Q. A continous time periodic signal $x(t)$ is real valued and has a fundamental period $T=8$. The nonzero FS coefficient for $x(t)$ are

$$X_1 = X_{-1} = 2 \qquad X_3 = X_{-3}^* = 4j$$

Express $x(t)$ in the form

$$x(t) = \sum_{n=0}^{\infty} A_n \cos(\omega_n t + \phi_n)$$

Sol: given $T_0 = 8$, so $\omega = \frac{2\pi}{T_0} = \frac{2\pi}{8} = \frac{\pi}{4}$

Now FS is given as

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} X_n e^{jn\frac{\pi}{4}t}$$

$$\Rightarrow x(t) = X_{-3} e^{-3j\frac{\pi}{4}t} + X_{-1} e^{-j\pi/4\, t} + X_1 e^{j\pi/4\, t} + X_3 e^{3j\pi/4\, t} \quad \text{--(i)}$$

~~$\Rightarrow x(t) =$~~ given $X_{-1} = X_1 = 2$

and $X_3 = 4j$

$X_{-3}^* = 4j$ so $X_{-3} = -4j$

Use these in eq (i)

$$x(t) = -4j\, e^{-3j\pi/4\, t} + 2e^{-j\pi/4\, t} + 2e^{j\pi/4\, t} + 4j\, e^{3j\pi/4\, t}$$

$$= 4j\left[e^{j3\pi/4\, t} - e^{-j3\pi/4\, t}\right] + 2\left[e^{j\pi/4\, t} + e^{-j\pi/4\, t}\right]$$

$$x(t) = 4j\left[2j \sin\tfrac{3\pi}{4}t\right] + 2\left[2\cos\tfrac{\pi}{4}t\right)$$

$$x(t) = -8\sin\tfrac{3\pi}{4}t + 4\cos\tfrac{\pi}{4}t$$

$$x(t) = 4\cos\tfrac{\pi}{4}t + 8\cos\left(\tfrac{3\pi}{4}t + \tfrac{\pi}{2}\right) \quad \text{Ans.}$$

Note:- $\cos(\pi/2 + \theta) = -\sin\theta$

Q. For The Continous time periodic signal

$$x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right)$$

Determine the fundamental freq. ω_0 and the FS Coefficient X_n such That

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

Sol) given $x(t) = 2 + \cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{5\pi}{3}t\right)$

$$T_{01} = \frac{2\pi}{2\pi/3} = 3 \text{ sec.}$$

$$T_{02} = \frac{2\pi}{5\pi/3} = \frac{6}{5} \text{ sec.}$$

$$\frac{T_{01}}{T_{02}} = \frac{3}{6/5} = \frac{15}{6} \quad \text{Rational}$$

Now $T_0 = \dfrac{\text{L.C.M of Numerator of } T_{01}, T_{02} \ldots}{\text{H.C.F of denominator of } T_{01}, T_{02} \ldots}$

$$T_0 = \frac{6}{1} = 6 \text{ sec.}$$

So Fundamental freq. $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{6} = \pi/3$

So $$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\pi/3\, t} \quad \ldots (i)$$

again given $x(t) = 2 + \cos\left(\frac{2\pi}{3}\right)t + 4\sin\left(\frac{5\pi}{3}\right)t$

Or $$x(t) = 2 + \frac{e^{j2\pi/3\, t} + e^{-j2\pi/3\, t}}{2} + 4\,\frac{e^{j5\pi/3\, t} - e^{-j5\pi/3\, t}}{2j}$$

Or $$x(t) = -\frac{4}{2j}e^{-j\frac{5\pi}{3}t} + \frac{1}{2}e^{-j\frac{2\pi}{3}t} + 2 + \frac{1}{2}e^{j2\pi/3\, t} + \frac{4}{2j}e^{j\frac{5\pi}{3}t} \quad \ldots (ii)$$

~~Comparing with eq-(i)~~ Now eq-(i) is

$$x(t) = X_{-5}\, e^{-j 5\pi/3\, t} + X_{-2}\, e^{-j 2\pi/3\, t} + X_0 + X_2\, e^{j \frac{2\pi}{3} t} + X_5\, e^{j 5\pi/3\, t}$$

Compare This with eq (ii)

$$\left.\begin{aligned} X_{-5} &= -\frac{2}{j} = 2j \\ X_{-2} &= \frac{1}{2} \\ X_2 &= 1/2 \\ X_5 &= \frac{2}{j} = -2j \\ X_0 &= 2 \end{aligned}\right] \rightarrow \text{Ans.}$$

Properties of Continous time Fourier series:-

(i) linearity:-

$$x(t) \xleftrightarrow{FSC} X_n$$

$$y(t) \xleftrightarrow{FSC} Y_n$$

Then $ax(t) + by(t) \xleftrightarrow{FSC} aX_n + bY_n$

(ii) Time Shifting:-

if $x(t) \xleftrightarrow{FSC} X_n$

Then $x(t-t_0) \longleftrightarrow e^{-jn\omega_0 t_0} X_n$

Proof

We know That $X_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$

let for $x(t-t_0)$ FSC is Y_n

So $Y_n = \frac{1}{T}\int_0^T x(t-t_0)\, e^{-jn\omega_0 t}\, dt$

Now let $t - t_0 = \tau$

$t = \tau + t_0$, $dt = d\tau$

So $$Y_n = \frac{1}{T}\int_{-t_0}^{T-t_0} x(\tau)\, e^{-jn\omega_0(\tau+t_0)}\, d\tau$$

$$= \frac{1}{T}\int_{-t_0}^{T-t_0} x(\tau)\, e^{-jn\omega_0 \tau}\, e^{-jn\omega_0 t_0}\, d\tau$$

$$Y_n = e^{-jn\omega_0 t_0}\left[\frac{1}{T}\int_{-t_0}^{T-t_0} x(\tau)\, e^{-jn\omega_0 \tau}\, d\tau\right]$$

$$\Rightarrow Y_n = e^{-jn\omega_0 t_0} X_n$$

So $x(t-t_0) \xleftrightarrow{FSC} Y_n = e^{-jn\omega_0 t_0} X_n$

(iii) Frequency Shifting:-

if $x(t) \longleftrightarrow X_n$

Than $e^{jm\omega_0 t} x(t) \longleftrightarrow X_{n-m}$

(iv) Time Reversal:-

if $x(t) \longleftrightarrow X_n$

Then $x(-t) \longleftrightarrow X_{-n}$

Proof We know That

$$X_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt \quad \cdots (i)$$

Now we have to Find out FSC of $x(-t)$
let it is Y_n

So $$Y_n = \frac{1}{T}\int_0^T x(-t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{T}\int_{-T/2}^{T/2} x(-t)\, e^{-jn\omega_0 t}\, dt$$

Now put $t = -\tau$

$dt = -d\tau$

$$Y_n = \frac{1}{T}\int_{T/2}^{-T/2} x(\tau)\, e^{jn\omega_0 \tau}\, (-d\tau)$$

$$Y_n = \frac{1}{T}\int_{-T/2}^{T/2} x(\tau)\, e^{-j(-n)\omega_0 \tau}\, d\tau$$

$$\boxed{Y_n = X_{-n}}$$

So $x(-t) \xleftrightarrow{FSC} X_{-n}$

(5) Time scaling:-

if $x(t) \longleftrightarrow X_n$

Then $x(at) \longleftrightarrow X_n$

Proof by Time Scaling operation, we can change the Time period of given fn.

let Time period of $x(t)$ is T_0

Then $$X_n = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$

and Then Time period of $x(at)$ is $\frac{T_0}{a}$

and $\omega = a\omega_0$

So $$Y_n = \frac{1}{(T_0/a)}\int_0^{T_0/a} x(at)\, e^{-jna\omega_0 t}\, dt$$

Now let $at = \tau$

$a\,dt = d\tau$

$$Y_n = \frac{1}{T_0/a}\int_0^{T_0} x(\tau)\, e^{-jn\omega_0 \tau}\, \frac{d\tau}{a}$$

$$Y_n = \frac{1}{T_0}\int_0^{T_0} x(\tau)\, e^{-jn\omega_0 \tau}\, d\tau = X_n$$

So $x(at) \xleftrightarrow{FSC} Y_n = X_n$

Periodic Convolution:-

if $x(t) \xleftrightarrow{FSC} X_n$

and $y(t) \xleftrightarrow{FSC} Y_n$

Then $x(t) \circledast y(t) \xLeftrightarrow{FSC} X_n Y_n = Z_n$

Proof We know that

$$x(t) \circledast y(t) = \frac{1}{T}\int_0^T x(\tau) y(t-\tau) d\tau$$

Now

$$Z_n = \frac{1}{T}\int_0^T [x(t) \circledast y(t)] e^{-jn\omega_0 t} dt$$

$$= \frac{1}{T}\int_{t=0}^{T} \left[\frac{1}{T}\int_{\tau=0}^{T} x(\tau) y(t-\tau) d\tau \right] e^{-jn\omega_0 t} dt$$

$$= \frac{1}{T}\int_{\tau=0}^{T} \left\{ \frac{1}{T}\int_{t=0}^{T} y(t-\tau) e^{-jn\omega_0 t} dt \right\} x(\tau) d\tau$$

Now use shifting property.

$$= \frac{1}{T}\int_{\tau=0}^{T} e^{-jn\omega_0 \tau} Y_n x(\tau) d\tau$$

$$= X_n Y_n$$ Hence proved.

Multiplication:-

if $x(t) \longleftrightarrow X_n$

and $y(t) \longleftrightarrow Y_n$

Then $x(t) y(t) \longleftrightarrow \sum_{k=-\infty}^{\infty} X_k Y_{n-k}$

(8) Differentiation :-

if $x(t) \xrightarrow{FSC} X_n$

Then $\frac{dx(t)}{dt} \xrightarrow{FSC} jn\omega_0 X_n$

Proof

$$\cancel{X_n} \quad x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j\omega_0 t n}$$

$$\frac{dx(t)}{dt} = \sum_{n=-\infty}^{\infty} (j\omega_0 n X_n) e^{j\omega_0 n t}$$

So $\frac{dx(t)}{dt} \xrightarrow{FSC} jn\omega_0 X_n$

(9) Integration:-

if $x(t) \xrightarrow{FSC} X_n$

Then $\int_{-\infty}^{t} x(t)\,dt \longrightarrow \frac{1}{jn\omega_0} X_n$

Proof

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

integrate both side

$$\int_{-\infty}^{t} x(t)\,dt = \sum_{n=-\infty}^{\infty} X_n \left(\frac{e^{jn\omega_0 t}}{jn\omega_0}\right)_{-\infty}^{t}$$

$$\Rightarrow \int_{-\infty}^{t} x(t)\,dt = \sum_{n=-\infty}^{\infty} \frac{X_n}{jn\omega_0} \left(e^{jn\omega_0 t} - 0\right)$$

$$\Rightarrow \int_{-\infty}^{t} x(t)\,dt = \sum_{n=-\infty}^{\infty} \frac{X_n}{jn\omega_0} e^{jn\omega_0 t}$$

So $\int_{-\infty}^{t} x(t)\,dt \xrightarrow{FSC} \frac{1}{jn\omega_0} X_n$

10. Conjugation and Conjugate Symmetry:-

if $x(t) \xrightarrow{FSC} X_n$

Then $x^*(t) \longrightarrow X^*_{-n}$

Case (i) if $x(t)$ is real i.e.

if $x^*(t) = x(t)$

Then $X^*_{-n} = X_n$

$X_{-n} = X^*_n$

So $X_{-n} = X^*_n = X_n$ (if $x(t)$ is real and even)

· That is, if $x(t)$ is real and even, Then so are its Fourier Series Coefficient.

Case (ii) if $x(t)$ is real and odd Then its coefficient are purely imaginary and odd.

$X_{-n} = X^*_n = -X_n$

11. Parseval's Theorem for power signals:-

if $x(t) \longrightarrow X_n$

Then $\frac{1}{T}\int_0^T |x(t)|^2 dt \longrightarrow \sum_{n=-\infty}^{\infty} |X_n|^2$