}$

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta_{AB}$$

$$\mathbf{A} \times \mathbf{B} = AB \sin \theta_{AB} \mathbf{a}_n$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$



(i) *Commutative law:*

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$$

(ii) *Distributive law:*

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$$

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2 = A^2$$

(iii)

Also note that

$$\mathbf{a}_x \cdot \mathbf{a}_y = \mathbf{a}_y \cdot \mathbf{a}_z = \mathbf{a}_z \cdot \mathbf{a}_x = 0$$

$$\mathbf{a}_x \cdot \mathbf{a}_x = \mathbf{a}_y \cdot \mathbf{a}_y = \mathbf{a}_z \cdot \mathbf{a}_z = 1$$

Note that the cross product has the following basic properties:

(i) It is not commutative:

$$\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$$

It is anticommutative:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

(ii) It is not associative:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$$

(iii) It is distributive:

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$$

(iv)

$$\mathbf{A} \times \mathbf{A} = 0$$

Multiplication of three vectors **A**, **B**, and **C** can result in either:

3. Scalar triple product: $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

or

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

4. Vector triple product: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Practice Problem #2 If $\mathbf{A} = \mathbf{a}_x + 3\mathbf{a}_z$ and $\mathbf{B} = 5\mathbf{a}_x + 2\mathbf{a}_y - 6\mathbf{a}_z$, find θ_{AB}.

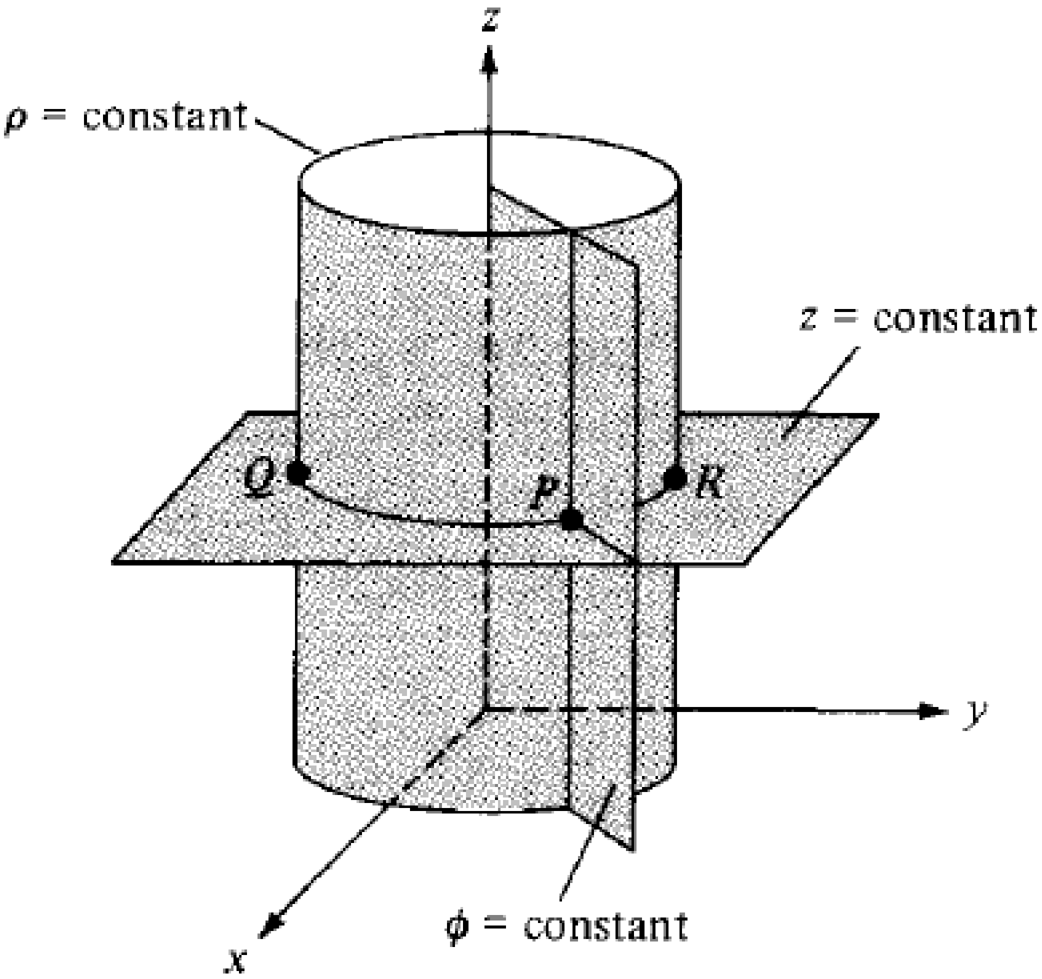
DEL OPERATOR

The del operator, written ∇, is the vector differential operator. In Cartesian coordinates,

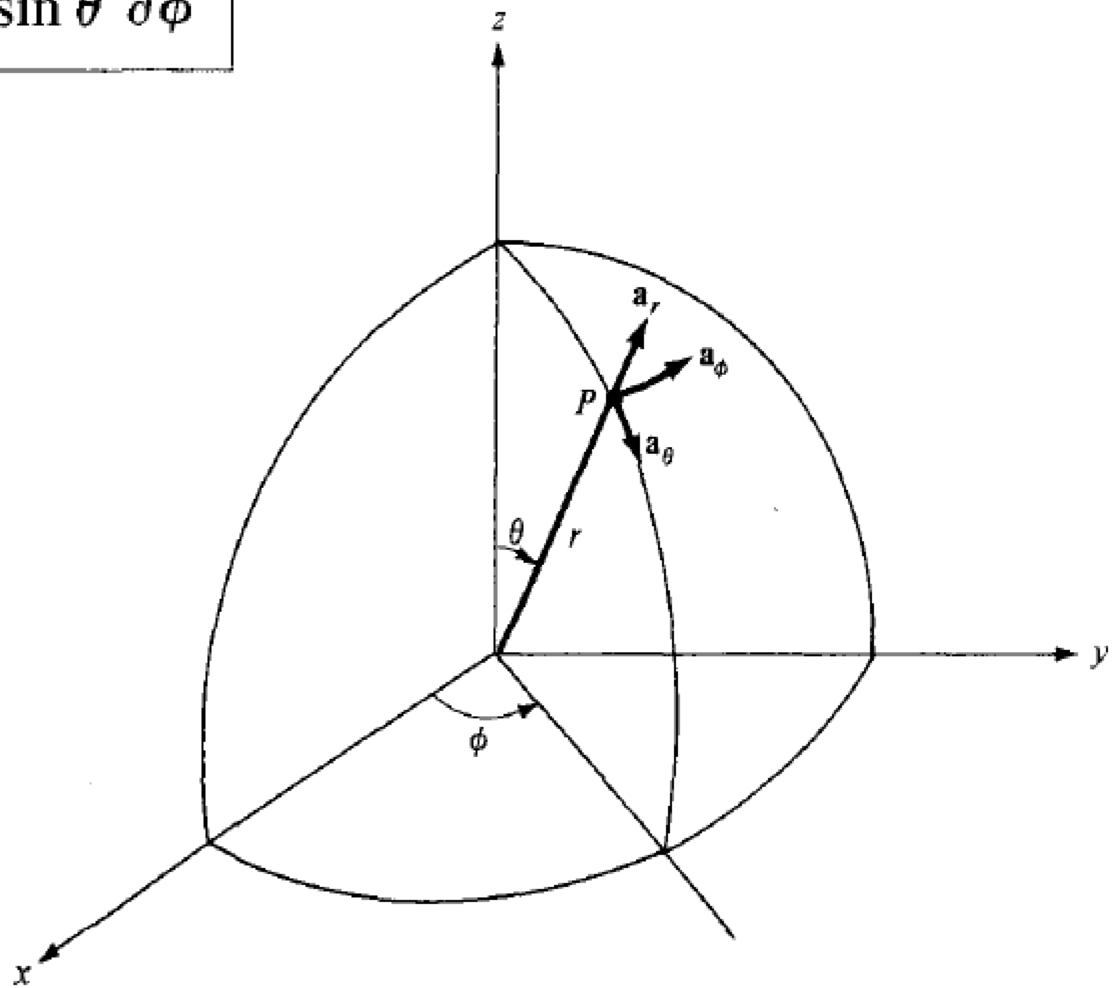
$$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z$$

1. The gradient of a scalar V, written as ∇V
2. The divergence of a vector $\mathbf{A}$, written as $\nabla \cdot \mathbf{A}$
3. The curl of a vector $\mathbf{A}$, written as $\nabla \times \mathbf{A}$
4. The Laplacian of a scalar V, written as $\nabla^2 V$

$$\nabla = \mathbf{a}_\rho \frac{\partial}{\partial \rho} + \mathbf{a}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \mathbf{a}_z \frac{\partial}{\partial z}$$



$$\nabla = \mathbf{a}_r \frac{\partial}{\partial r} + \mathbf{a}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{a}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}$$



GRADIENT OF A SCALAR

The **gradient** of a scalar field V is a vector that represents both the magnitude and the direction of the maximum space rate of increase of V.

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

for cylindrical coordinates,

$$\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z$$

and for spherical coordinates,

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi$$

Properties of the gradient of scalar

(a) $\nabla(V + U) = \nabla V + \nabla U$

(b) $\nabla(VU) = V\nabla U + U\nabla V$

(c) $\nabla\left[\frac{V}{U}\right] = \frac{U\nabla V - V\nabla U}{U^2}$

(d) $\nabla V^n = nV^{n-1}\nabla V$

where U and V are scalars and n is an integer.

Practice Problem- # 3

Given $W = x^2y^2 + xyz$, compute ∇W and the direction derivative dW/dl in the direction $3\mathbf{a}_x + 4\mathbf{a}_y + 12\mathbf{a}_z$ at $(2, -1, 0)$.

Solution:

$$\nabla W = \frac{\partial W}{\partial x}\mathbf{a}_x + \frac{\partial W}{\partial y}\mathbf{a}_y + \frac{\partial W}{\partial z}\mathbf{a}_z$$
$$= (2xy^2 + yz)\mathbf{a}_x + (2x^2y + xz)\mathbf{a}_y + (xy)\mathbf{a}_z$$

At $(2, -1, 0)$: $\nabla W = 4\mathbf{a}_x - 8\mathbf{a}_y - 2\mathbf{a}_z$

Hence,

$$\frac{dW}{dl} = \nabla W \cdot \mathbf{a}_l = (4, -8, -2) \cdot \frac{(3, 4, 12)}{13} = -\frac{44}{13}$$

DIVERGENCE OF A VECTOR

The **divergence** of **A** at a given point P is the *outward* flux per unit volume as the volume shrinks about P.

$$\text{div}\,\mathbf{A} = \nabla \cdot \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v}$$

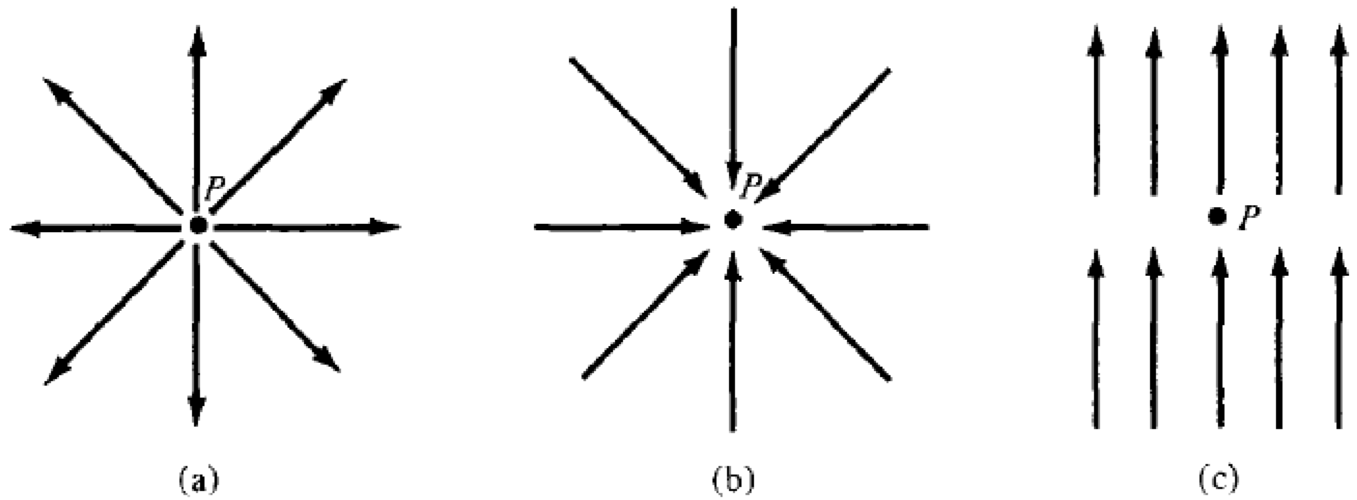


Figure 3.14 Illustration of the divergence of a vector field at P; **(a)** positive divergence, **(b)** negative divergence, **(c)** zero divergence.

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

1. It produces a scalar field (because scalar product is involved).
2. The divergence of a scalar V, div V, makes no sense.
3. $\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$
4. $\nabla \cdot (V\mathbf{A}) = V\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla V$

Practice Problem- # 4

Determine the divergence of the following vector fields and evaluate them at the specified points.

(a) $\mathbf{A} = yz\mathbf{a}_x + 4xy\mathbf{a}_y + y\mathbf{a}_z$ at $(1, -2, 3)$

(b) $\mathbf{B} = \rho z \sin\phi\, \mathbf{a}_\rho + 3\rho z^2 \cos\phi\, \mathbf{a}_\phi$ at $(5, \pi/2, 1)$

(c) $\mathbf{C} = 2r\cos\theta\cos\phi\, \mathbf{a}_r + r^{1/2}\mathbf{a}_\phi$ at $(1, \pi/6, \pi/3)$

(a) $\nabla \bullet \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0 + 4x + 0 = \underline{\underline{4x}}.$

At $(1,-2,3)$, $\nabla \bullet \boldsymbol{A} = \underline{\underline{4}}.$

(b)

$$\nabla \bullet \boldsymbol{B} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho B_\rho) + \frac{1}{\rho}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial \rho}$$

$$= \frac{1}{\rho} 2\rho z \sin\phi - \frac{1}{\rho} 3\rho z^2 \sin\phi = 2z\sin\phi - 3z^2 \sin\phi$$

$$= \underline{\underline{(2-3z)z\sin\phi}}.$$

$$At\,(5, \frac{\pi}{2}, 1), \quad \nabla \bullet \boldsymbol{B} = (2-3)(1) = \underline{\underline{-1}}.$$

(c)

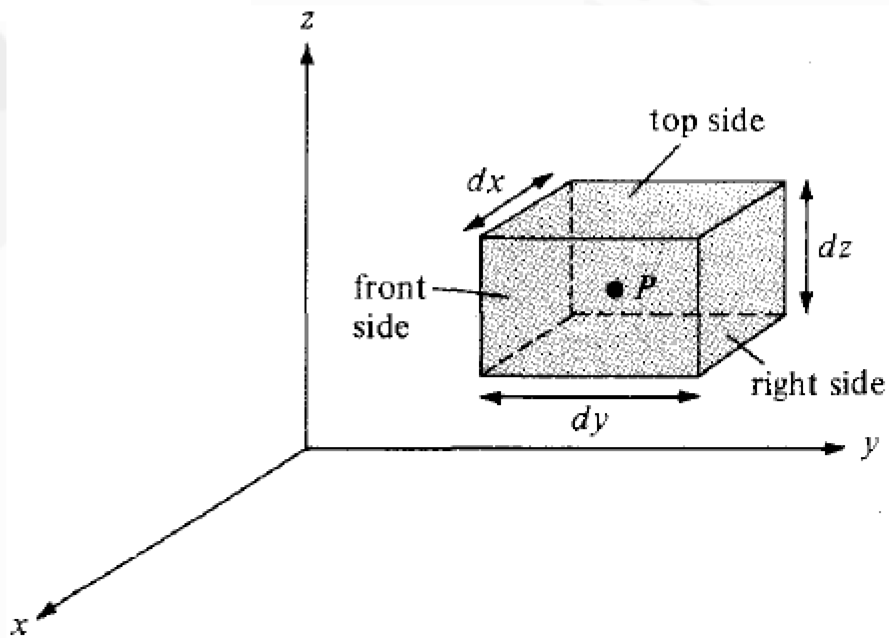
$$\nabla \bullet \boldsymbol{C} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 C_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(C_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial C_\phi}{\partial \phi}$$

$$= \frac{1}{r^2} 6r^2 \cos\theta\cos\phi$$

$$= \underline{\underline{6\cos\theta\cos\phi}}$$

$$At\,(1, \frac{\pi}{6}, \frac{\pi}{3}), \quad \nabla \bullet \boldsymbol{C} = 6\cos\frac{\pi}{6}\cos\frac{\pi}{3} = \underline{\underline{2.598}}.$$

$$\text{div}\,\mathbf{A} = \nabla \cdot \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v}$$

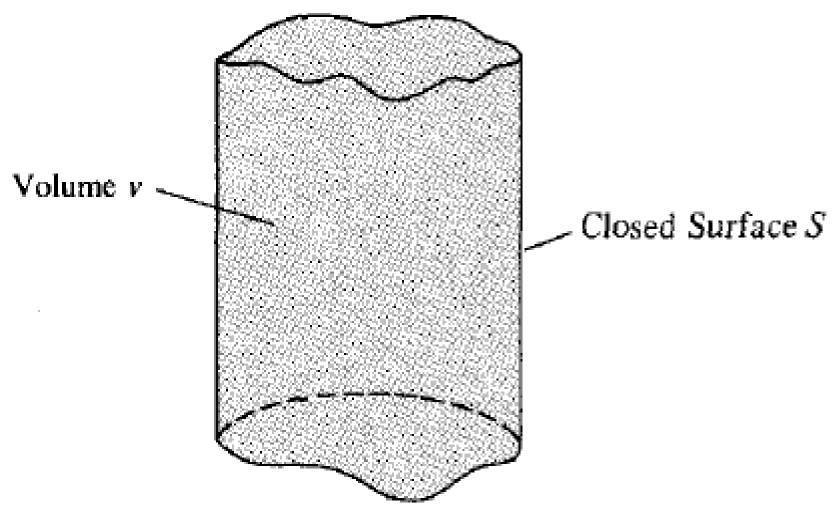


$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \left(\int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}} \right) \mathbf{A} \cdot d\mathbf{S}$$

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{A} \, dv$$

This is called the *divergence theorem*, otherwise known as the *Gauss–Ostrogradsky theorem.*

The **divergence theorem** states that the total outward flux of a vector field **A** through the *closed* surface *S* is the same as the volume integral of the divergence of **A**.



Practice Problem- # 5

If $\mathbf{G}(r) = 10e^{-2z}(\rho\mathbf{a}_\rho + \mathbf{a}_z)$, determine the flux of $\mathbf{G}$ out of the entire surface of the cylinder $\rho = 1, 0 \le z \le 1$. Confirm the result using the divergence theorem.

Solution:

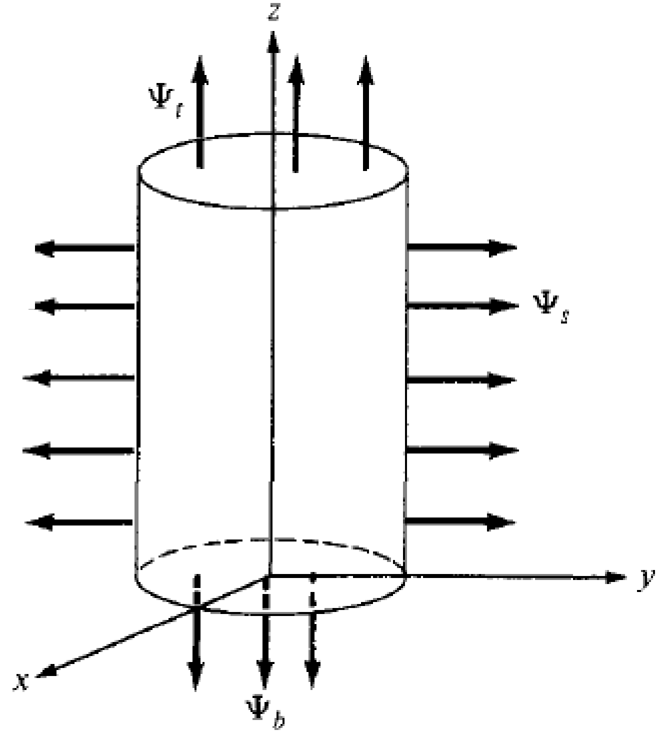
$$\Psi = \oint \mathbf{G} \cdot d\mathbf{S} = \Psi_t + \Psi_b + \Psi_s$$

where Ψ_t, Ψ_b, and Ψ_s are the fluxes through the top, bottom, and sides (curved surface) of the cylinder as in Figure 3.17.

For Ψ_t, $z = 1$, $d\mathbf{S} = \rho\, dp\, d\phi\, \mathbf{a}_z$. Hence,

$$\Psi_t = \int \mathbf{G} \cdot d\mathbf{S} = \int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} 10e^{-2}\rho\, d\rho\, d\phi = 10e^{-2}(2\pi) \left.\frac{\rho^2}{2}\right|_0^1$$

$$= 10\pi e^{-2}$$



For Ψ_b, $z = 0$ and $d\mathbf{S} = \rho\, d\rho\, d\phi(-\mathbf{a}_z)$. Hence,

$$\Psi_b = \int_b \mathbf{G} \cdot d\mathbf{S} = \int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} 10e^0 \rho\, d\rho\, d\phi = -10(2\pi) \frac{\rho^2}{2} \Big|_0^1$$
$$= -10\pi$$

For Ψ_s, $\rho = 1$, $d\mathbf{S} = \rho\, dz\, d\phi\, \mathbf{a}_\rho$. Hence,

$$\Psi_s = \int_s \mathbf{G} \cdot d\mathbf{S} = \int_{z=0}^{1} \int_{\phi=0}^{2\pi} 10e^{-2z} \rho^2\, dz\, d\phi = 10(1)^2(2\pi) \frac{e^{-2z}}{-2} \Big|_0^1$$
$$= 10\pi(1 - e^{-2})$$

$$\Psi = \Psi_t + \Psi_b + \Psi_s = 10\pi e^{-2} - 10\pi + 10\pi(1 - e^{-2}) = 0$$

Alternatively, since S is a closed surface, we can apply the divergence theore

$$\Psi = \oint_S G \cdot d\mathbf{S} = \int_v (\nabla \cdot \mathbf{G})\, dv$$

But

$$\nabla \cdot \mathbf{G} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho G_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \phi} G_\phi + \frac{\partial}{\partial z} G_z$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho^2 10 e^{-2z}) - 20 e^{-2z} = 0$$

showing that **G** has no source. Hence,

$$\Psi = \int_v (\nabla \cdot G)\, dv = 0$$

Practice Problem- # 6

Determine the flux of $\mathbf{D} = \rho^2 \cos^2\phi\, \mathbf{a}_\rho + z \sin\phi\, \mathbf{a}_\phi$ over the closed surface of the cylinder $0 \le z \le 1$, $\rho = 4$. Verify the divergence theorem for this case.

$$\Psi = \oint_S \boldsymbol{D} \bullet d\boldsymbol{S} = \Psi_t + \Psi_b + \Psi_c$$

$\Psi_t = 0 = \Psi_b$ since $\boldsymbol{D}$ has no z-component

$$\Psi_c = \iint \rho^2 \cos^2\phi \rho d\phi dz = \rho^3 \int_{\phi=0}^{\phi=2\pi} \cos^2\phi d\phi \int_{z=0}^{z=1} dz \Big|_{\rho=4}$$

$$= (4)^3 \pi(1) = 64\pi$$

$$\Psi = 0 + 0 + 64\pi = \underline{\underline{64\pi}}$$

By the divergence theorem,

$$\oint_S \boldsymbol{D} \bullet d\boldsymbol{S} = \oint_V \nabla \bullet \boldsymbol{D} dv$$

$$\nabla \bullet \boldsymbol{D} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^3 \cos^2\phi) + \frac{1}{\rho}\frac{\partial}{\partial\phi} z \sin\phi + \frac{\partial A_z}{dz}$$

$$= 3\rho \cos^2\phi + \frac{z}{\rho}\cos\phi.$$

$$\Psi = \int_V \nabla \bullet \boldsymbol{D} dv = \int_V (3\rho\cos^2\phi + \frac{z}{\rho}\cos\phi)\rho d\phi dz d\rho$$

$$= 3\int_0^4 \rho^2 d\rho \int_0^{2\pi} \cos^2\phi d\phi \int_0^1 dz + \int_0^4 d\rho \int_0^{2\pi} \cos\phi d\phi \int_0^1 z dz$$

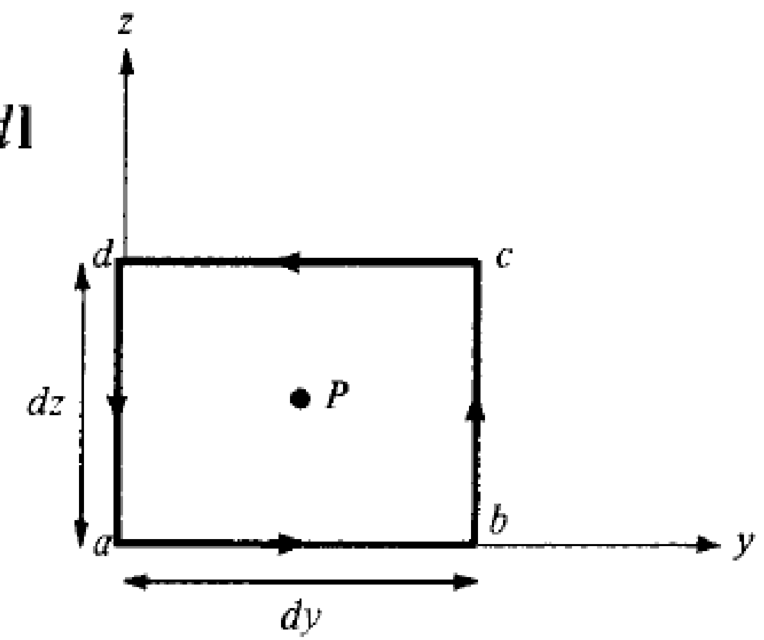
$$= 3(\frac{4^3}{3})\pi(1) = \underline{\underline{64\pi}}.$$

CURL OF A VECTOR AND STOKES'S THEOREM

The **curl of A** is an axial (or rotational) vector whose magnitude is the maximum circulation of **A** per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.[2]

$$\text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \left(\lim_{\Delta S \to 0} \frac{\oint_L \mathbf{A} \cdot d\mathbf{l}}{\Delta S} \right)_{\max} \mathbf{a}_n$$

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = \left(\int_{ab} + \int_{bc} + \int_{cd} + \int_{da} \right) \mathbf{A} \cdot d\mathbf{l}$$



$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \times \mathbf{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\mathbf{a}_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\mathbf{a}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\mathbf{a}_z$$

$$\nabla \times \mathbf{A} = \frac{1}{\rho}\begin{vmatrix} \mathbf{a}_\rho & \rho\,\mathbf{a}_\phi & \mathbf{a}_z \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$$\nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \mathbf{a}_r & r\,\mathbf{a}_\theta & r\sin\theta\,\mathbf{a}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta\,A_\phi \end{vmatrix}$$

1. The curl of a vector field is another vector field.
2. The curl of a scalar field V, $\nabla \times V$, makes no sense.
3. $\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$
4. $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$
5. $\nabla \times (V\mathbf{A}) = V\nabla \times \mathbf{A} + \nabla V \times \mathbf{A}$
6. The divergence of the curl of a vector field vanishes, that is, $\nabla \cdot (\nabla \times \mathbf{A}) = 0$.
7. The curl of the gradient of a scalar field vanishes, that is, $\nabla \times \nabla V = 0$.

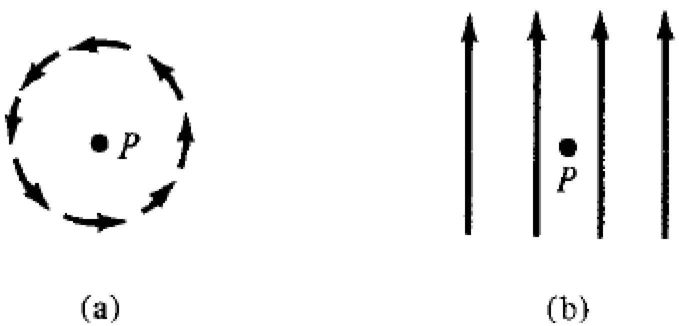


Figure 3.19 Illustration of a curl: **(a)** curl at P points out of the page; **(b)** curl at P is zero.

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

The **Laplacian** of a scalar field V, written as $\nabla^2 V$, is the divergence of the gradient of V.

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

A scalar field V is said to be *harmonic* in a given region if its Laplacian vanishes in that region. In other words, if

$$\nabla^2 V = 0$$

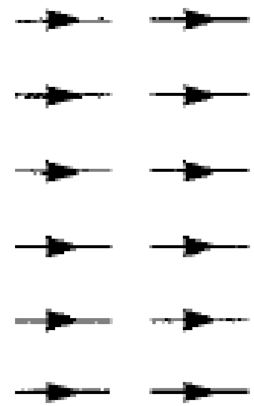
$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$$

(a) $\nabla \cdot \mathbf{A} = 0, \nabla \times \mathbf{A} = 0$

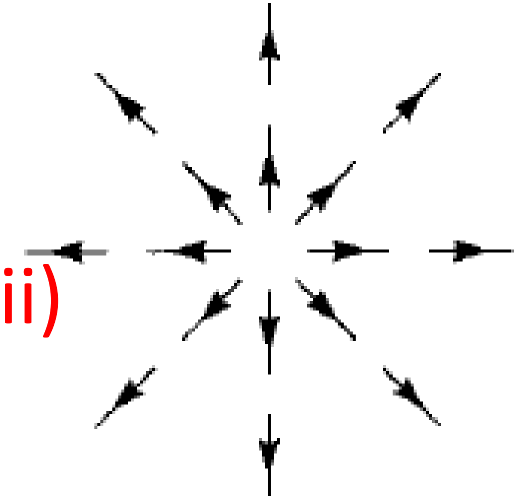
(b) $\nabla \cdot \mathbf{A} \neq 0, \nabla \times \mathbf{A} = 0$

(c) $\nabla \cdot \mathbf{A} = 0, \nabla \times \mathbf{A} \neq 0$

(d) $\nabla \cdot \mathbf{A} \neq 0, \nabla \times \mathbf{A} \neq 0$



(i)



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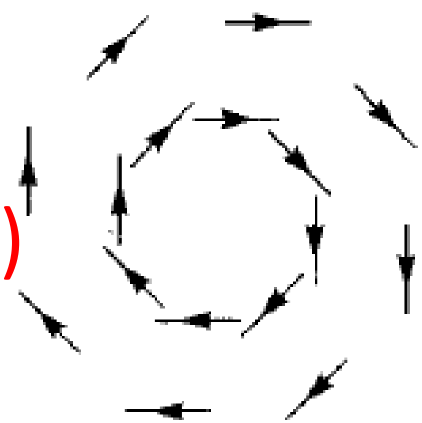

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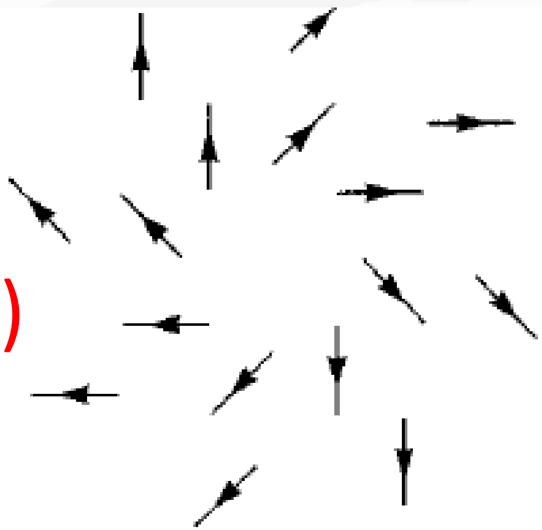


(iv)

Faraday's law

Faraday discovered that the **induced emf,** V_{emf} (in volts), in any closed circuit is equal to the time rate of change of the magnetic flux linkage by the circuit.

$$V_{emf} = -\frac{d\lambda}{dt} = -N\frac{d\Psi}{dt} \qquad (9.1)$$

Lenz's law states that the direction of the induced current is such that the magnetic field produced by it opposes the change in the original magnetic field (which is the cause of the induction)

stationary charges	→ electrostatic fields
steady currents	→ magnetostatic fields
time-varying currents	→ electromagnetic fields (or waves)

Transformer and motional electromotive forces

$$\boxed{V_{emf} = -\frac{d\psi}{dt}} \qquad (9.4)$$

In terms of E and B, eq. (9.4) can be written as

$$V_{emf} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \qquad (9.5)$$

S is the surface area of the circuit bounded by the closed path *L*. It is clear from eq. (9.5) that in a time-varying situation, both electric and magnetic fields are present and are interrelated. Note that *dl* and S in eq. (9.5) are in accordance with the right-hand rule as well as Stokes's theorem.

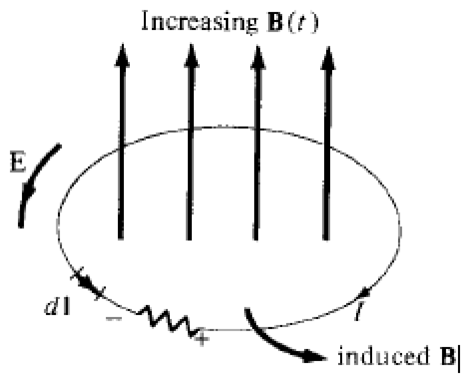
The variation of flux with time may be caused in three ways:
1. By having a stationary loop in a time-varying B field
2. By having a time-varying loop area in a static B field
3. By having a time-varying loop area in a time-varying B field

A. Transformer EMF (stationary loop in a time-varying B field)

$$V_{\mathrm{emf}} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \qquad (9.6b)$$

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (9.8)$$



This is one of the Maxwell's equations for time-varying fields.

Transformer and motional electromotive forces

B. Motional EMF (Moving Loop in Static B Field)

When a conducting loop is moving in a static B field, an emf is induced in the loop.
The force on a moving charge with velocity **u** in a magnetic field **B**:

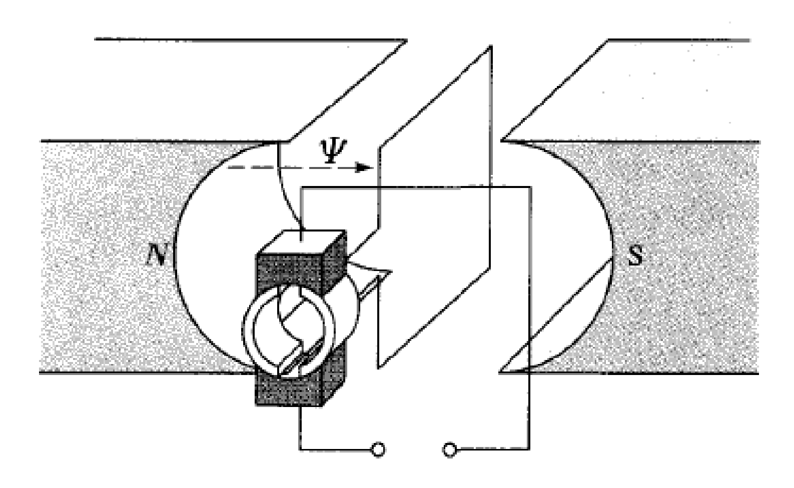
$$\mathbf{F}_m = Q\mathbf{u} \times \mathbf{B} \qquad (8.2)$$

Motional electric field:

$$\mathbf{E}_m = \frac{\mathbf{F}_m}{Q} = \mathbf{u} \times \mathbf{B} \qquad (9.9)$$

This motional emf exists in motors and generators

$$V_{emf} = \oint_L \mathbf{E}_m \cdot d\mathbf{l} = \oint_L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$



By applying Stokes's theorem to eq. (9.10)

$$\int_S (\nabla \times \mathbf{E}_m) \cdot d\mathbf{S} = \int_S \nabla \times (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{S}$$

or

$$\boxed{\nabla \times \mathbf{E}_m = \nabla \times (\mathbf{u} \times \mathbf{B})}$$

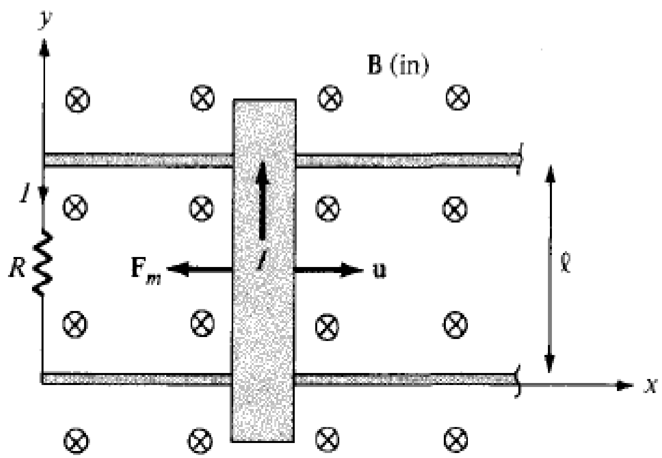


Figure 9.5 Induced emf due to a moving loop in a static **B** field.

$$\mathbf{F}_m = I\boldsymbol{\ell} \times \mathbf{B}$$

$$F_m = I\ell B$$

$$V_{\text{emf}} = uB\ell$$

Transformer and motional electromotive forces

C. Moving Loop in Time-Varying Field

Both transformer and motional emfs exist in this case:

$$\boxed{V_{emf} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \oint_L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}} \qquad (9.15)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathrm{B}}{\partial t} + \nabla \times (\mathbf{u} \times \mathbf{B}) \qquad (9.16)$$

Practice Problem- # 7

A conducting bar can slide freely over two conducting rails as shown in Figure 9.6. Calculate the induced voltage in the bar

(a) If the bar is stationed at $y = 8$ cm and $\mathbf{B} = 4 \cos 10^6 t\, \mathbf{a}_z$ mWb/m^2

(b) If the bar slides at a velocity $\mathbf{u} = 20\mathbf{a}_y$ m/s and $\mathbf{B} = 4\mathbf{a}_z$ mWb/m^2

(c) If the bar slides at a velocity $\mathbf{u} = 20\mathbf{a}_y$ m/s and $\mathbf{B} = 4 \cos (10^6 t - y)\, \mathbf{a}_z$ mWb/m^2

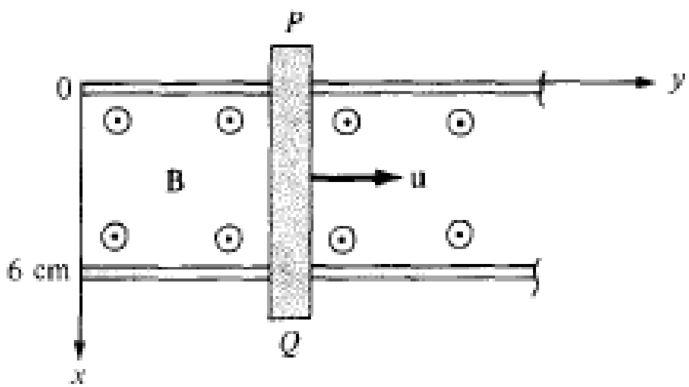


Figure 9.6 For Example 9.1.

(a) In this case, we have transformer emf given by

$$V_{\text{emf}} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = \int_{y=0}^{0.08} \int_{x=0}^{0.06} 4(10^{-3})(10^{6}) \sin 10^{6}t \, dx \, dy$$

$$= 4(10^{3})(0.08)(0.06) \sin 10^{6}t$$

$$= 19.2 \sin 10^{6}t \text{ V}$$

(b) This is the case of motional emf:

$$V_{emf} = \int (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = \int_{x=\ell}^{0} (u\mathbf{a}_y \times B\mathbf{a}_z) \cdot dx\mathbf{a}_x$$
$$= -uB\ell = -20(4.10^{-3})(0.06)$$
$$= -4.8 \text{ mV}$$

(c) Both transformer emf and motional emf are present in this case. This problem can be solved in two ways.

Method 1: Using eq. (9.15)

$$V_{emf} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \int (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \tag{9.1.1}$$

$$= \int_{x=0}^{0.06} \int_{0}^{y} 4.10^{-3}(10^6) \sin(10^6 t - y')dy'\, dx$$

$$+ \int_{0.06}^{0} [20\mathbf{a}_y \times 4.10^{-3} \cos(10^6 t - y)\mathbf{a}_z] \cdot dx\, \mathbf{a}_x$$

$$= 240 \cos(10^6 t - y')\Big|_0^y - 80(10^{-3})(0.06) \cos(10^6 t - y)$$

$$= 240 \cos(10^6 t - y) - 240 \cos 10^6 t - 4.8(10^{-3}) \cos(10^6 t - y)$$

$$\simeq 240 \cos(10^6 t - y) - 240 \cos 10^6 t \tag{9.1.2}$$

because the motional emf is negligible compared with the transformer emf. Using trigonometric identity

$$\cos A - \cos B = -2 \sin \frac{A + B}{2} \sin \frac{A - B}{2}$$

$$V_{emf} = 480 \sin\left(10^6 t - \frac{y}{2}\right) \sin \frac{y}{2} \text{ V} \tag{9.1.3}$$

Method 2: Alternatively we can apply eq. (9.4), namely,

$$V_{emf} = -\frac{\partial \Psi}{\partial t}$$

where,

$$\begin{aligned}\Psi &= \int \mathbf{B} \cdot d\mathbf{S} \\ &= \int_{y=0}^{y} \int_{x=0}^{0.06} 4 \cos(10^6 t - y)\, dx\, dy \\ &= -4(0.06) \sin(10^6 t - y) \Big|_{y=0}^{y} \\ &= -0.24 \sin(10^6 t - y) + 0.24 \sin 10^6 t \text{ mWb}\end{aligned}$$

But,

$$\frac{dy}{dt} = u \rightarrow y = ut = 20t$$

Hence,

$$\Psi = -0.24 \sin(10^6 t - 20t) + 0.24 \sin 10^6 t \text{ mWb}$$

$$\begin{aligned}V_{emf} &= -\frac{\partial \Psi}{\partial t} = 0.24(10^6 - 20) \cos(10^6 t - 20t) - 0.24(10^6) \cos 10^6 t \text{ mV} \\ &\simeq 240 \cos(10^6 t - y) - 240 \cos 10^6 t \text{ V}\end{aligned}$$

Displacement current

For static EM fields

$$\nabla \times \mathbf{H} = \mathbf{J}$$

divergence of the curl of any vector field is identically zero

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J}$$

The continuity of current in

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \neq 0$$

Thus eqs. these are obviously incompatible for time-varying conditions. We must modify the eq. To do this, we add a term to eq. $\nabla \times \mathbf{H} = \mathbf{J}$ so that it becomes

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_d \tag{9.20}$$

where $\mathbf{J}_d$ is to be determined and defined. Again, the divergence of the curl of any vector is zero. Hence:

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} + \nabla \cdot \mathbf{J}_d \tag{9.21}$$

In order for eq. (9.21) to agree with eq. (9.19),

$$\nabla \cdot \mathbf{J}_d = -\nabla \cdot \mathbf{J} = \frac{\partial \rho_v}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) = \nabla \cdot \frac{\partial \mathbf{D}}{\partial t} \tag{9.22a}$$

or

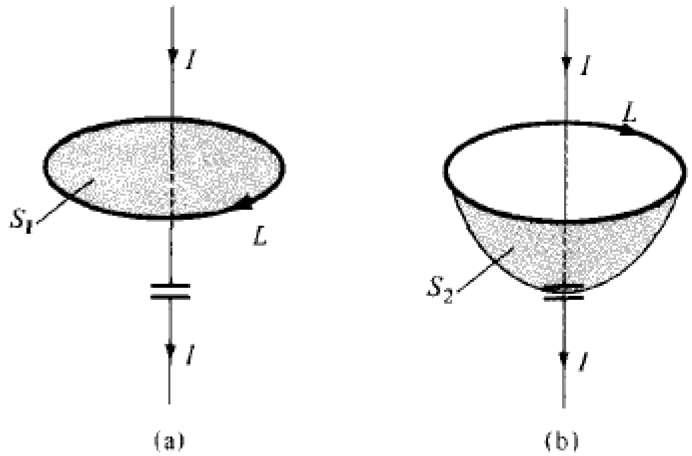
$$\boxed{\mathbf{J}_d = \frac{\partial \mathbf{D}}{dt}} \tag{9.22b}$$

Substituting eq. (9.22b) into eq. (9.20) results in

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}} \tag{9.23}$$

This is Maxwell's equation (based on Ampere's circuit law) for a time-varying field. The term $\mathbf{J}_d = \partial \mathbf{D}/\partial t$ is known as *displacement current density* and $\mathbf{J}$ is the conduction current density ($\mathbf{J} = \sigma\mathbf{E}$).[3] The insertion of $\mathbf{J}_d$ into eq. (9.17) was one of the major contributions of Maxwell. Without the term $\mathbf{J}_d$, electromagnetic wave propagation (radio or TV waves, for example) would be impossible. At low frequencies, $\mathbf{J}_d$ is usually neglected compared with $\mathbf{J}$. However, at radio frequencies, the two terms are comparable. At the time of Maxwell, high-frequency sources were not available and eq. (9.23) could not be verified experimentally. It was years later that Hertz succeeded in generating and detecting radio waves thereby verifying eq. (9.23). This is one of the rare situations where mathematical argument paved the way for experimental investigation.

Displacement current



Based on the displacement current density, we define the *displacement current* as

$$I_d = \int \mathbf{J}_d \cdot d\mathbf{S} = \int \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S} \tag{9.24}$$

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_{S_1} \mathbf{J} \cdot d\mathbf{S} = I_{\text{enc}} = I \tag{9.25}$$

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_{S_2} \mathbf{J} \cdot d\mathbf{S} = I_{\text{enc}} = 0 \tag{9.26}$$

The total current density is $\mathbf{J} + \mathbf{J}_d$.

eq. (9.25), $\mathbf{J}_d = 0$ so that the equation remains valid. In eq. (9.26), $\mathbf{J} = 0$ so that

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_{S_2} \mathbf{J}_d \cdot d\mathbf{S} = \frac{d}{dt}\int_{S_2} \mathbf{D} \cdot d\mathbf{S} = \frac{dQ}{dt} = I \tag{9.27}$$

① **Gauss law:** $\psi_{out} = Q_{enclosed}$

$$\oint E \cdot ds = \frac{Q_{enclosed}}{\varepsilon_0}$$

$$\oint \varepsilon_0 E \cdot ds = Q_{enclosed}$$

$\therefore \; D = \varepsilon_0 E$ — electric flux density or electric displacement

$$\oint D \cdot ds = Q_{enclosed}$$

Apply divergence theorem

$$\int_v (\nabla \cdot D)\, dv = \int_v \rho_v \, dv$$

$$\boxed{\nabla \cdot D = \rho_v}$$ — 1st Maxwell's equation for electrostatic

$\nabla \rightarrow 1/m$, $D \rightarrow C/m^2$, $\rho_v \rightarrow \frac{C}{m^3}$

The total output magnetic flux through any closed loop surface is zero.

$$\phi_{net} = 0$$

$$\oint B \cdot ds = 0$$

Apply divergence eq^n

$$\int_V (\nabla \cdot \vec{B})\, dV = 0$$

$$\boxed{\nabla \cdot \vec{B} = 0}$$

⇑

2nd maxwell's equation for magnetostatics

Maxwell's equations in final forms

Table 9.1 Generalized Forms of Maxwell's Equations

Differential Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v dV$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of isolated magnetic charge*
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t}\int_S \mathbf{B} \cdot d\mathbf{S}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}$	Ampère's circuit law

TABLE 7.2 Maxwell's Equations for Static EM Fields

Differential (or Point) Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of magnetic monopole
$\nabla \times \mathbf{E} = 0$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = 0$	Conservativeness of electrostatic field
$\nabla \times \mathbf{H} = \mathbf{J}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S}$	Ampere's law

Boundary conditions

If the field exists in a region consisting of two different media, the conditions that the field must satisfy at the interface separating the media are called *boundary conditions*.

These conditions are helpful in determining the field on one side of the boundary if the field on the other side is known.

Obviously, the conditions will be dictated by the types of material the media are made of. We shall consider the boundary conditions at an interface separating

- dielectric (er1) and dielectric (er2)
- conductor and dielectric
- conductor and free space

Boundary conditions

To determine the boundary conditions, we need to use Maxwell's equations:

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \tag{5.52}$$

and

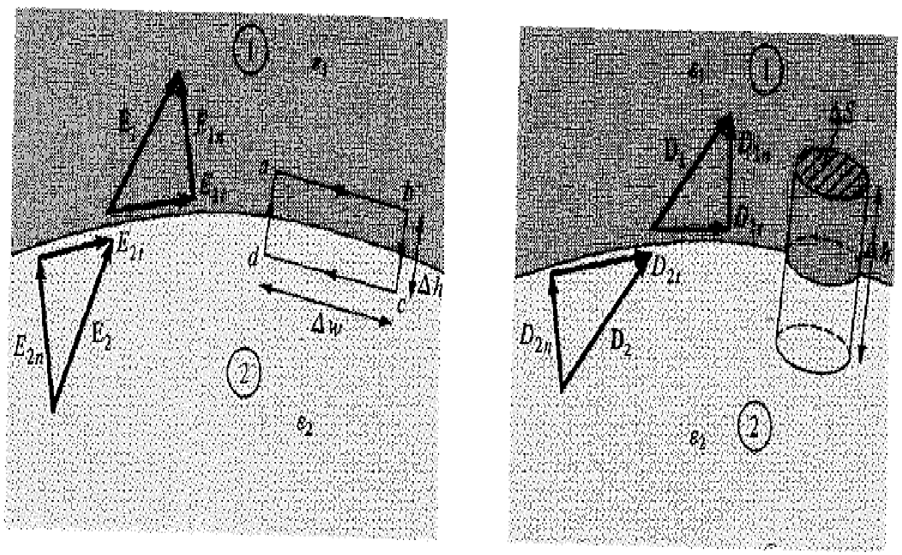
$$\oint \mathbf{D} \cdot d\mathbf{S} = Q_{\text{enc}} \tag{5.53}$$

Also we need to decompose the electric field intensity **E** into two orthogonal components:

$$\mathbf{E} = \mathbf{E}_t + \mathbf{E}_n \tag{5.54}$$

where $\mathbf{E}_t$ and $\mathbf{E}_n$ are, respectively, the tangential and normal components of **E** to the interface of interest. A similar decomposition can be done for the electric flux density **D**.

Boundary conditions



(a) (b)

We apply eq. (5.52) to the closed path *abcda* of Figure 5.10(a) assuming that the path i very small with respect to the variation of **E**. We obtain

$$0 = E_{1t}\,\Delta w - E_{1n}\frac{\Delta h}{2} - E_{2n}\frac{\Delta h}{2} - E_{2t}\,\Delta w + E_{2n}\frac{\Delta h}{2} + E_{1n}\frac{\Delta h}{2} \qquad (5.56$$

where $E_t = |\mathbf{E}_t|$ and $E_n = |\mathbf{E}_n|$. As $\Delta h \rightarrow 0$, eq. (5.56) becomes

$$\boxed{E_{1t} = E_{2t}} \qquad (5.57$$

Thus the tangential components of **E** are the same on the two sides of the boundary. In other words, $\mathbf{E}_t$ undergoes no change on the boundary and it is said to be *continuous* across the boundary. Since $\mathbf{D} = \varepsilon\mathbf{E} = \mathbf{D}_t + \mathbf{D}_n$, eq. (5.57) can be written as

$$\frac{D_{1t}}{\varepsilon_1} = E_{1t} = E_{2t} = \frac{D_{2t}}{\varepsilon_2}$$

or

$$\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2} \qquad (5.58)$$

that is, D_t undergoes some change across the interface. Hence D_t is said to be *discontinuous* across the interface.

$$\varepsilon_1 = \varepsilon_o\varepsilon_{r1} \text{ and } \varepsilon_2 = \varepsilon_o\varepsilon_{r2}$$

$$\mathbf{E}_1 = \mathbf{E}_{1t} + \mathbf{E}_{1n}$$

$$\mathbf{E}_2 = \mathbf{E}_{2t} + \mathbf{E}_{2n}$$

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0$$

Boundary conditions

Similarly, we apply eq. (5.53) to the pillbox (Gaussian surface) of Figure 5.10(b). Allowing $\Delta h \rightarrow 0$ gives

$$\Delta Q = \rho_S \, \Delta S = D_{1n} \, \Delta S - D_{2n} \, \Delta S$$

or

$$\boxed{D_{1n} - D_{2n} = \rho_S} \qquad (5.59)$$

where ρ_S is the free charge density placed deliberately at the boundary. It should be borne in mind that eq. (5.59) is based on the assumption that **D** is directed from region 2 to region 1 and eq. (5.59) must be applied accordingly. If no free charges exist at the interface (i.e., charges are not deliberately placed there), $\rho_S = 0$ and eq. (5.59) becomes

$$\boxed{D_{1n} = D_{2n}} \qquad (5.60)$$

Thus the normal component of **D** is continuous across the interface; that is, D_n undergoes no change at the boundary. Since $\mathbf{D} = \varepsilon\mathbf{E}$, eq. (5.60) can be written as

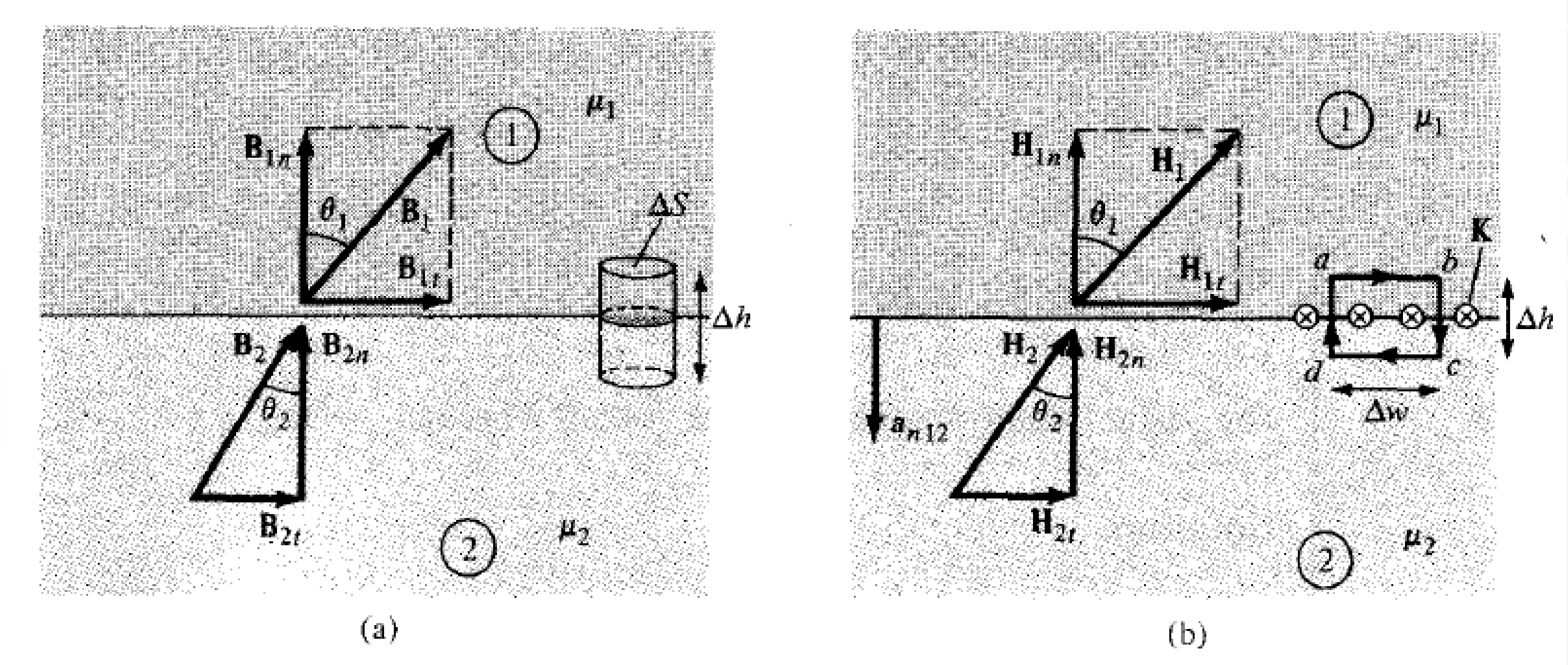
$$\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n} \qquad (5.61)$$

Boundary conditions

MAGNETIC BOUNDARY CONDITIONS

$$\oint \mathbf{B} \cdot d\mathbf{S} = 0$$

$$\oint \mathbf{H} \cdot d\mathbf{l} = I$$



Consider the boundary between two magnetic media 1 and 2, characterized, respectively, by μ_1 and μ_2 as in Figure 8.16. Applying eq. (8.38) to the pillbox (Gaussian surface) of Figure 8.16(a) and allowing $\Delta h \to 0$, we obtain

$$B_{1n}\,\Delta S - B_{2n}\,\Delta S = 0 \tag{8.40}$$

Thus

$$\boxed{\mathbf{B}_{1n} = \mathbf{B}_{2n}} \qquad \text{or} \qquad \mu_1\mathbf{H}_{1n} = \mu_2\mathbf{H}_{2n} \tag{8.41}$$

since $\mathbf{B} = \mu\mathbf{H}$. Equation (8.41) shows that the normal component of **B** is continuous at the boundary. It also shows that the normal component of **H** is discontinuous at the boundary; **H** undergoes some change at the interface.

Similarly, we apply eq. (8.39) to the closed path *abcda* of Figure 8.16(b) where surface current K on the boundary is assumed normal to the path. We obtain

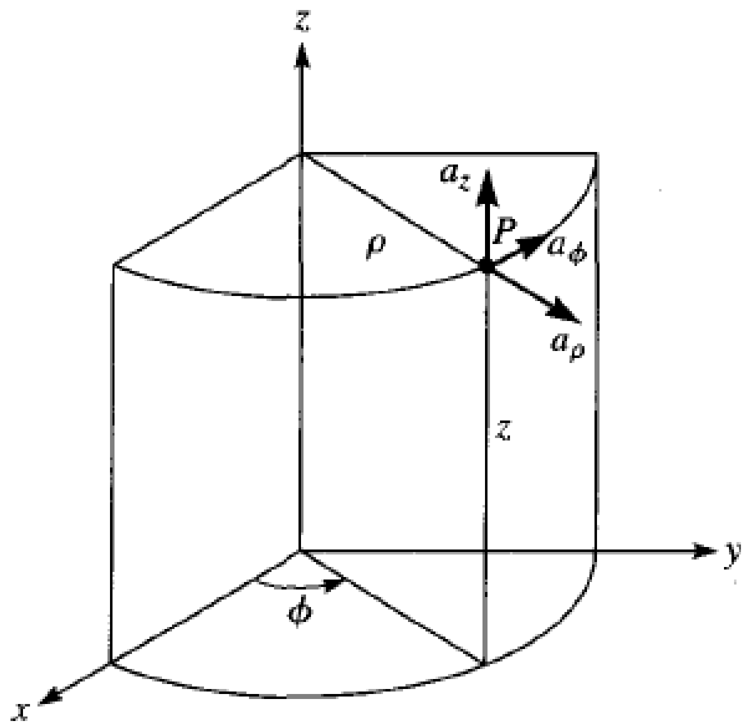
$$\begin{aligned} K \cdot \Delta w = H_{1t} \cdot \Delta w + H_{1n} \cdot \frac{\Delta h}{2} + H_{2n} \cdot \frac{\Delta h}{2} \\ -H_{2t} \cdot \Delta w - H_{2n} \cdot \frac{\Delta h}{2} - H_{1n} \cdot \frac{\Delta h}{2} \end{aligned} \tag{8.42}$$

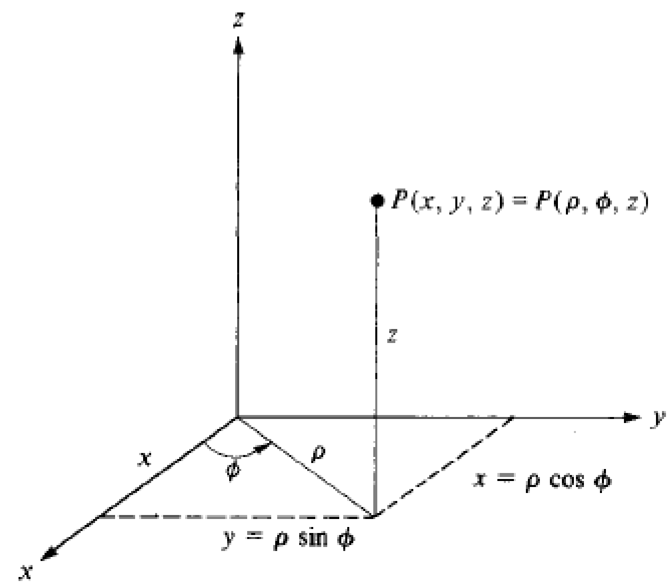
As $\Delta h \to 0$, eq. (8.42) leads to

$$H_{1t} - H_{2t} = K \tag{8.43}$$

Coordinate systems

CYLINDRICAL COORDINATES:





$$\rho = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}, \qquad z = z$$

$$x = \rho \cos \phi, \qquad y = \rho \sin \phi, \qquad z = z$$

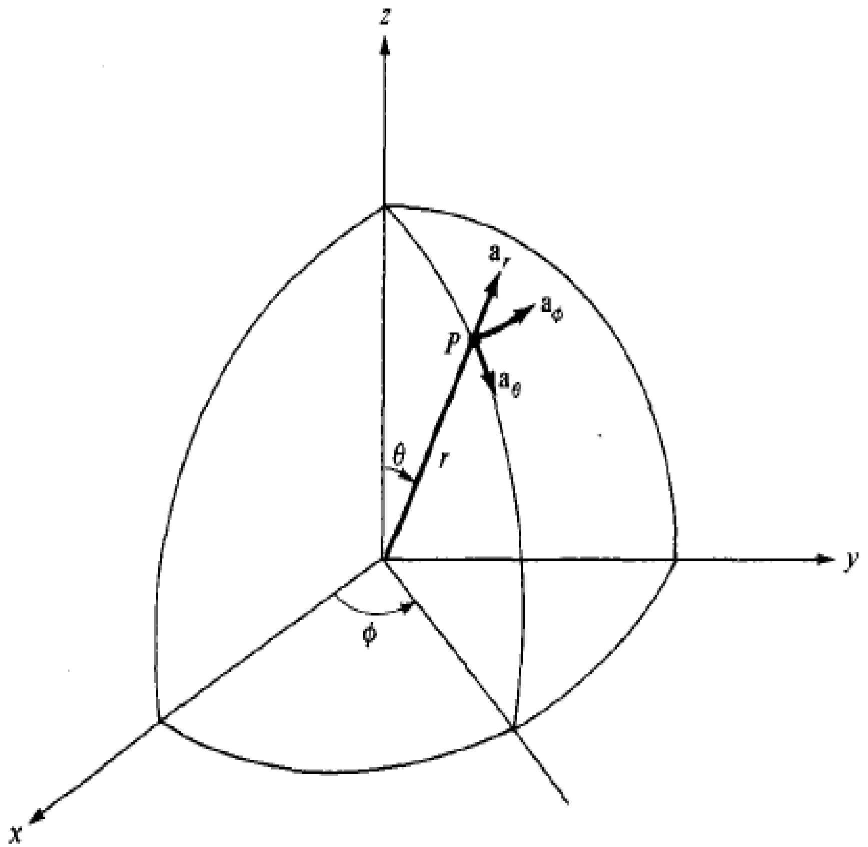
Transformation from cylindrical to rectangular and vice versa

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

SPHERICAL COORDINATES

The spherical coordinate system is most appropriate when dealing with problems having a degree of spherical symmetry.



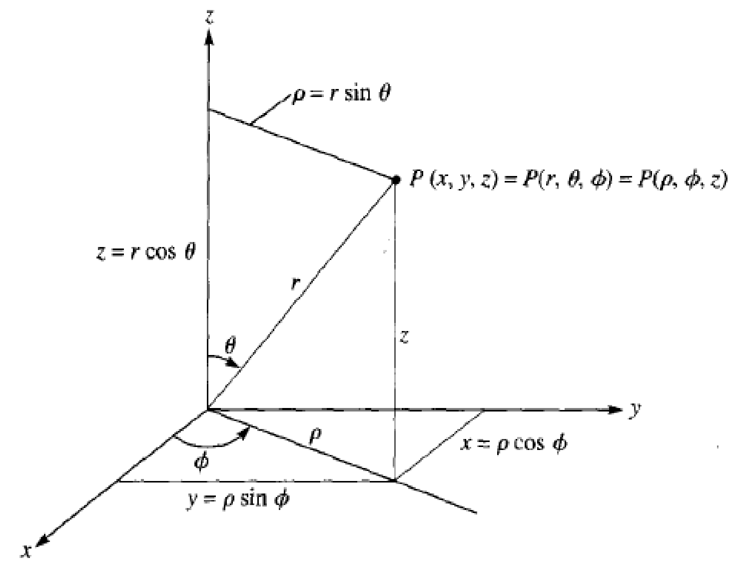


Figure 2.5 Relationships between space variables (x, y, z), (r, θ, ϕ), and (ρ, ϕ, z).

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}, \qquad \phi = \tan^{-1}\frac{y}{x}$$

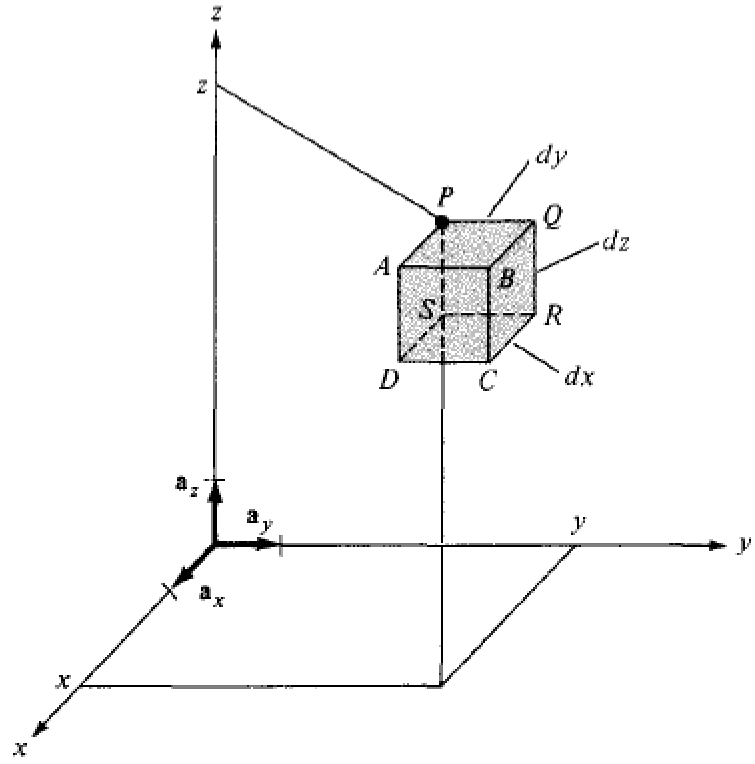
$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

Transformation

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -\cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

Differential normal areas in Cartesian coordinates



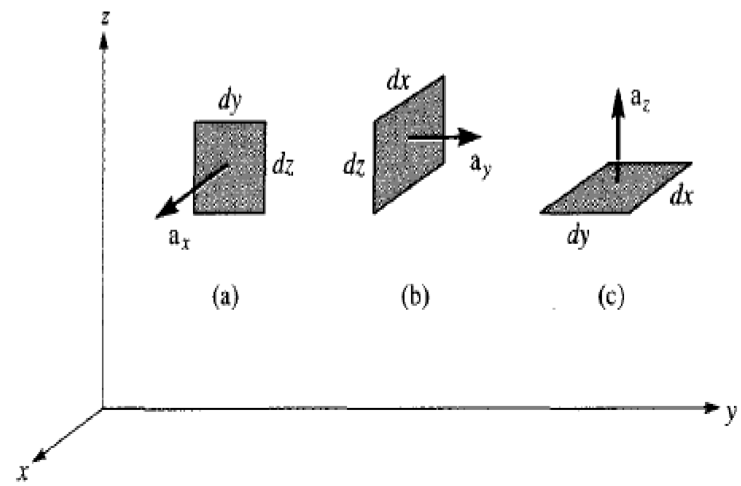
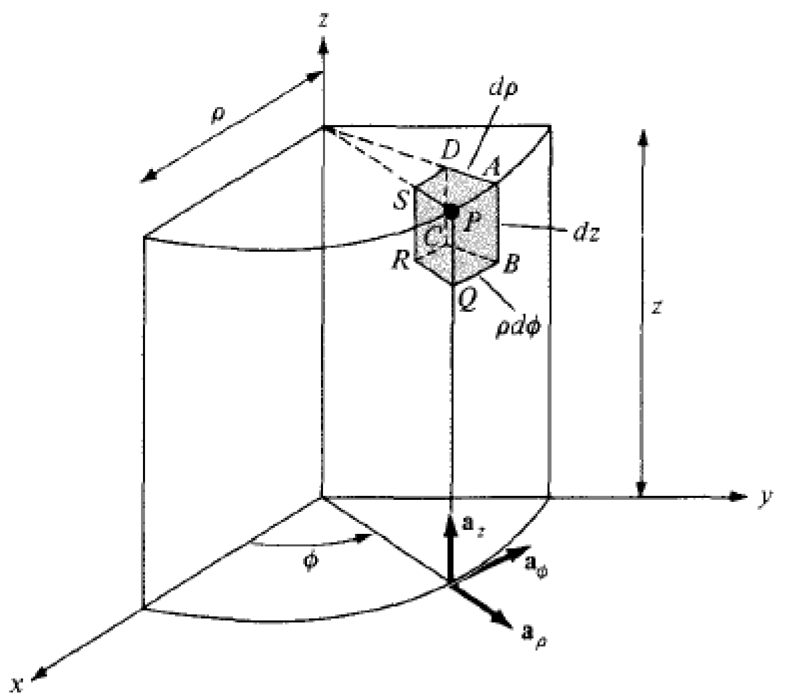


Figure 3.2 Differential normal areas in Cartesian coordinates: **(a)** $d\mathbf{S} = dy\,dz\,\mathbf{a}_x$, **(b)** $d\mathbf{S} = dx\,dz\,\mathbf{a}_y$, **(c)** $d\mathbf{S} = dx\,dy\,\mathbf{a}_z$

Cylindrical Coordinates



(1) Differential displacement is given by

$$d\mathbf{l} = d\rho\, \mathbf{a}_\rho + \rho\, d\phi\, \mathbf{a}_\phi + dz\, \mathbf{a}_z$$

(2) Differential normal area is given by

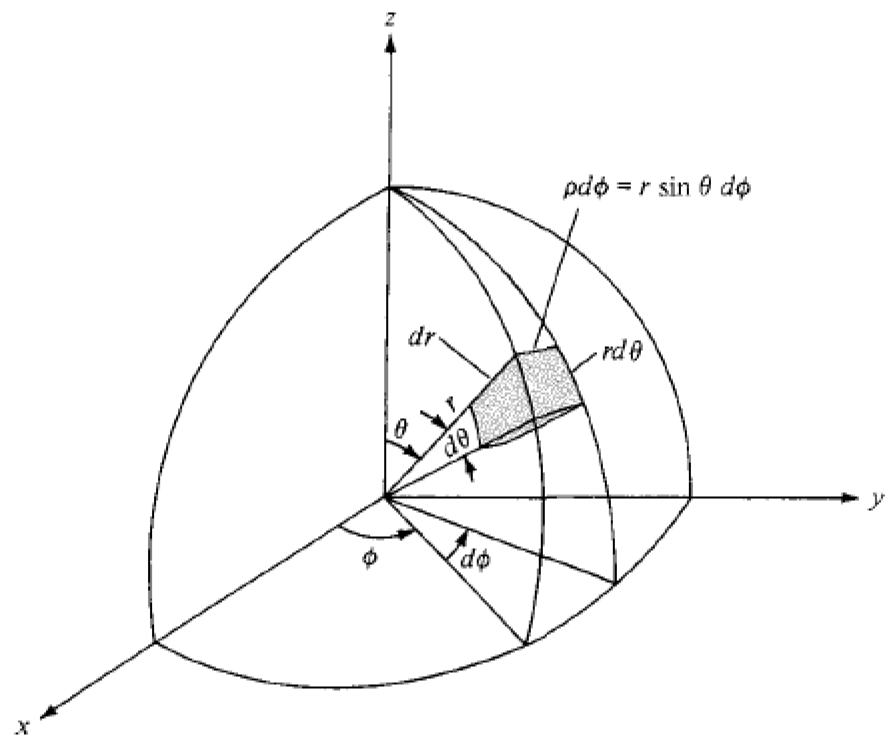
$$d\mathbf{S} = \begin{array}{l} \rho\, d\phi\, dz\, \mathbf{a}_\rho \\ d\rho\, dz\, \mathbf{a}_\phi \\ \rho\, d\phi\, d\rho\, \mathbf{a}_z \end{array}$$

and illustrated in Figure 3.4.

(3) Differential volume is given by

$$dv = \rho\, d\rho\, d\phi\, dz$$

Spherical Coordinates



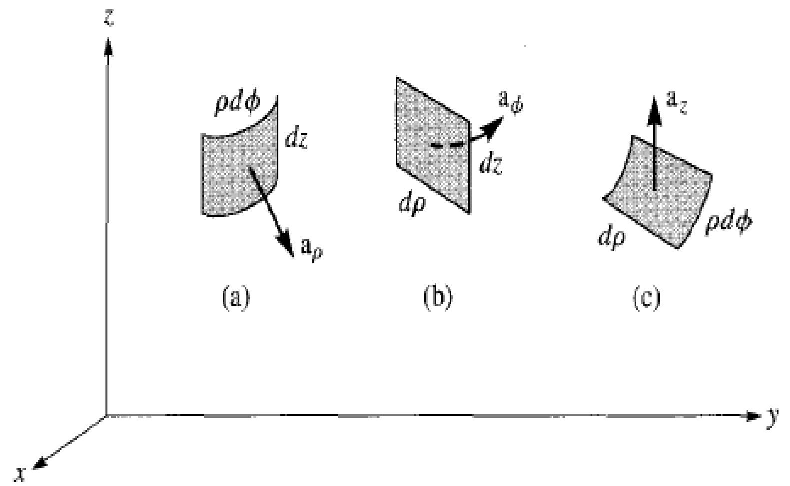


Figure 3.4 Differential normal areas in cylindrical coordinates: **(a)** $d\mathbf{S} = \rho\, d\phi\, dz\, \mathbf{a}_\rho$, **(b)** $d\mathbf{S} = d\rho\, dz\, \mathbf{a}_\phi$, **(c)** $d\mathbf{S} = \rho\, d\phi\, d\rho\, \mathbf{a}_z$