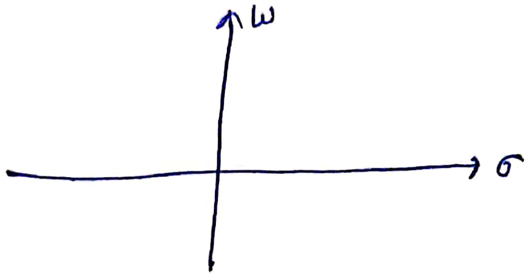
Laplace Transform

laplace Transform is a mathermatical tool which is used to Convert a time domain signal in to S-domain.

where $s = \sigma + j\omega$

σ is a real Part of s

and ω is a imaginary part of s



Laplace Transform is used for Continous time signal. and it is also applicable for Those signal whose Fourier Transform is Not exist.

laplace Transform Comes in two varieties:-

(i) Bilateral (ii) Unilateral

The bilateral laplace Transform offers insight into Nature of system charac. such as stability, causality and Frequency Response.

The Unilateral LT is a convenient tool for Solving differential equations with initial Conditions.

The Bilateral laplace Transform

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$$

e^{st} is called eigenfunction of LTI system

and $H(s)$ is called eigenvalues of LTI system.

Now Inverse laplace

$$h(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} H(s)\, e^{st}\, ds$$

Region of Convergence (Roc) for laplace Transform:-

laplace Transform is guaranteed to Converge if $x(t)\, e^{-\sigma t}$ is absolutely integrable.

$$\int_{-\infty}^{\infty} \left| x(t)\, e^{-\sigma t} \right| dt < \infty$$

The Range of $R\{s\} = \sigma$ for which the laplace Transform Converges is termed the Region of Convergence.

Q. Find the laplace Transform of

$$x(t) = e^{-at} u(t)$$

Sol:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} e^{-at} u(t) e^{-st} dt$$

Now $u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$

$$X(s) = \int_{0}^{\infty} e^{-at} e^{-st} dt$$

$$= \int_{0}^{\infty} e^{-t(s+a)} dt$$

$$= \left[\frac{e^{-t(s+a)}}{-(s+a)} \right]_0^{\infty}$$

$$= -\frac{1}{s+a} [0-1] = \frac{1}{s+a}$$

Now ROC:- $s+a > 0$

$s > -a$

Now $s = \sigma + j\omega$

So $\sigma + j\omega > -a$

or $Re\{s\} > -a$

So $L[e^{-at} u(t)] = \frac{1}{s+a}$ $\quad Re\{s\} > -a$

Q. Find laplace Transform of

$$x(t) = -e^{-at} u(-t)$$

Sol:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$= -\int_{-\infty}^{\infty} e^{-at} u(-t) e^{-st} dt$$

$$u(-t) = \begin{cases} 1 & t \le 0 \\ 0 & t > 0 \end{cases}$$

So

$$X(s) = -\int_{-\infty}^{0} e^{-at} e^{-st} dt + 0$$

$$= -\int_{-\infty}^{0} e^{t(-s-a)} dt$$

$$= -\left[\frac{e^{t(-s-a)}}{-s-a}\right]_{-\infty}^{0}$$

$$= \frac{1}{s+a}[1-0] \qquad -s-a > 0$$

$$= \frac{1}{s+a} \qquad s < -a$$

So

$$X(s) = \frac{1}{s+a} \qquad Re\{s\} < -a$$

Q. Find laplace of

$$x(t) = e^{-|t|}$$

Sol:

$$|t| = \begin{matrix} t & t \geq 0 \\ -t & t < 0 \end{matrix}$$

So $x(t) = \begin{matrix} e^{-t} & t \geq 0 \\ e^{t} & t < 0 \end{matrix}$

$$X(s) = \int_{-\infty}^{0} e^{t} e^{-st} dt + \int_{0}^{\infty} e^{-t} e^{-st} dt$$

$$= \int_{-\infty}^{0} e^{t(1-s)} dt + \int_{0}^{\infty} e^{-t(1+s)} dt$$

$$= \left(\frac{e^{t(1-s)}}{1-s}\right)\Bigg|_{-\infty}^{0} + \left[\frac{e^{-t(1+s)}}{-(1+s)}\right]_{0}^{\infty}$$

$$\Rightarrow \frac{1}{1-s}[1-0] - \frac{1}{1+s}[0-1]$$

$$\Rightarrow \frac{1}{1-s} + \frac{1}{1+s} = \frac{2}{1-s^2}$$

Roc: $1-s > 0$ and $1+s > 0$

$s < 1$ and $s > -1$

So $-1 < s < 1$

or $\boxed{-1 < Re\{s\} < 1}$

Q. Find laplace of

(a) $x(t) = \sin(\omega_0 t)u(t)$

(b) $x(t) = \cos(\omega_0 t)u(t)$

Sol: we know that

$$e^{i\omega_0 t} = \cos\omega_0 t + i\sin\omega_0 t$$

So $\cos\omega_0 t \Rightarrow$ Real Part of $e^{i\omega_0 t}$

And $\sin\omega_0 t \Rightarrow$ Imag. Part of $e^{i\omega_0 t}$

Now again we know That

$$L[e^{-at}u(t)] = \frac{1}{s+a} \quad Re\{s\} > -a$$

So $$L[e^{j\omega_0 t}u(t)] = \frac{1}{s-j\omega_0} \quad Re\{s\} > 0$$

$$\Rightarrow L[\cos(\omega_0 t)u(t) + j\sin(\omega_0 t)u(t)] = \frac{s+j\omega_0}{s^2+\omega_0^2}$$

$$L[\cos(\omega_0 t)u(t)] = \frac{s}{s^2+\omega_0^2} \quad Re\{s\} > 0$$

And $$L[\sin(\omega_0 t)u(t)] = \frac{\omega_0}{s^2+\omega_0^2} \quad Re\{s\} > 0$$

Q. Find laplace Transform of

$x(t) = e^{-2t} u(t) - e^{-3t} u(t)$

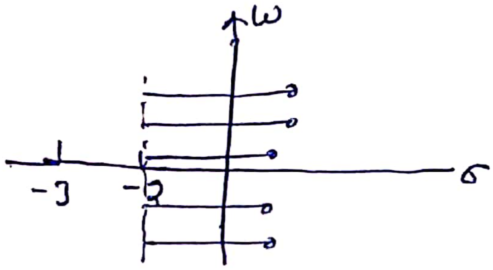
Sol: $L[x(t)] = L[e^{-2t} u(t) - e^{-3t} u(t)]$

$$= \frac{1}{s+2} - \frac{1}{s+3}$$

$Re\{s\} > -2 \qquad Re\{s\} > -3$

So $X(s) = \dfrac{s+3-s-2}{(s+2)(s+3)} = \dfrac{1}{s^2+5s+6}$

Roc:



(intersection of both)

So $X(s) = \dfrac{1}{s^2+5s+6} \qquad Re\{s\} > -2$

Q Determine laplace Transform of

$$x(t) = \begin{cases} e^{-at} & 0 < t < T \\ 0 & \text{otherwise.} \end{cases}$$

Sol: $L[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

$$\Rightarrow \int_{-\infty}^{0} x(t) e^{-st} dt + \int_0^T x(t) e^{-st} dt + \int_T^{\infty} x(t) e^{-st} dt$$

$$\Rightarrow 0 + \int_0^T e^{-at} e^{-st} dt + 0$$

$$\Rightarrow \int_0^T e^{-t(s+a)} dt$$

$$= \left[\frac{e^{-t(s+a)}}{-(s+a)} \right]_0^T$$

$$= -\frac{1}{s+a}\left[e^{-T(s+a)} - 1 \right]$$

$$\boxed{X(s) = \frac{1}{s+a}\left[1 - e^{-(s+a)T}\right]}$$ Ans.

The given signal is a Finite duration signal. So Roc is the entire s-plane.

Now at $s=-a$, $X(s) = \frac{0}{0}$ which is Undefined.

So by L'Hospital Rule

$$\lim_{s\to -a} X(s) = \lim_{s\to -a}\left[\frac{0 + Te^{-(s+a)T}}{1}\right]$$

$$\boxed{X(-a) = T}$$ Ans

$$X(s) = \frac{1}{s+a}\left[1 - e^{-(s+a)T}\right] \quad \text{Roc: entire splane}$$

$$T \quad \text{at } s=-a \quad \text{Roc: entire splane.}$$

Q Find inverse laplace of

$$X(s) = \frac{-3}{(s+2)(s-1)}$$

ROC: (a) $Re\{s\} > 1$

(b) $Re\{s\} < -2$

(c) $-2 < Re\{s\} < 1$

Sol: We know that

$$\frac{1}{s+a} \rightarrow e^{-at} u(t) \quad \text{--(i)}$$

$Re\{s\} > -a$

and $\frac{1}{s+a}$ $Re\{s\} < -a \rightarrow -e^{-at} u(-t)$ --(ii)

We use these result to find out ILT of $X(s)$

Now given $X(s) = \frac{-3}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1}$

$$-3 = A(s-1) + B(s+2)$$

$s=1$ $\quad -3 = B(3)$

$$B = -1$$

And at $s = -2$ $\quad -3 = A(-3)$

$$A = 1$$

So $X(s) = \frac{1}{s+2} - \frac{1}{s-1}$

(a) $Re\{s\} > 1$

from formula (i)

$$x(t) = e^{-2t} u(t) - e^{t} u(t)$$

(b) $Re\{s\} < -2$

use formula (ii)

$$\boxed{x(t) = -e^{-2t}u(-t) + e^{t}u(-t)}$$ Ans.

(c) $-2 < Re\{s\} < 1$

for greater Than use formula (i) and for less Than use formula (i)

$$\boxed{x(t) = e^{-2t}u(t) + e^{t}u(-t)}$$ Ans.

Properties of laplace Transform:-

(i) linearity:-

if $x_1(t) \longleftrightarrow X_1(s)$ $ROC: R_1$

$x_2(t) \longleftrightarrow X_2(s)$ $ROC: R_2$

Then $a x_1(t) + b x_2(t) \longrightarrow a X_1(s) + b X_2(s)$ $R_1 \cap R_2$

(ii) Time shifting:-

if $x(t) \longleftrightarrow X(s)$ $ROC: R$

Then $x(t-t_0) \longleftrightarrow X(s)\, e^{-st_0}$ $ROC: R$

(iii) Shifting in s-domain:-

if $x(t) \longleftrightarrow X(s)$ $ROC: R$

Then $x(t)\, e^{s_0 t} \longrightarrow X(s-s_0)$ $ROC: R + Re\{s_0\}$

Q. Find laplace Transform of

(a) $x_1(t) = e^{-at} \cos(\omega_0 t) u(t)$

(b) $x_2(t) = e^{-at} \sin(\omega_0 t) u(t)$

Sol: (a) $x_1(t) = e^{-at} \cos(\omega_0 t) u(t)$

we know That

$$L[\cos(\omega_0 t) u(t)] = \frac{s}{s^2+\omega_0^2} \quad Re\{s\} > 0$$

So by shifting property

$$L[e^{-at} \cos(\omega_0 t) u(t)] = \frac{s+a}{(s+a)^2+\omega_0^2} \quad Re\{s\} > 0-a$$

So

$$L[e^{-at} \cos(\omega_0 t) u(t)] = \frac{s+a}{(s+a)^2+\omega_0^2} \quad Re\{s\} > -a$$

(b) $x_2(t) = e^{-at} \sin(\omega_0 t) u(t)$

we know that

$$L[\sin(\omega_0 t) u(t)] = \frac{\omega_0}{s^2+\omega_0^2} \quad Re\{s\} > 0$$

Now by s- domain shifting property.

$$L[e^{-at} \sin(\omega_0 t) u(t)] = \frac{\omega_0}{(s+a)^2+\omega_0^2} \quad Re\{s\} > -a$$

(4) Time scaling Property:-

if $x(t) \longleftrightarrow X(s)$ ROC: R

Then $x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{s}{a}\right)$ ROC: aR

Proof we know That

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

Case 1: For a positive real constant 'a'

$$L[x(at)] = \int_{-\infty}^{\infty} x(at)\, e^{-st}\, dt$$

let $at = \tau$, $dt = \frac{d\tau}{a}$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-s/a\,\tau}\, \frac{d\tau}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(\tau)\, e^{-\frac{s}{a}\tau}\, d\tau$$

$$= \frac{1}{a} X(s/a)$$

Case 2:- if a is Negative real value.

$$L[x(-at)] = \int_{-\infty}^{\infty} x(-at)\, e^{-st}\, dt$$

let $-at = \tau$

$-a\,dt = d\tau$

$dt = -1/a\, d\tau$

$$\Rightarrow \int_{\infty}^{-\infty} x(\tau)\, e^{s/a\,\tau} \left(-\frac{d\tau}{a}\right)$$

$$= -\frac{1}{a} \int_{\infty}^{-\infty} x(\tau)\, e^{-(-s/a)\tau}\, d\tau$$

$$= 1/a \int_{-\infty}^{\infty} x(\tau)\, e^{-(-s/a)\tau}\, d\tau$$

$$= 1/a\, X(-s/a)$$

Combining two

$$L[x(at)] = \frac{1}{|a|} X\left(\frac{s}{a}\right)$$

Q. Determine laplace Transform of

(a) $g_1(t) = \delta(2t-3)$

(b) $g_2(t) = u(2t-1)$

(c) $g_3(t) = r(t/3 - 2)$

Sol: (a) $g_1(t) = \delta(2t-3)$

Fist we Find out LT of $\delta(t)$

$$L[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-st} dt$$

$$= \delta(0) = 1$$

So $L[\delta(t)] = 1$ ROC: entire s plane

Now by time shifting property

$$\delta(t-3) \rightarrow e^{-3s}(1) = e^{-3s}$$

Now again by time scalling property

$$\delta(2t-3) \rightarrow \frac{1}{2} e^{-3s}$$ ROC: entire s plane.

Ans.

(b) $g_2(t) = u(2t-1)$

Sol: $u(t) \longrightarrow \frac{1}{s}$ $Re\{s\} > 0$

Now by shifting

$u(t-1) \rightarrow e^{-s}\frac{1}{s}$ $Re\{s\} > 0$

by scalling

$u(2t-1) \rightarrow \frac{1}{2} e^{-s/2} \frac{1}{(s/2)}$

$= \frac{1}{s} e^{-s/2}$ $Re\{s\} > 0$ Ans.

(c) $g_3(t) = r\left[\frac{t}{3} - 2\right]$

Sol: $r(t) = t\,u(t)$

$r(t) \xrightarrow{S} \frac{1}{s^2}$ $Re\{s\} > 0$

Now by shifting

$r(t-2) \longrightarrow \frac{1}{s^2} e^{-2s}$ $Re\{s\} > 0$

Now again by time scaling

$r(t/3 - 2) \longrightarrow 3 \frac{1}{(3s)^2} e^{-6s}$ $Re\{s\} > 0$

$= \frac{1}{3s^2} e^{-6s}$ $Re\{s\} > 0$

Ans.

⑤ Time Reversal:-

if $x(t) \longleftrightarrow X(s)$ ROC: R

Then $x(-t) \longleftrightarrow X(-s)$ ROC: -R

⑥ Differentiation in the time domain

if $x(t) \longleftrightarrow X(s)$ ROC: R

Then $\frac{dx(t)}{dt} \longleftrightarrow sX(s)$ ROC: R

Proof

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\, e^{st}\, ds$$

differentiate both side w.r.t t

$$\frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\, (\cancel{\tfrac{d}{dt}}\, s e^{st})\, ds$$

$$\Rightarrow \frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} (sX(s))\, e^{st}\, ds$$

So $\frac{dx(t)}{dt} = L^{-1}[sX(s)]$

or $L\left[\frac{dx(t)}{dt}\right] = sX(s)$

⑦ Differentiation in the s-domain:-

if $x(t) \longrightarrow X(s)$ ROC: R

Then $t\,x(t) \longrightarrow -\frac{dX(s)}{ds}$ ROC: R

Proof we know that

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

differentiating both side w.r.t ~~t~~ s

$$\frac{dX(s)}{ds} = \int_{-\infty}^{\infty} x(t)\,(-t e^{-st})\, dt$$

$$\frac{dX(s)}{ds} = \int_{-\infty}^{\infty} (-t x(t))\, e^{-st}\, dt$$

So $\quad -t x(t) \xrightarrow{L} \frac{dX(s)}{ds}$

Or $\quad t x(t) \xrightarrow{L} -\frac{dX(s)}{ds}$

Q. Find laplace Transform of

$$x(t) = t e^{-at} u(t)$$

Sol: we know That

$$L[e^{-at} u(t)] = \frac{1}{s+a} \qquad Re\{s\} > -a$$

Now by diff. Property

$$L[t e^{-at} u(t)] = -\frac{d}{ds}\left[\frac{1}{s+a}\right]$$

$$= \frac{1}{(s+a)^2} \qquad Re\{s\} > -a$$

8 Convolution Property:-

if $x_1(t) \xleftrightarrow{L} X_1(s)$ ROC = R_1

and $x_2(t) \xleftrightarrow{L} X_2(s)$ ROC = R_2

Then $x_1(t) * x_2(t) \xleftrightarrow{L} X_1(s)\, X_2(s)$ ROC : $R_1 \cap R_2$

Proof

we know that

$$x_1(t) * x_2(t) = \int_{\tau=-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

take laplace of both side

$$L[x_1(t) * x_2(t)] = \int_{t=-\infty}^{\infty} \left\{ \int_{\tau=-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \right\} e^{-st}\, dt$$

$$= \int_{\tau=-\infty}^{\infty} \left\{ \int_{t=-\infty}^{\infty} x_2(t-\tau)\, e^{-st}\, dt \right\} x_1(\tau)\, d\tau$$

$$= \int_{\tau=-\infty}^{\infty} e^{-s\tau}\, X_2(s)\, x_1(\tau)\, d\tau$$ [by using time shifting]

$$= X_2(s) \int_{\tau=-\infty}^{\infty} x_1(\tau)\, e^{-s\tau}\, d\tau$$

$$= X_2(s)\, X_1(s)$$ Ans.

9. Multiplication Property:-

if $x_1(t) \xleftrightarrow{L} X_1(s)$ ROC = R_1

and $x_2(t) \xleftrightarrow{L} X_2(s)$ ROC = R_2

Then

$$x_1(t)\,x_2(t) \xleftrightarrow{L} \frac{1}{2\pi j}\left[X_1(s) * X_2(s)\right]$$

10. Integration in the time domain:-

if $x(t) \xleftrightarrow{L} X(s)$ if ROC = R

Then $\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{L} \frac{X(s)}{s}$ ROC ⇒ $R \cap \{Re(s) > 0\}$

11. Conjugation Property:-

if $x(t) \longleftrightarrow X(s)$

Then $x^*(t) \longleftrightarrow X^*(s^*)$

Causality:- For a causal LTI system, the impulse response $h(t) = 0$ for $t < 0$. it means signal is right sided.

So for a system with a rational system f/n, causality of the system is equivalent to the ROC being the right Half plane to the right of the rightmost pole.

Stability:- An LTI system is stable if and only if the ROC of its system f/n $H(s)$ includes the $j\omega$-axis.

Stable and Causal System

A causal system with rational system f/n $H(s)$ is stable if and only if all the poles of $H(s)$ lie in the left-Half of s-plane.

Q. Find the Inverse laplace of

$$H(s) = \frac{1}{s^2 - s - 6}$$

(a) if system is causal and unstable

(b) if system is stable and Non causal.

Sol:

$$H(s) = \frac{1}{s^2 - 3s + 2s - 6} = \frac{1}{s(s-3) + 2(s-3)}$$

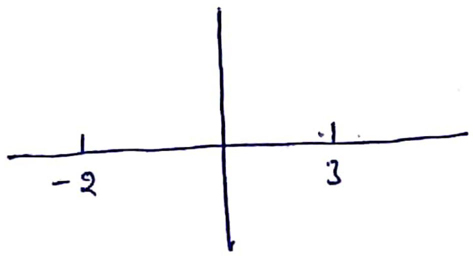
$$= \frac{1}{(s+2)(s-3)}$$

$$H(S) = \frac{1}{(S+2)(S-3)} = \frac{A}{S+2} + \frac{B}{S-3}$$

$$A = -1/5$$

$$B = 1/5$$

So $H(S) = -\frac{1}{5}\frac{1}{S+2} + \frac{1}{5}\frac{1}{S-3}$



(a) System is Causal and Unstable

Than $Re\{s\} > 3$

So $h(t) = -\frac{1}{5} e^{-2t} u(t) + \frac{1}{5} e^{3t} u(t)$ Ans.

(b) System is stable and Non Causal

Than $-2 < Re\{s\} < 3$

So $h(t) = -\frac{1}{5} e^{-2t} u(t) - \frac{1}{5} e^{3t} u(-t)$ Ans.