State Variable Analysis

State:-

The state of a dynamical system is a minimal set of variable and the knowledge of these variable at $t = t_0$ together with input for $t \geq t_0$ completely determine the behaviour of the system for any time t.

State variables:-

The state variables of a dynamic system are the minimal set of variables which determine the dynamics of a linear system.

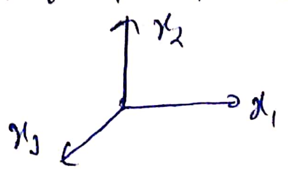
State Vector:-

The 'n' set of state variables used for describing the dynamic equation of a linear system can be considered the n component of the state vector $x(t)$.

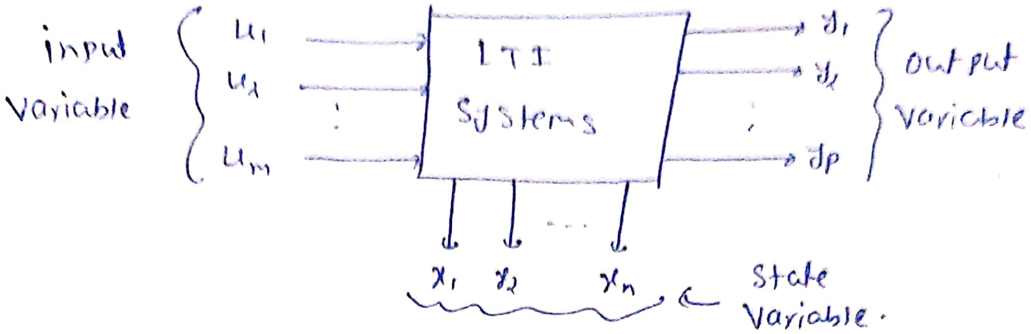
$$x(t) = [x_1, x_2, x_3 \cdots x_n]$$

State space:-

~~Tt~~ The "state space" is the n- dimensional space whoose coordinate axis casists of x_1- axis, x_2- axis, ... x_n axis. Any state can be uniquely represented by a point in the state space.



State model Representation of a Linear System :-



Here $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}_{m \times 1}$ input Vector

$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}_{n \times 1}$ State Vector

$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}$ output vector.

Now the state model Representation is given as

$$\boxed{\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y &= C\,x(t) + D\,u(t) \end{aligned}}$$

Writing State Space Equation for Electrical System:-

Step1:- Assign all inductor currents and Capacitor voltage as state variables.

Step2:- For inductor select a set of loop current and write the Relationship b/w state variable and ~~Theri~~ Their derivative.

Step3:- Eliminate all the other variable ~~&~~ other than state variable and input variable.

Step4:- For Capacitor select a set of suitable Node and apply kcl then write the Relationship b/w state variable and their derivative.

Step5:- Eliminate all the other variable other than state variable and input.

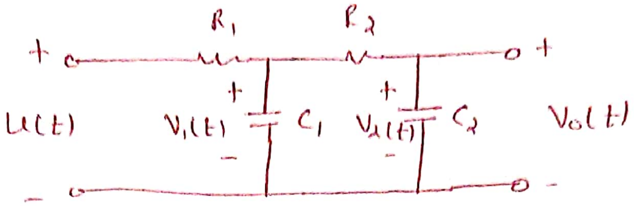
Step6: Write above eq. in the form

$$\dot{x} = Ax + Bu$$

Step7: Write output eq. and eliminate all the other variable (Except state variable and input)

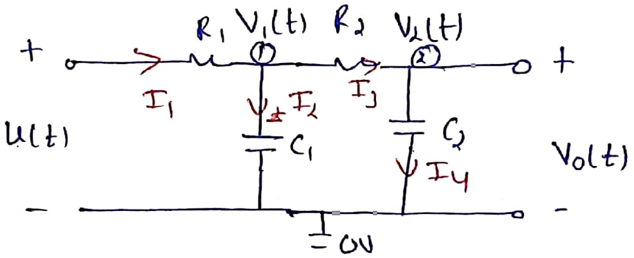
put in the form $y = Cx + Du$

Q. Write down state eq:-



Sol: Step (i) State Variable are $V_1(t)$ and $V_2(t)$.

Step (ii)



KCL at Node-1 $\quad I_1 = I_2 + I_3$

$$\frac{u(t) - V_1(t)}{R_1} = C_1 \frac{dV_1(t)}{dt} + \frac{V_1(t) - V_2(t)}{R_2}$$

$$\boxed{\frac{dV_1(t)}{dt} = \frac{-V_1(t)}{R_1 C_1} - \frac{V_1(t)}{R_2 C_1} + \frac{V_2(t)}{R_2 C_1} + \frac{u(t)}{R_1 C_1}}$$

Now KCL at Node-2

$$I_3 = I_4$$

$$\frac{V_1(t) - V_2(t)}{R_2} = C_2 \frac{dV_2(t)}{dt}$$

$$\boxed{\frac{dV_2(t)}{dt} = \frac{1}{R_2 C_2} V_1(t) - \frac{1}{R_2 C_2} V_2(t)}$$

$$\begin{bmatrix} \dot{V}_1(t) \\ \dot{V}_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1C_1}-\frac{1}{R_2C_1} & \frac{1}{R_2C_1} \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix} \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1C_1} \\ 0 \end{bmatrix} u(t)$$

Now output

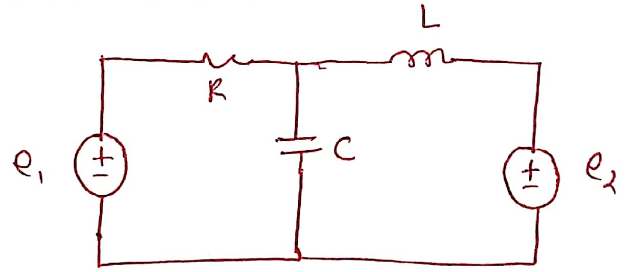
$$V_0(t) = V_2(t)$$

$$V_0(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix}$$

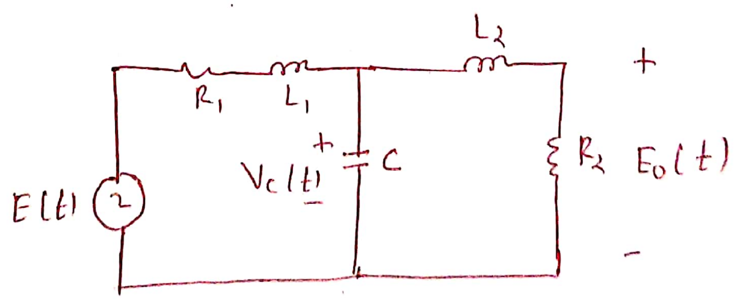
Ans.

Practice Question

Q.1



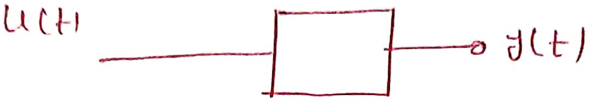
Q.2



State Space Representation of n^{th} order differential equation:-

Q. Construct the State model for a system given by following diff. eq.

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y\cancel{\frac{dy}{dt}} = u$$



where $u(t)$ is input and $y(t)$ is output.

Sol: No. of State Variable = order of eq = 3

let $y = z_1$

$$\frac{dy}{dt} = \dot{z}_1 = z_2$$

$$\frac{d^2y}{dt^2} = \dot{z}_2 = z_3$$

$$\frac{d^3y}{dt^3} = \dot{z}_3$$

use all in given eq.

$$\dot{z}_3 + 6z_3 + 11z_2 + 6z_1 = u$$

$$\dot{z}_3 = -6z_1 - 11z_2 - 6z_3 + u$$

$$\dot{z}_2 = z_3$$

$$\dot{z}_1 = z_2$$

and output $y = z_1$

So state model is

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 11 & 6 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

Q. $$\frac{d^3y}{dt^3} + 9\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 15y = \frac{d}{dt}u(t) + u(t)$$

Sol: In given Question derivative of input is also present. So we apply the following procedure.

Step (i) First take LT of given eq.

$$s^3 y(s) + 9s^2 y(s) + 11 s y(s) + 15\, y(s) = s u(s) + u(s)$$

$$T(s) = \frac{y(s)}{u(s)} = \frac{s+1}{s^3 + 9s^2 + 11s + 15}$$

step (ii) multiply Numerator and denominator by $w(s)$

$$\frac{y(s)}{u(s)}\,\frac{w(s)}{w(s)} = \frac{s+1}{s^3 + 9s^2 + 11s + 15}$$

Step (iii) Now ~~write~~ take $$\frac{y(s)}{w(s)} = \cancel{\frac{1}{s^3+9s^2+11s+15}}\,(s+1)$$

and $$\frac{w(s)}{u(s)} = \cancel{s+1}\,\frac{1}{s^3 + 9s^2 + 11s + 15}$$

Step (iv)

take ILT of both

$$\frac{W(s)}{U(s)} = \frac{1}{s^3+9s^2+11s+15}$$

$\Rightarrow$ $s^3 W(s) + 9s^2 W(s) + 11s W(s) + 15 W(s) = U(s)$

$\Rightarrow$ $\frac{d^3w}{dt^3} + 9\frac{d^2w}{dt^2} + 11\frac{dw}{dt} + 15w = u(t)$ (by taking ILT)

Now let $w = z_1$ -- (i)

$\frac{dw}{dt} = \dot{z}_1 = z_2$ -- (ii)

$\frac{d^2w}{dt^2} = \dot{z}_2 = z_3$ -- (iii)

$\frac{d^3w}{dt^3} = \dot{z}_3$ -- (iv)

So $\dot{z}_3 + 9z_3 + 11z_2 + 15z_1 = u(t)$

$\dot{z}_3 = -15z_1 - 11z_2 - 9z_3 + u(t)$

and $\dot{z}_2 = z_3$

$\dot{z}_1 = z_2$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -15 & -11 & -9 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Step (v) Now take $\frac{Y(s)}{W(s)} = s+1$

$Y(s) = sW(s) + W(s)$

take ILT

$y(t) = \frac{dw(t)}{dt} + w(t) = z_2 + z_1$ (by using (i) and ~~(iii)~~ (ii))

$y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ Ans.

State space Representation of discrete time System :-

$$\begin{bmatrix} z_1(n+1) \\ z_2(n+1) \\ \vdots \\ z_m(n+1) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \\ a_{n1} & a_{n2} & \cdots \end{bmatrix} \begin{bmatrix} z_1(n) \\ z_2(n) \\ \vdots \\ z_m(n) \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots \\ b_{21} & \cdots \\ b_{31} & \cdots \\ b_{nn} & \cdots \end{bmatrix} u(n)$$

$$y(n) = \begin{bmatrix} c_{11} & c_{12} & \cdots \\ c_{21} & c_{22} & \cdots \\ \vdots & \vdots & \\ c_{n1} & c_{n2} & \cdots \end{bmatrix} \begin{bmatrix} z_1(n) \\ \vdots \\ z_m(n) \end{bmatrix} + \begin{bmatrix} D_{11} & \cdots \\ D_{21} & \cdots \\ \vdots & \end{bmatrix} u(n)$$

$$\boxed{\begin{aligned} z(n+1) &= A z(n) + B u(n) \\ y &= C z(n) + D u(n) \end{aligned}}$$

Q. Find State eq. of given discrete time System

$$y(n) - \frac{7}{4} y(n-1) + \frac{1}{8} y(n-2) = x(n)$$

Sol: No. of State Variable = 2

let $y(n-2) = z_1(n)$ --- (i)

$y(n-1) = z_2(n) = z_1(n+1)$ --- (ii)

$y(n) = z_2(n+1)$ --- (iii)

So given eq. become

$$z_2(n+1) - \frac{3}{4} z_2(n) + \frac{1}{8} z_1(n) = x(n)$$

Or

$$\boxed{\begin{aligned} z_2(n+1) &= \frac{3}{4} z_2(n) - \frac{1}{8} z_1(n) + x(n) \quad \text{--(iv)} \\ z_1(n+1) &= z_2(n) \end{aligned}}$$

and [from (iii)]

$$\begin{bmatrix} z_1(n+1) \\ z_2(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{8} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} z_1(n) \\ z_2(n) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(n)$$

Ans.

Now output

$$y(n) = z_2(n+1)$$

Now use eq (iv)

$$y(n) = \frac{3}{4} z_2(n) - \frac{1}{8} z_1(n) + x(n)$$

So

$$y(n) = \begin{bmatrix} -\frac{1}{8} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} z_1(n) \\ z_2(n) \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} x(n)$$

Ans.

Q.

Find the state eq. of given discrete time system

$$y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = x(n) + \frac{1}{2} x(n-1) + x(n-2)$$

Sol: First take z transform

$$Y(z) - \frac{3}{4} z^{-1} Y(z) + \frac{1}{8} z^{-2} Y(z) = X(z) + \frac{1}{2} z^{-1} X(z) + z^{-2} X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{2} z^{-1} + z^{-2}}{1 - 3/4\, z^{-1} + \frac{1}{8} z^{-2}}$$

Now $$\frac{Y(z)}{X(z)} \frac{W(z)}{W(z)} = \frac{1 + \frac{1}{2} z^{-1} + z^{-2}}{1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}}$$

So $$\frac{W(z)}{X(z)} = \frac{1}{1 - 3/4\, z^{-1} + 1/8\, z^{-2}}$$ and $$\frac{Y(z)}{W(z)} = 1 + \frac{1}{2} z^{-1} + z^{-2}$$

First take $$\frac{W(z)}{X(z)} = \frac{1}{1 - 3/4\, z^{-1} + 1/8\, z^{-2}}$$

$$W(z) - \frac{3}{4} z^{-1} W(z) + \frac{1}{8} z^{-2} W(z) = X(z)$$

take IZT

$$\boxed{w(n) - \frac{3}{4} w(n-1) + \frac{1}{8} w(n-2) = x(n)} \quad \text{--- (i)}$$

No. of state variable = 2

So let $w(n-2) = z_1(n)$ -- (ii)

$w(n-1) = z_1(n+1) = z_2(n)$ -- (iii)

$w(n) = z_2(n+1)$ -- (iv)

Use These in eq (i)

$$z_2(n+1) - \frac{3}{4} z_2(n) + \frac{1}{8} z_1(n) = x(n)$$

$$\boxed{z_2(n+1) = -\frac{1}{8} z_1(n) + \frac{3}{4} z_2(n) + x(n)} \quad \text{-- (v)}$$

Now take $\dfrac{Y(z)}{W(z)} = 1 + \frac{1}{2} z^{-1} + z^{-2}$

$$Y(z) = W(z) + \frac{1}{2} z^{-1} W(z) + z^{-2} W(z)$$

take IZT

$$y(n) = w(n) + \frac{1}{2} w(n-1) + w(n-2)$$

Use (ii), (iii) and (iv)

$$y(n) = z_2(n+1) + \frac{1}{2} z_2(n) + z_1(n)$$

Now use eq (v)

$$y(n) = -\frac{1}{8} z_1(n) + \frac{3}{4} z_2(n) + x(n) + \frac{1}{2} z_2(n) + z_1(n)$$

$$y(n) = \frac{7}{8} z_1(n) + \frac{5}{4} z_2(n) + x(n)$$

~~y(n) =~~

so state vecto eq. are

$$z_1(n+1) = z_2(n)$$

$$z_2(n+1) = -\frac{1}{8} z_1(n) + \frac{3}{4} z_2(n) + x(n)$$

and out put vector.

$$y(n) = \frac{7}{8} z_1(n) + \frac{5}{4} z_2(n) + x(n)$$

$$\begin{bmatrix} z_1(n+1) \\ z_2(n+1) \end{bmatrix} = \begin{bmatrix} \cancel{-\frac{1}{8}} & \cancel{\frac{3}{4}} \\ 0 & 1 \\ -1/8 & 3/4 \end{bmatrix} \begin{bmatrix} z_1(n) \\ z_2(n) \end{bmatrix} + \begin{bmatrix} \cancel{1} \\ 0 \\ 1 \end{bmatrix} x(n)$$

$$y = \begin{bmatrix} \frac{7}{8} & \frac{5}{4} \end{bmatrix} \begin{bmatrix} z_1(n) \\ z_2(n) \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} x(n)$$

Ans.

Transfer fn:-

We know That for a General State model

$$\dot{x} = Ax + Bu \quad \text{--(i)}$$

$$y = Cx + Du \quad \text{--(ii)}$$

taking laplace Transform of both the eq. assuming zero initial Condition

$$sX(s) = AX(s) + BU(s) \quad \text{--(iii)}$$

$$Y(s) = CX(s) + DU(s) \quad \text{--(iv)}$$

Now from (iii)

$$X(s)[sI - A] = BU(s)$$

$$X(s) = [sI - A]^{-1} BU(s)$$

Use This in eq. (iv)

$$Y(s) = C[sI - A]^{-1} BU(s) + DU(s)$$

So $$\boxed{T(s) = \frac{Y(s)}{U(s)} = C[sI - A]^{-1} B + D}$$

Q. Obtain the transfer f/n if State model is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Sol: given $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

Now $T(s) = C[sI - A]^{-1} B + D$

$$[sI - A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

Now $[sI - A]^{-1} = \dfrac{Adj[sI - A]}{|sI - A|}$

$$= \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

So $T(s) = \dfrac{1}{s^2+3s+2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= \dfrac{1}{s^2+3s+2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} = \dfrac{1}{s^2+3s+2}$ Ans.

Practice Question

Find $T(s)$ of

(i)
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -12 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 8 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(ii)
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -\frac{1}{2} & 2 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Resolvant and State Transition Matrix

matrix $\phi(s) = [sI - A]^{-1}$ is known as Resolvant matrix and its ILT i.e. $\phi(t)$ is known as state Transition matrix.

Now $\phi(s) = \frac{1}{sI - A}$

So $\phi(t) = e^{At}$

Q Find State Transition Matrix

if $\phi(s) = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$

Sol:

given $\phi(s) = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$

Now $\frac{s}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$

$A = \frac{-1}{(1)} = -1, \quad B = \frac{-2}{(-1)} = 2$

So $\frac{s}{(s+1)(s+2)} = \frac{-1}{s+1} + \frac{2}{s+2}$

take ILT

$L^{-1}\left[\frac{s}{(s+1)(s+2)}\right] = -e^{-t} + 2e^{-2t}$

Now take

$$\frac{1}{(s+1)(s+2)} = \frac{C}{s+1} + \frac{D}{s+2}$$

$$C = 1, \quad D = -1$$

So

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$L^{-1}\left[\frac{1}{(s+1)(s+2)}\right] = e^{-t} - e^{-2t}$$

Now

$$\phi(t) = \begin{bmatrix} -e^{-t}+2e^{-2t}+3e^{-t}-3e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix}$$

$$\phi(t) = \begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix}$$

Ans.