Fourier Series

The Fourier series is a mathematical tool That allows the Representation of any Periodic signal as the Sum of Harmonically Related Sinusoids.

The FS is named after the French Physicist Joseph Fourier, who was the First to suggest That periodic signals could be represented by a sum of Sinusoids.

Existance of Fourier Series:-

A f/n $x(t)$ can be Represented by FS if it follow the following Dirichlet Condition.

(i) it is discontinous, ~~Thre~~ There are a Finite no. of discontinuities in the period T.

(ii) it has a Finite avg. value over the period T.

(iii) it has a Finite no. of positive and Negative maxima in the period T.

Types of Fourier Series:-

(i) Trigonometric Fourier series.

(ii) Exponential Fourier series.

(i) Trigonometric Fourier Series:-

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right]$$

Where a_0, a_n and b_n are the Trigonometric FS Coefficients.

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos(n\omega_0 t)\,dt = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(n\omega_0 t)\,dt$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin(n\omega_0 t)\,dt = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(n\omega_0 t)\,dt$$

Symmetry in Fourier Series:-

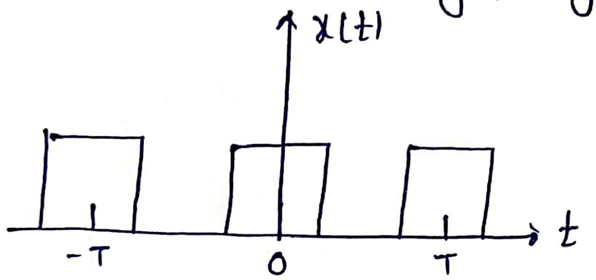
(a) Even Symmetry.

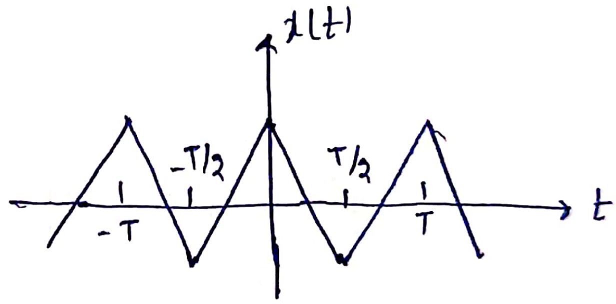
(b) Odd Symmetry

(c) Half wave Symmetry.

(a) Even Symmetry:- $x(-t) = x(t)$

if a waveform is symmetrical about vertical axis Than it have even symmetry.





$$a_0 = \frac{2}{T}\int_0^{T/2} x(t)\,dt$$

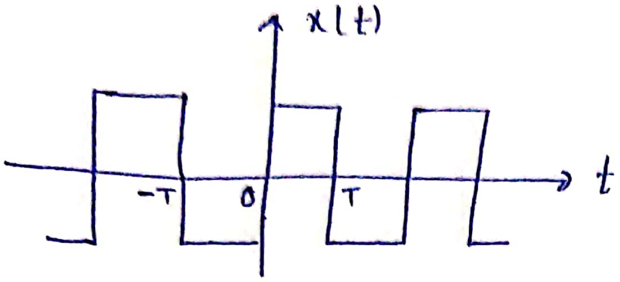
$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\omega_0 t\,dt$$

$$b_n = 0$$

Odd Symmetry:-

$x(-t) = -x(-t)$

If a waveform is symmetrical about center, Then it have odd symmetry.



$a_0 = 0$

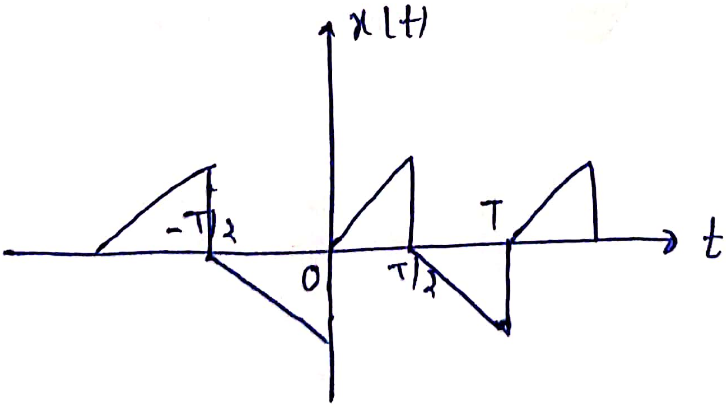
$a_n = 0$

$$b_n = \frac{4}{T}\int_0^{T/2} x(t)\sin(n\omega_0 t)\,dt$$

Half wave Symmetry:-

If a waveform is increasing for Half duration and decreasing for Remaining Half duration, Then it have Half wave Symmetry.

$$x(t) = -x\left(t \pm \frac{T}{2}\right)$$

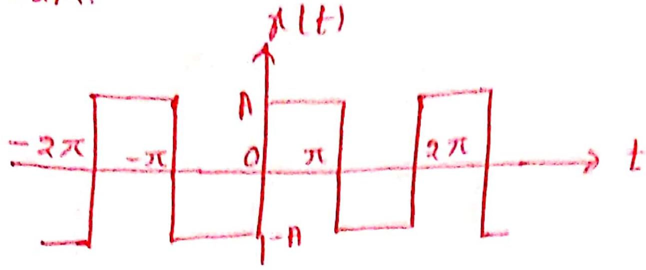


$a_0 = 0,\quad a_n = b_n = 0 \qquad n$ even

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos(n\omega_0 t)\,dt \qquad n \text{ odd}$$

$$b_n = \frac{4}{T}\int_0^{T/2} x(t)\sin(n\omega_0 t)\,dt \qquad n \text{ odd}$$

Q. Find The Trigonometric Fourier series for the square wave given in fig and Plot the line Spectrum.



Sol: Step(i) First Find out Time Period

$$T_0 = 2\pi \quad , \quad \omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{2\pi} = 1$$

Step(ii) Symmetry

given wave form has odd symmetry

So $a_0 = 0$

$a_n = 0$

$$b_n = \frac{4}{T_0}\int_0^{T_0/2} x(t)\sin(n\omega_0 t)\,dt$$

Or $$b_n = \frac{4}{2\pi}\int_0^{\pi} x(t)\sin(nt)\,dt$$

Step(iii) Find $x(t)$

Here we have to Find out $x(t)$ for

$0 < t < \pi$

So by given wave form

$$\boxed{x(t) = A \qquad 0 < t < \pi}$$

So

$$b_n = \frac{4}{2\pi} \int_0^{\pi} A \sin nt \, dt$$

$$= \frac{2A}{\pi} \left[- \frac{\cos nt}{n} \right]_0^{\pi}$$

$$= -\frac{2A}{n\pi} \left[\cos n\pi - \cos 0 \right]$$

$$b_n = -\frac{2A}{n\pi} \left[(-1)^n - 1 \right]$$

So
$$b_n = \begin{cases} 4A/n\pi & n = 1, 3, 5 \ldots \quad \text{(odd)} \\ 0 & n = 2, 4, 6 \ldots \quad \text{(even)} \end{cases}$$

Now Fourier series

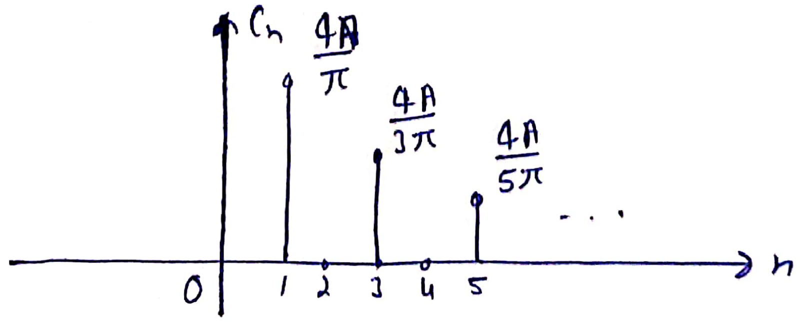
$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right]$$

$$\Rightarrow x(t) = 0 + \sum_{n=1}^{\infty} \left[0 + b_n \sin(nt) \right] \qquad \omega_0 = 1$$

$$\Rightarrow x(t) = \sum_{n=1}^{\infty} b_n \sin(nt)$$

$$= \frac{4A}{\pi} \sin t + \frac{4A}{3\pi} \sin 3t + \frac{4A}{5\pi} \sin 5t + \cdots \quad \text{Ans.}$$

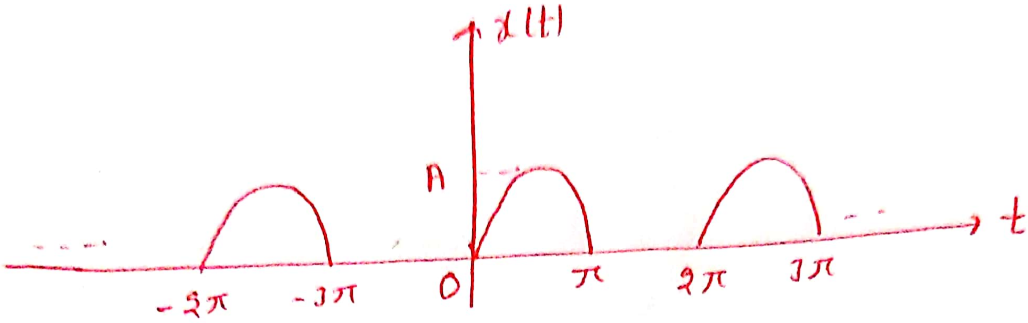
Now $C_n = \sqrt{a_n^2 + b_n^2} = \sqrt{0 + b_n^2} = |b_n|$



Ans.

Q.1 Find Trigometric FS for given Half wave Rectified Sine wave as Show in fig. and sketch the line Spectrum.

RTU 2019 (15 marks)



Sol. Step (i) Time Period

$T_0 = 2\pi$ So $\omega_0 = \frac{2\pi}{2\pi} = 1$

Step (ii) symmetry (Nither even nor odd)

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi} x(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega_0 t\,dt = \frac{2}{2\pi}\int_0^{2\pi} x(t)\cos nt\,dt$$

$$b_n = \frac{2}{T}\int_0^t x(t)\sin n\omega_0 t\,dt = \frac{2}{2\pi}\int_0^{2\pi} x(t)\sin nt\,dt$$

Step (iii) Find $x(t)$ $(0<t<2\pi)$

$$x(t) = \begin{cases} A\sin\omega t & 0<t<\pi \\ 0 & \pi<t<2\pi \end{cases}$$

Now $\omega = 1$

So

$$x(t) = \begin{cases} A\sin t & 0<t<\pi \\ 0 & \pi<t<2\pi \end{cases}$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x(t)\,dt$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} A\sin t\,dt + 0\right]$$

$$= \frac{A}{2\pi}\left[-\cos t\right]_0^{\pi}$$

$$= -\frac{A}{2\pi}\left[\cos\pi - \cos 0\right]$$

$$\boxed{a_0 = -\frac{A}{2\pi}\left[-1-1\right] = \frac{A}{\pi}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x(t)\cos nt\,dt$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} A\sin t\cos nt\,dt + 0\right]$$

$$= \frac{A}{2\pi}\int_0^{\pi}(2\sin t\cos nt\,dt)$$

$$a_n = \frac{A}{2\pi}\int_0^{\pi}\left\{\sin(1+n)t + \sin(1-n)t\right\}dt \quad \text{--(i)}$$

$$= \frac{A}{2\pi}\left[-\frac{\cos(1+n)t}{1+n} - \frac{\cos(1-n)t}{1-n}\right]_0^{\pi}$$

$$= -\frac{A}{2\pi}\left[\left\{\frac{\cos(1+n)\pi}{1+n} + \frac{\cos(1-n)\pi}{1-n}\right] - \left\{\frac{1}{1+n} + \frac{1}{1-n}\right\}\right]$$

$$= -\frac{A}{2\pi}\left[\left\{\frac{-(-1)^n}{1+n} + \frac{-(-1)^n}{1-n}\right\} - \left\{\frac{1}{1+n} + \frac{1}{1-n}\right\}\right)$$

$$= -\frac{A}{2\pi}\left[-(-1)^n\left\{\frac{1}{1+n} + \frac{1}{1-n}\right\} - \left\{\frac{1}{1+n} + \frac{1}{1-n}\right\}\right)$$

$$= \frac{A}{2\pi}\;\frac{2}{1-n^2}\;\left((-1)^n + 1\right)$$

$$\boxed{a_n = \frac{A}{\pi(1-n^2)}\left\{(-1)^n + 1\right\} \qquad n \neq 1}$$

at $n=1$ a_n become ∞ So we have to Find out Value of a_n at $n=1$ seperately.

So put $n=1$ in eq (i)

$$a_1 = \frac{A}{2\pi}\int_0^{\pi}\{\sin 2t + 0\}\,dt$$

$$= -\frac{A}{2\pi}\left[\frac{\cos 2t}{2}\right]_0^{\pi} = -\frac{A}{4\pi}\left[\cos 2\pi - \cos 0\right]$$

$$\boxed{a_1 = 0}$$

Now $b_n = \frac{1}{\pi}\int_0^{2\pi} x(t)\sin nt\,dt$

$$= \frac{1}{\pi}\left[\int_0^{\pi} A\sin t \sin nt\,dt + 0\right]$$

$$= \frac{A}{2\pi}\int_0^{\pi}\left[2\sin t\sin nt\,dt\right]$$

$$= \frac{A}{2\pi}\int_0^{\pi}\{\cos(1-n)t - \cos(1+n)t\}\,dt \quad \text{--(ii)}$$

$$= \frac{A}{2\pi}\left[\frac{\sin(1-n)t}{1-n} - \frac{\sin(1+n)t}{1+n}\right]_0^{\pi}$$

$$b_n = \frac{A}{2\pi}[0-0] \qquad n \neq 1$$

$$\boxed{b_n = 0 \qquad n \neq 1}$$

from (ii) $(n=1)$

$$b_1 = \frac{A}{2\pi}\int_0^{\pi}\{1-\cos 2t\}\,dt$$

$$= \frac{A}{2\pi}\left[(t)_0^{\pi} - \left(\frac{\sin 2t}{2}\right)_0^{\pi}\right]$$

$$\boxed{b_1 = \frac{A}{2\pi}\left[\pi - (0-0)\right] = A/2}$$

Now FS is

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$x(t) = a_0 + a_1 \cos t + \sum_{n=2}^{\infty} a_n \cos nt + b_1 \sin t + \sum_{n=2}^{\infty} b_n \sin nt$$

$$x(t) = \frac{A}{\pi} + 0 + \sum_{n=2}^{\infty} \frac{A}{\pi(1-n^2)} \{(-1)^n + 1\} \cos nt + \cancel{0}\,\frac{A}{2} \sin t + 0$$

So

$$\boxed{x(t) = \frac{A}{\pi} + \frac{A}{\pi} \sum_{n=2}^{\infty} \frac{1}{1-n^2} \{(-1)^n + 1\} \cos nt + \frac{A}{2} \sin t}$$ Ans.

Now line spectrum

$a_0 = A/\pi$

$a_1 = 0$ $\quad b_1 = A/2$

$a_2 = \frac{2A}{-3\pi}$ $\quad b_2 = 0$

$a_3 = 0$ $\quad b_3 = 0$

$a_4 = \frac{2A}{-15\pi}$ $\quad b_4 = 0$

$\vdots \qquad \vdots$

$C_n = \sqrt{a_n^2 + b_n^2}$ and $C_0 = a_0 = \frac{A}{\pi}$

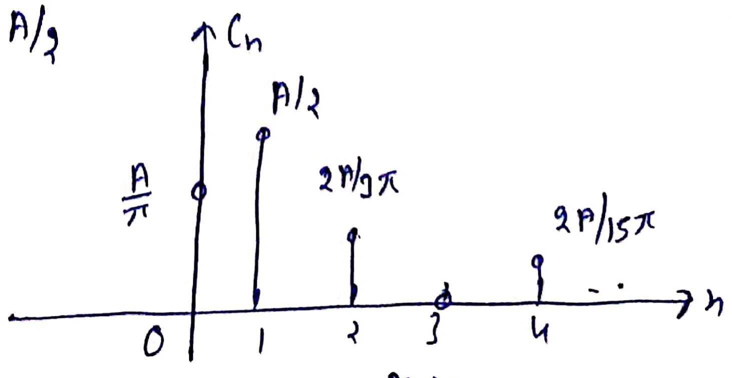
$C_1 = \sqrt{0 + A^2/4} = A/2$

$C_2 = 2A/3\pi$

$C_3 = 0$

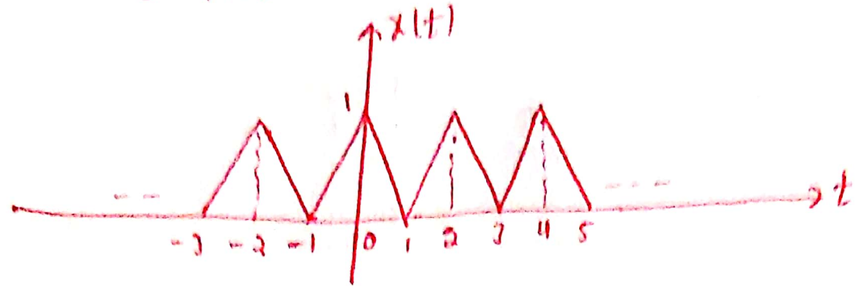
$C_4 = \frac{2A}{15\pi}$

$\vdots$



Ans.

Q. Find Trigonometric FS for the given Triangular wave and Plot the line spectrum.



Sol: Step(i) Time Period $T_0 = 2$

$$\omega_0 = \frac{2\pi}{2} = \pi$$

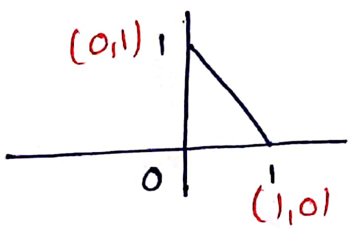
Step(ii) Symmetry $\Rightarrow$ Even

$$a_0 = \frac{2}{T_0}\int_0^{T_0/2} x(t)\,dt = \frac{2}{2}\int_0^{1} x(t)\,dt$$

$$a_n = \frac{4}{T_0}\int_0^{T_0/2} x(t)\cos n\omega_0 t\,dt = \frac{4}{2}\int_0^{1} x(t)\cos n\pi t\,dt$$

$b_n = 0$

Step(iii) $x(t)$ $\quad 0 < t < 1$



it is a ~~strio~~ straight line so

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - 0 = \frac{1-0}{0-1}(x-1)$$

$$y = -x + 1$$

or

$$x(t) = -t + 1 \qquad 0 < t < 1$$

So

$$a_0 = \int_0^1 x(t)\,dt$$

$$= \int_0^1 (1-t)\,dt$$

$$= (t)_0^1 - \left(\frac{t^2}{2}\right)_0^1$$

$$= (1-0) - (1/2 - 0)$$

$$\boxed{a_0 = \frac{1}{2}}$$

$$a_n = 2\int_0^1 x(t)\cos(n\pi t)\,dt$$

$$= 2\int_0^1 \underset{\mathrm{I}}{(-t+1)}\ \underset{\mathrm{II}}{\cos(n\pi t)}\,dt$$

$$= 2\left[\left\{(-t+1)\frac{\sin n\pi t}{n\pi}\right\}_0^1 - \int_0^1 (-1+0)\frac{\sin(n\pi t)}{n\pi}\,dt\right]$$

$$= \frac{2}{n\pi}\left[\{0-0\} + \left(-\frac{\cos n\pi t}{n\pi}\right)_0^1\right]$$

$$= -\frac{2}{n^2\pi^2}\left[\cos n\pi - \cos 0\right]$$

$$= -\frac{2}{n^2\pi^2}\left[(-1)^n - 1\right] = \frac{2}{n^2\pi^2}\left[1-(-1)^n\right]$$

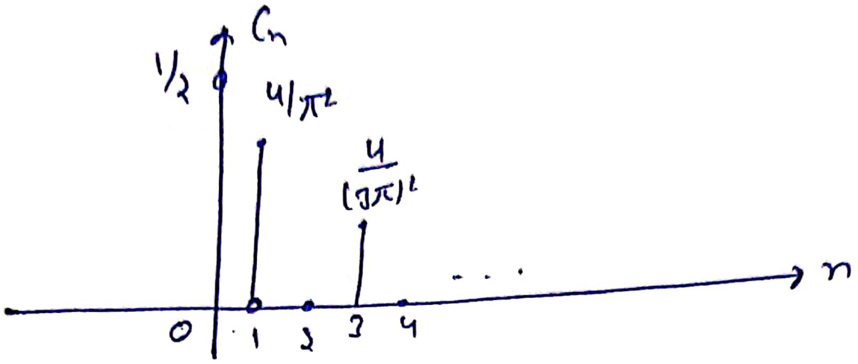
$$\boxed{a_n = \begin{cases} \dfrac{4}{n^2\pi^2} & n = 1, 3, 5 \ldots \ \text{odd} \\ 0 & n = 2, 4, 6 \ldots \ \text{even} \end{cases}}$$

So

$$\boxed{x(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left\{1-(-1)^n\right\}\cos nt}$$ Ans.

or

$$x(t) = \frac{1}{2} + \frac{4}{\pi^2}\cos(\pi t) + \frac{4}{(3\pi)^2}\cos(3\pi t) + \cdots$$



$$a_0 = 1/2 \qquad a_1 = \frac{4}{\pi^2}, \quad a_2 = 0, \quad a_3 = \frac{4}{(3\pi)^2}$$

So $C_n = \sqrt{a_n^2 + b_n^2} = \sqrt{a_n^2 + 0} = |a_n|$

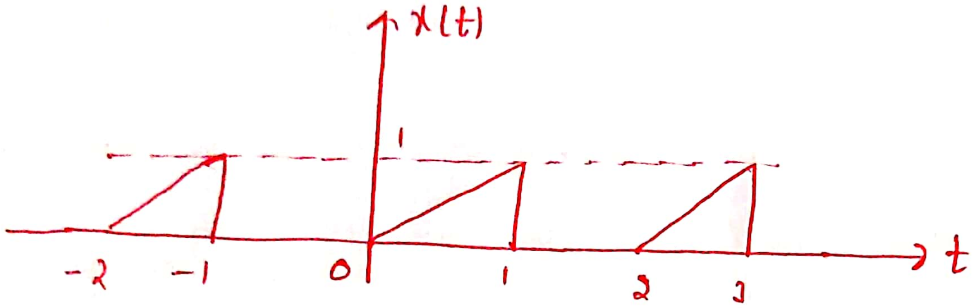
Exponential Fourier series:-

The Exponential FS is Expressed as

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

where $$X_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$$

Q: Find the Exponential Fourier series of



Sol:

(i) Time Period = 2

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{2} = \pi$$

(ii) $x(t)$ $\quad 0 < t < 2$

$$x(t) = \begin{cases} t & 0 < t < 1 \\ 0 & 1 < t < 2 \end{cases}$$

So $$X_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$$

$$X_n = \frac{1}{2}\int_0^2 x(t)\, e^{-jn\pi t}\, dt \quad \text{-- } \cancel{(i)}$$

$$= \frac{1}{2}\left[\int_0^1 t\, e^{-jn\pi t}\, dt + 0\right] \quad \text{-- (i)}$$

$$= \frac{1}{2}\left[\left(t\,\frac{e^{-jn\pi t}}{-jn\pi}\right)_0^1 - \int_0^1 (1)\,\frac{e^{-jn\pi t}}{-jn\pi}\, dt\right]$$

$$= \frac{1}{2}\left(\frac{1}{-jn\pi}\right)\left[\left(e^{-jn\pi} - 0\right) - \left(\frac{e^{-jn\pi t}}{-jn\pi}\right)_0^1\right]$$

$$= \frac{1}{2}\left(-\frac{1}{jn\pi}\right)\left[\left(\cos n\pi - j\sin n\pi\right) + \frac{1}{jn\pi}\left\{e^{-jn\pi} - e^{-j0}\right\}\right]$$

$$= -\frac{1}{2jn\pi}\left[(-1)^n + \frac{1}{jn\pi}\left\{\cos n\pi - j\sin n\pi - 1\right\}\right]$$

$$x_n = -\frac{1}{2jn\pi}\left[(-1)^n + \frac{1}{jn\pi}\left\{(-1)^n - 1\right\}\right] \qquad n \neq 0$$

Put $n=0$ in eq (i)

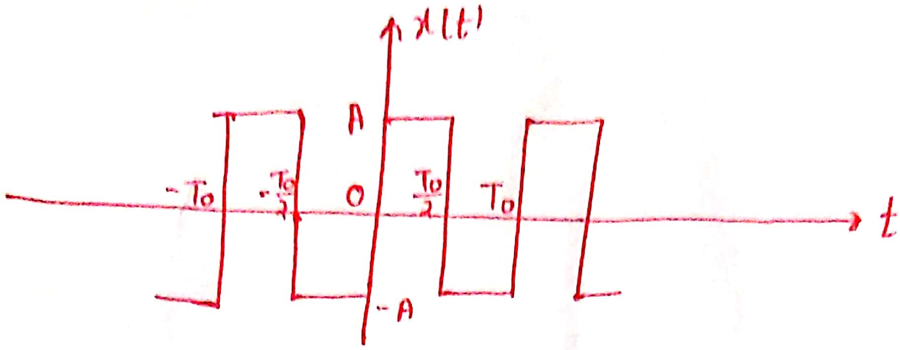
$$x_0 = \frac{1}{2}\left[\int_0^1 t\,dt\right]$$

$$= \frac{1}{2}\left[\frac{t^2}{2}\right]_0^1 = \frac{1}{4}[1-0] = 1/4$$

Now series is

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{jn\omega_0 t}$$

Q. Find Exponential Fourier series of given waveform and also draw its line spectrum.



Sol: (i) $T_0 = T_0$

$\omega_0 = \frac{2\pi}{T_0}$

(ii) $X_n = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jn\frac{2\pi}{T_0}t}\, dt$

$x(t) = ?$ $\quad 0 < t < T_0$

$x(t) = \begin{cases} A & 0 < t < T_0/2 \\ -A & T_0/2 < t < T_0 \end{cases}$

So $X_n = \frac{1}{T_0}\left[\int_0^{T_0/2} A\, e^{-jn\frac{2\pi}{T_0}t}\, dt + \int_{T_0/2}^{T_0} (-A)\, e^{-jn\frac{2\pi}{T_0}t}\, dt\right]$ ---(i)

$= \frac{A}{T_0}\left[\left(\frac{e^{-jn\frac{2\pi}{T_0}t}}{-jn\frac{2\pi}{T_0}}\right)_0^{T_0/2} - \left(\frac{e^{-jn\frac{2\pi}{T_0}t}}{-jn\frac{2\pi}{T_0}}\right)_{T_0/2}^{T_0}\right]$

$= \frac{A}{-j2n\pi}\left[\left(e^{-jn\pi} - e^{-0}\right) - \left(e^{-j2\pi n} - e^{-jn\pi}\right)\right]$

$= \frac{-A}{j2n\pi}\left[2e^{-jn\pi} - (1) - (1)\right]$

$\because e^{-0} = 1$ and $e^{-j2n\pi} = \cos 2n\pi - j\sin 2n\pi = 1 - 0 = 1$

$\boxed{X_n \Rightarrow \frac{-A}{jn\pi}\left[(-1)^n - 1\right] = \frac{A}{jn\pi}\left[1 - (-1)^n\right]}$

$n \neq 0$

Put $n=0$ in (i)

$$x_0 = \frac{1}{T_0}\left[\int_0^{T_0/2} A\,dt + \int_{T_0/2}^{T_0} (-A)\,dt\right]$$

$$= \frac{A}{T_0}\left[(t)_0^{T_0/2} - (t)_{T_0/2}^{T_0} \right]$$

$$= \frac{A}{T_0}\left[(T_0/2 - 0) - (T_0 - T_0/2) \right]$$

$$= 0$$

So $X_n = \frac{2A}{jn\pi}$ $n = 1, 3, 5 \ldots$ or $n = -1, -3, -5 \ldots$ (odd)

0 $n \to$ even

$X_0 = 0$

So $x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$

$$\ldots X_{-3} e^{-3j\omega_0 t} + X_{-2} e^{-2j\omega_0 t} + X_{-1} e^{-j\omega_0 t} + X_0 + X_1 e^{j\omega_0 t} + \ldots$$

$$\Rightarrow \cancel{\ldots - \frac{2A}{j3\pi} e^{-3j\omega_0 t} - \frac{2A}{j\pi} + \frac{2A}{j\pi}}$$

$$\Rightarrow \ldots - \frac{2A}{j3\pi} e^{-3j\omega_0 t} + 0 + \frac{2A}{-j\pi} e^{-j\omega_0 t} + 0 + \frac{2A}{j\pi} e^{j\omega_0 t} + \ldots$$

Ans.

Now line spectrum

$$C_n = \begin{cases} 2|X_n| & n \geq 1 \\ X_0 & n = 0 \end{cases}$$

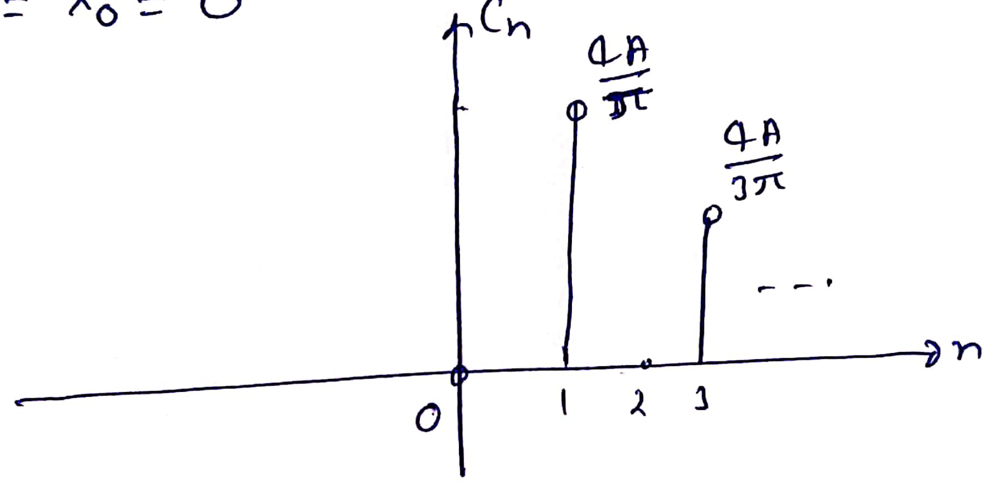
So $C_1 = 2|X_1| = \frac{4A}{\pi}$

$C_2 = 0$

$C_3 = 2|X_3| = \frac{4A}{3\pi}$

$\vdots$

$C_0 = X_0 = 0$

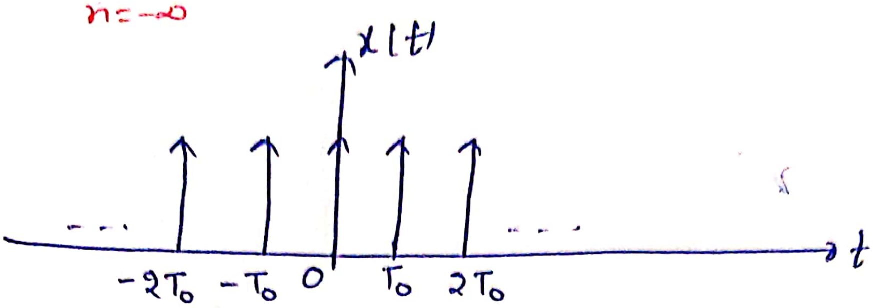


Ans.

Q. Find the Exponential FS of

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

Sol:



Here $x(t) = \delta(t) \quad -T_0/2 < t < T_0/2$

So $X_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jn\omega_0 t} dt$

$$= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) e^{-jn\frac{2\pi}{T_0}t} dt$$

or $\boxed{X_n = \frac{1}{T_0}}$

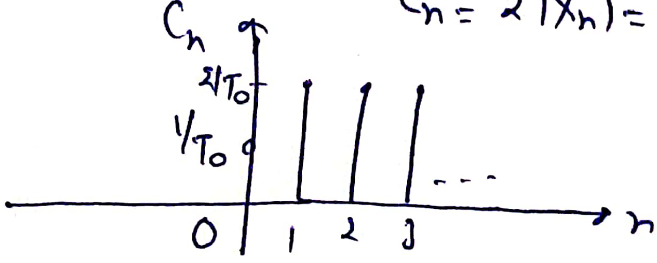
So FS is $x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \cancel{dt}$

$$\boxed{x(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}}$$

or $x(t) = \dots \frac{1}{T_0} e^{-j\omega_0 t} + \frac{1}{T_0} + \frac{1}{T_0} e^{j\omega_0 t} + \dots$

Now $C_0 = X_0 = \frac{1}{T_0}$

$C_n = 2|X_n| = \frac{2}{T_0} \quad n \geq 1$



Ans.