Z- Transform

⇒ It is discrete-time counterpart of the laplace Transform.

⇒ Z Transform is used to convert a discrete time signal in to Z-domain.

$$Z[x(n)] = X(z)$$

$x(n)$ ⇒ discrete time signal

$X(z)$ ⇒ Z-domain Representation of $x(n)$

here $z = e^{jw}$

⇒ Now To Convert This a mathematical formula is used which is given below

$$\boxed{Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}}$$

and the Inverse Z-Transform of $X(z)$ is

$$\boxed{x(n) = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz}$$

Region of Convergence for Z-transforms:-

The range of values of Z for which the Z-transform converges is termed the Region of Convergence.

Property of ROC:-

(1) ROC may be outside of a circle.

(2) ROC may be Inside of a circle.

(3) ROC may be b/w of two circle.

(4) if $X(z)$ is rational, Then the ROC must not Contain any poles.

(5) ROC may be Entire Z-plane.

(6) ROC may be Entire Z-plane, Except $Z=0$.

(7) ROC may be Entire Z-plane, Except $Z=\infty$.

(8) ROC may be Entire Z plane, Except $Z=0$ and $Z=\infty$.

Q. Determine the z-transform of the causal signal $x(n) = a^n u(n)$ and depict the RoC and the locations of poles and zeros in the z-plane.

Sol: By definition

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

Now $u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$

$$X(z) = \sum_{n=-\infty}^{-1} a^n u(n) z^{-n} + \sum_{n=0}^{\infty} a^n u(n) z^{-n}$$

$$= 0 + \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

$$X(z) = 1 + (az^{-1}) + (az^{-1})^2 + \dots$$

This series Represent Infinite G.P. and sum of Infinite G.P. is given by

$$S = \frac{a}{1-r} \quad \text{when } |r| < 1$$

where a = first element

r = Common Ratio

~~Here~~

So $X(z) = \dfrac{1}{1-az^{-1}}$ $\qquad |az^{-1}| < 1$

or $\boxed{X(z) = \dfrac{z}{z-a}}$ $\qquad \dfrac{|a|}{|z|} < 1$

or $|z| > |a|$

$$\boxed{\text{So } \; z[a^n u(n)] = \frac{z}{z-a} \quad \text{with ROC } |z| > |a|}$$

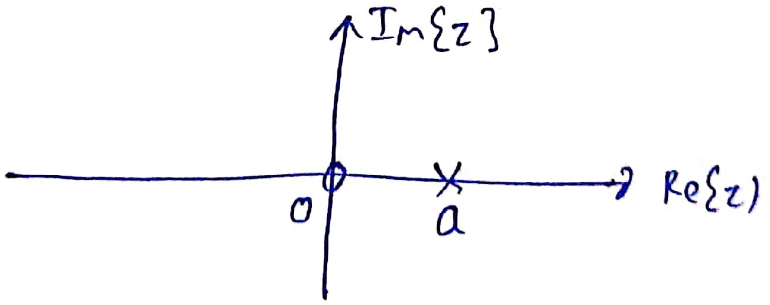
Now Poles and Zeros :-

The Roots of Numerator are Called Zeros and the Roots of denominator are Called Poles.

So in this Question ~~Zers~~ Zeros is

$z = 0$ (denoted by small circle o)

and Poles is $z = a$. (denote by cross x)



Now we have to draw RoC

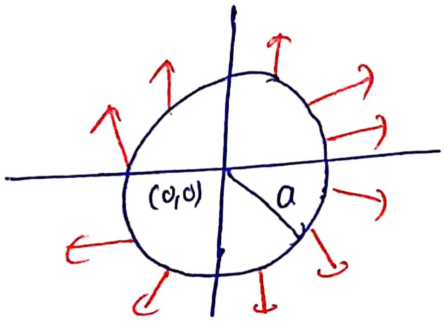
$|z| > |a|$

$\Rightarrow |x+iy| > |a|$

$\Rightarrow \sqrt{x^2+y^2} > |a|$

take square both side

$$\boxed{x^2+y^2 > a^2}$$



outside of circle.

Q: Determine z transform of

$$x(n) = -a^n u(-n-1)$$

and depict the RoC and the location of poles and zeros in the z-plane.

Sol:

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u(-n-1) z^{-n}$$

Now $u(-n-1) = 1 \quad n \le -1$

$0 \quad n > -1$

So $$X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n} + 0$$

$$X(z) = -\sum_{n=-\infty}^{-1} (a^{-1}z)^{-n}$$

$$= -\left[\cdots\cdots (a^{-1}z)^3 + (a^{-1}z)^2 + (a^{-1}z) \right]$$

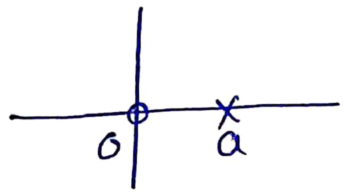
$$X(z) = -\left[\frac{a^{-1}z}{1-a^{-1}z}\right] \qquad |a^{-1}z|<1$$

$$\Rightarrow \quad X(z) = -\frac{z}{a-z} \qquad \frac{|z|}{|a|}<1$$

$$\Rightarrow \quad \boxed{X(z) = \frac{z}{z-a} \qquad |z|<|a|}$$

Now poles at $z=a$

and zeros at $z=0$

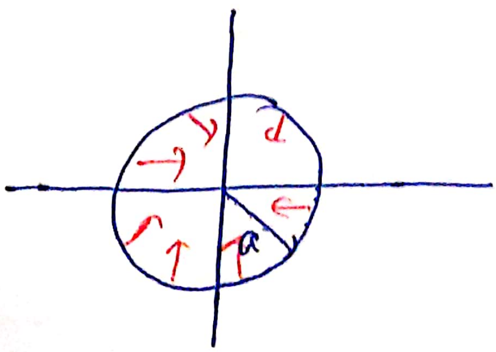


Now to draw ROC $|z|<|a|$

$$|x+iy|<|a|$$

$$\sqrt{x^2+y^2}<|a|$$

$$\Rightarrow \quad x^2+y^2<a^2$$



inside of circle.

Q: Find z-transform of

$x(n) = \left(\frac{1}{2}\right)^n u(n) + 2^n u(-n-1)$

Sol: given $x(n) = \left(\frac{1}{2}\right)^n u(n) + 2^n u(-n-1)$

$$\underbrace{\left(\frac{1}{2}\right)^n u(n)}_{x_1(n)} + \underbrace{2^n u(-n-1)}_{x_2(n)} \quad \text{-- (i)}$$

Now we know That

$$z[a^n u(n)] = \frac{z}{z-a} \quad |z| > |a|$$

and $$z[-a^n u(-n-1)] = \frac{z}{z-a} \quad |z| < |a|$$

So for $x_1(n) = \left(\frac{1}{2}\right)^n u(n)$

$$X_1(z) = \frac{z}{z - 1/2} \quad |z| > \frac{1}{2} \rightarrow R_1$$

and for $x_2(n) = 2^n u(-n-1)$

$$X_2(z) = -\frac{z}{z-2} \quad |z| < 2 \rightarrow R_2$$

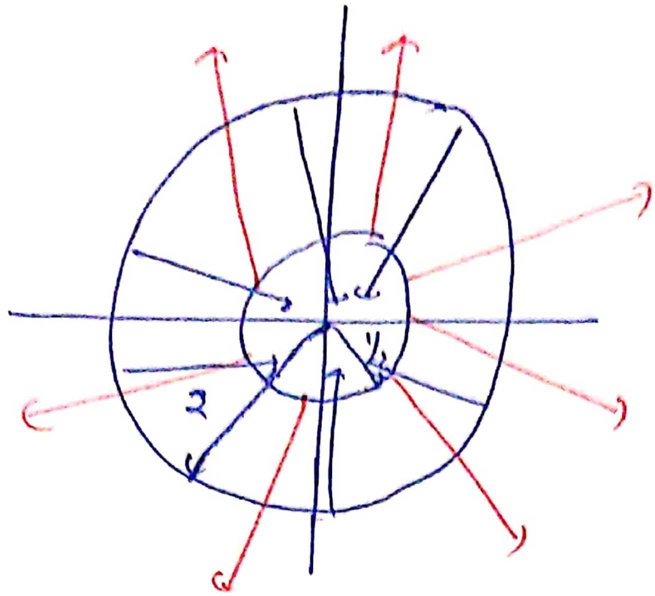
Now by z-transform of (i)

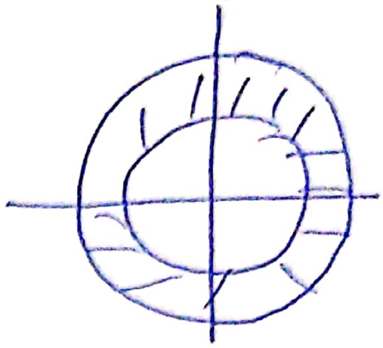
$$X(z) = X_1(z) + X_2(z) \quad \text{RoC}: R_1 \cap R_2$$

$$X(z) = \frac{z}{z - 1/2} - \frac{z}{z-2} \quad \frac{1}{2} < |z| < 2$$

$$= z\left[\frac{z - 2 - z + 1/2}{(z - 1/2)(z-2)}\right]$$

$$= \frac{-3/2\, z}{z^2 - \frac{5}{2}z + 1} \quad \frac{1}{2} < |z| < 2$$





Roc: b/w of two circle.

Q. Find the Z- Transform of

(a) $x(n) = \delta(n)$

Sol: $z[x(n)] = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n}$

$= \delta(0) z^{-0}$

$= 1$

So $z[\delta(n)] = 1$ Roc: Entire z plane.

Q: Determine the z-transform and ROC of the following finite duration signals:

(a) $x_1(n) = \{\underset{\uparrow}{1}, 2, 6, -2, 0, 3\}$

(b) $x_2(n) = \{1, 2, 6, -2, 0, \underset{\uparrow}{3}\}$

(c) $x_3(n) = \{1, 2, \underset{\uparrow}{6}, -2, 0, 3\}$

Sol:

(a) $$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n}$$

$$= \sum_{n=0}^{5} x_1(n) z^{-n}$$

$$= x_1(0) + x_1(1) z^{-1} + x_1(2) z^{-2} + x_1(3) z^{-3} + x_1(4) z^{-4} + x_1(5) z^{-5}$$

$$\boxed{X_1(z) = 1 + 2z^{-1} + 6z^{-2} - 2z^{-3} + 3z^{-5}}$$

ROC: Entire z plane Except $z=0$

(b) $$X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$= \sum_{n=-5}^{0} x_2(n) z^{-n}$$

$$= x_2(-5) z^{5} + x_2(-4) z^{4} + x_2(-3) z^{3} + x_2(-2) z^{2} + x_2(-1) z + x_2(0)$$

$$\boxed{X_2(z) = z^{5} + 2z^{4} + 6z^{3} - 2z^{2} + 3}$$

ROC: Entire z plane Except $z = \infty$.

(c) $x_3(n) = \{1, 2, \underset{\uparrow}{6}, -2, 0, 3\}$

$$X_3(z) = \sum_{n=-2}^{3} x_3(n) z^{-n}$$

$$= x_3(-2) z^2 + x_3(-1) z + x_3(0) + x_3(1) z^{-1} + x_3(2) z^{-2} + x_3(3) z^{-3}$$

$$= z^2 + 2z + 6 - 2z^{-1} + 3z^{-3}$$

Roc: Entire z plane Except $z = 0$ And $z = \infty$.

Properties of z-transform:-

(1) linearity:-

if $x_1(n) \xleftrightarrow{z} X_1(z)$ ROC: R_1

and $x_2(n) \xleftrightarrow{z} X_2(z)$ ROC: R_2

Then $a x_1(n) + b x_2(n) \xleftrightarrow{z} a X_1(z) + b X_2(z)$

ROC: $R_1 \cap R_2$

(2) Time Shifting:-

if $x(n) \longleftrightarrow X(z)$ ROC: R

Then $x(n-n_0) \longleftrightarrow z^{-n_0} X(z)$ with ROC = R Except for the possible addition or deletion of $z=0$ and $z=\infty$

proof We know that

$$z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

So $$z[x(n-n_0)] = \sum_{n=-\infty}^{\infty} x(n-n_0) z^{-n}$$

let $n - n_0 = k$

$$\Rightarrow z[x(n-n_0)] = \sum_{k=-\infty}^{\infty} x(k) z^{-(k+n_0)}$$

$$= \sum_{k=-\infty}^{\infty} z^{-n_0} x(k) z^{-k}$$

$$= z^{-n_0} X(z)$$ Hence proved.

Q: Find z transform of

(a) $a^{n-1} u(n-1)$ (b) $-a^{n-1} u(-n)$

Sol: (a) $a^{n-1} u(n-1)$

We know That $z[a^n u(n)] = \frac{z}{z-a}$ $|z| > |a|$

here $x(n) = a^n u(n)$

and $X(z) = \frac{z}{z-a}$

Now by Time Shifting

$$z[x(n-1)] = z^{-1} X(z)$$

$$\Rightarrow z[a^{n-1} u(n-1)] = z^{-1} \frac{z}{z-a}$$

$$\Rightarrow \boxed{z[a^{n-1} u(n-1)] = \frac{1}{z-a} \quad |z| > |a|}$$

(b) $-a^{n-1} u(-n)$

since we know That

$$z[-a^n u(-n-1)] = \frac{z}{z-a} \quad |z| < |a|$$

Here $x(n) = -a^n u(-n-1)$

and $X(z) = \frac{z}{z-a}$

Now by shifting

$$z[x(n-1)] = z^{-1} X(z)$$

$$\Rightarrow Z\left[-a^{n-1}u(-n)\right] = z^{-1}\frac{z}{z-a} \qquad |z|<|a|$$

$$\Rightarrow \boxed{Z\left[-a^{n-1}u(-n)\right] = \frac{1}{z-a} \qquad |z|<|a|}$$

③ Scaling in the z-domain:-

if $x(n) \longleftrightarrow X(z)$ ROC $= R$

Then $a^n x(n) \longleftrightarrow X(z/a)$ ROC $= |a|R$

Proof We know That

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\text{So } Z[a^n x(n)] = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (a^{-1}z)^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (z/a)^{-n}$$

$$= X(z/a)$$

So $Z[a^n x(n)] = X(z/a)$ Hence Proved.

4. Time Reversal property :-

if $x(n) \longleftrightarrow X(z)$ ROC : R

Then $x(-n) \longleftrightarrow X\left(\frac{1}{z}\right)$ ROC : $\frac{1}{R}$

Proof $z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

Now $z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$

Replace n by -m

So $z[x(-n)] = \sum_{m=\infty}^{-\infty} x(m) z^{m}$

$$= \sum_{m=-\infty}^{\infty} x(m) (z^{-1})^{-m}$$

$$= \sum_{m=-\infty}^{\infty} x(m) \left(\frac{1}{z}\right)^{-m}$$

$$= X\left(\frac{1}{z}\right)$$

So $\boxed{x(-n) \xleftrightarrow{z} X\left(\frac{1}{z}\right)}$

(8) Differentiation in Z-domain Property:-

if $x(n) \longleftrightarrow X(z)$ ROC: R

Then $n x(n) \longleftrightarrow -z \frac{d}{dz} X(z)$ ROC: R

Proof

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

differentiate both side w.r.t z

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} x(n) (-n z^{-n-1})$$

$$\Rightarrow \frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} (-n x(n)) \frac{z^{-n}}{z}$$

$$\Rightarrow z \frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} (-n x(n)) z^{-n}$$

$\Rightarrow$ so $Z[n x(n)] = -z \frac{dX(z)}{dz}$ Hence Proved

Q. Find the z transform of

(a) $x_1(n) = n a^n u(n)$

(b) $x_2(n) = n^2 a^n u(n)$

Sol: (a) $x_1(n) = n a^n u(n)$

here $x(n) = a^n u(n)$

and $X(z) = \frac{z}{z-a} \quad |z|>|a|$

Now by diff. property we know that

$$Z[n x(n)] = -z \frac{d}{dz} X(z)$$

$$\Rightarrow Z[n a^n u(n)] = -z \frac{d}{dz}\left(\frac{z}{z-a}\right) \quad |z|>|a|$$

$$= -z\left[\frac{(z-a)(1) - z(1-0)}{(z-a)^2}\right] \quad |z|>|a|$$

$$\boxed{X(z) = \frac{az}{(z-a)^2} \quad |z|>|a|}$$

So $\boxed{Z[n a^n u(n)] = \frac{az}{(z-a)^2} \quad |z|>|a|}$

(b) $x_2(n) = n^2 a^n u(n)$

Sol: $x_2(n) = n(n a^n u(n))$

Here $x(n) = n a^n u(n)$

So $X(z) = \frac{az}{(z-a)^2} \quad |z|>|a|$

Now by diff. Property

$$z\left[n^2 a^n u(n)\right] = -z\frac{d}{dz}\left(\frac{az}{(z-a)^2}\right) \qquad |z|>|a|$$

$$= \frac{az^2 + a^2 z}{(z-a)^3}$$

So $$\boxed{z\left[n^2 a^n u(n)\right] = \frac{az^2+a^2 z}{(z-a)^3} \qquad |z|>|a|}$$

Q. Find $x(n)$ whose z - Transform is

$$X(z) = \log(1+az^{-1}) \qquad |z|>|a|$$

Sol: by derivative Property we know that

$$z[n x(n)] = -z\frac{dX(z)}{dz}$$

or $$z^{-1}\left[-z\frac{dX(z)}{dz}\right] = n x(n)$$

$$\Rightarrow z^{-1}\left[-z\frac{d}{dz}\log(1+az^{-1})\right] = n x(n)$$

$$\Rightarrow z^{-1}\left[-z\frac{1}{1+az^{-1}}\{0-az^{-2}\}\right] = n x(n)$$

$$\Rightarrow z^{-1}\left[\frac{az^{-1}}{1+az^{-1}}\right] = n x(n)$$

$$\Rightarrow z^{-1}\left[\frac{a}{z+a}\right] = n x(n)$$

$$\Rightarrow a(-a)^{n-1}u(n-1) = n x(n)$$

or $\boxed{x(n) = \frac{a(-a)^{n-1} u(n-1)}{n}}$

(6) Convolution Property:-

if $x_1(n) \longleftrightarrow X_1(z)$ ROC: R_1

and $x_2(n) \longleftrightarrow X_2(z)$ ROC: R_2

Then $x_1(n) * x_2(n) \longleftrightarrow X_1(z) X_2(z)$

ROC: $R_1 \cap R_2$

Proof

we know that

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

take z transform of both side

$$z[x_1(n) * x_2(n)] = \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k) \right\} z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) \sum_{n=-\infty}^{\infty} x_2(n-k) z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) \left\{ z^{-k} X_2(z) \right\}$$

Note: by use of shifting Property.

$$= X_2(z) \sum_{k=-\infty}^{\infty} x_1(k) z^{-k}$$

$$= X_2(z) X_1(z)$$ Hence Proved.

(7) Correlation Property:-

if $x_1(n) \longleftrightarrow X_1(z)$ ROC: R_1

and $x_2(n) \longleftrightarrow X_2(z)$ ROC: R_2

Then $$z[\gamma_{x_1 x_2}(m)] = z\left[\sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n-m)\right]$$

$$= X_1(z)\, X_2\left(\frac{1}{z}\right)$$

and ROC $R_1 \cap \frac{1}{R_2}$

(8) Conjugation and Conjugate Symmetry:-

if $x(n) \longleftrightarrow X(z)$ ROC: R

Then $x^*(n) \longleftrightarrow X^*(z^*)$ ROC: R

Proof

$$z[x(n)] = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

take complex conjugate both side

so $$z[x^*(n)] = \sum_{n=-\infty}^{\infty} x^*(n)\, z^{-n}$$

$$= \left[\sum_{n=-\infty}^{\infty} x(n)\, (z^*)^{-n}\right]^*$$

$$= [X(z^*)]^*$$

$$= X^*(z^*)$$ Hence proved.

(9) The Initial Value theorem

if $x(n) = 0, \; n < 0$ Then

$$\boxed{x(0) = \lim_{z \to \infty} X(z)}$$

This is called initial value theorem.

Proof

We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Now $x(n) = 0$ for $n < 0$

$$\text{So } X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

~~Now take limit $z \to \infty$ both side~~

$$\lim_{z \to \infty} \cancel{X(z) = \ominus\, x(n)}$$

$$X(z) = x(0) + x(1) z^{-1} + x(2) z^{-2} + \cdots$$

$$\Rightarrow \quad X(z) = x(0) + \frac{1}{z} x(1) + \frac{1}{z^2} x(2) + \cdots$$

take limit $z \to \infty$ both side

$$\lim_{z \to \infty} X(z) = x(0) + 0$$

$$\text{So} \quad \boxed{x(0) = \lim_{z \to \infty} X(z)}$$

10. The Final Value Theorem:-

if $x(n) = 0$ for $n < 0$

Then $x(\infty) = \lim_{z \to 1} (z-1)\, X(z)$