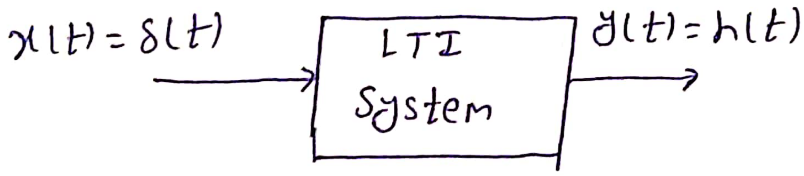
UNIT-2

LTI Systems:- A system which follow both linearity and time invariant property is called LTI system.

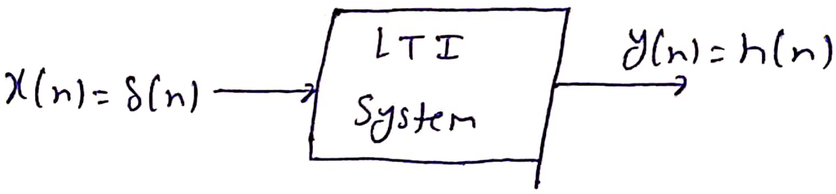
The Response of LTI system characterised by its impulse response.

Unit impulse Response:-

The impulse response is defined as the output of an LTI system due to a unit impulse signal applied as input at time $t=0$

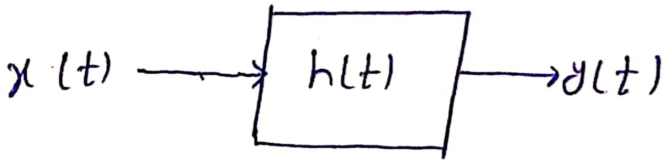


similarly for discrete time system



Convolution integral:-

Now take an LTI system with input $x(t)$, output $y(t)$ and impulse response $h(t)$



Now According to shifting Property of δ f/n.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

Now for LTI System if

$$x(t) \longrightarrow y(t)$$

$$\delta(t) \longrightarrow h(t)$$

$$\delta(t-\tau) \longrightarrow h(t-\tau)$$

$$x(\tau)\,\delta(t-\tau) \longrightarrow x(\tau)\,h(t-\tau)$$

$$\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \longrightarrow \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

So $$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

The integral Relationship given in above eq. is called the Convolution integral of signals.

This also Represented as

$$\boxed{y(t) = x(t) * h(t)}$$

we Can also write

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

Similarly for discrete time LTI system

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

or $$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

Q. Find the Convolution of

$f(t) = 3\cos 2t$ and $g(t) = e^{-|t|}$

Sol: We know That

$$y(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau \quad --(i)$$

Now $f(\tau) = 3\cos 2\tau$

and $g(t-\tau) = e^{-|t-\tau|}$

$$\text{Now } |t-\tau| = \begin{cases} t-\tau & t-\tau \geq 0 \text{ or } \tau \leq t \\ -(t-\tau) & t-\tau < 0 \text{ or } \tau > t \end{cases}$$

$$\text{so } g(t-\tau) = \begin{cases} e^{-(t-\tau)} & \tau \leq t \\ e^{+t-\tau} & \tau > t \end{cases}$$

so from (i)

$$y(t) = \int_{-\infty}^{t} 3\cos 2\tau\, e^{-t+\tau}\, d\tau + \int_{t}^{\infty} 3\cos 2\tau\, e^{t-\tau}\, d\tau$$

$$= 3e^{-t} \int_{-\infty}^{t} e^{\tau} \cos 2\tau\, d\tau + 3e^{t} \int_{t}^{\infty} e^{-\tau} \cos 2\tau\, d\tau \quad \cdot\cdot(ii)$$

Now we know Than

$$\int e^{at} \cos bt\, dt = \frac{e^{at}}{a^2+b^2} \left\{ a\cos bt + b\sin bt \right\} + c$$

So Eq. (ii) become

$$= 3e^{-t}\left[\frac{e^{\tau}}{1+4}\left\{\cos 2\tau + 2\sin 2\tau\right\}\right]_{-\infty}^{t} + 3e^{t}\left[\frac{e^{-\tau}}{1+4}\left\{-\cos 2\tau + 2\sin 2\tau\right\}\right]_{t}^{\infty}$$

$$= 3e^{-t}\left[\frac{e^{t}}{5}\left\{\cos 2t + 2\sin 2t\right\} - 0\right] + 3e^{t}\left[0 - \frac{e^{-t}}{5}\left\{-\cos 2t + 2\sin 2t\right\}\right]$$

$$= \frac{3}{5}\left[\cos 2t + 2\sin 2t + \cos 2t - 2\sin 2t\right]$$

$$= \frac{3}{5}\left[2\cos 2t\right]$$

$$= \frac{6}{5}\cos 2t \quad \text{Ans.}$$

Q. For an LTI system

$x(t) = e^{5t}u(t)$ and $h(t) = u(t+5)$

Find $y(t)$

Sol: We know That

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Now $x(\tau) = e^{5\tau}u(\tau)$

$h(t-\tau) = u(t-\tau+5)$

So $$y(t) = \int_{-\infty}^{\infty} e^{5\tau}u(\tau)\, u(t-\tau+5)\, d\tau$$

$u(\tau) = 1 \qquad \tau \geq 0$

and $u(t-\tau+5) = 1 \qquad t-\tau+5 \geq 0$

or $\tau \leq (t+5)$

$$y(t) = \int_{\tau=0}^{t+5} e^{5\tau}\, d\tau = \left[\frac{e^{5\tau}}{5}\right]_0^{t+5} \qquad t+5 \geq 0$$

$$= \frac{1}{5}\left[e^{5(t+5)} - e^0\right] \qquad t \geq -5$$

$$= \frac{1}{5}\left[e^{5(t+5)} - 1\right] \quad \text{Ans.}$$

Q. Find output of given LTI System

$$x(t) = e^{2t} u(-t)$$

$$h(t) = u(t-3)$$

Sol:

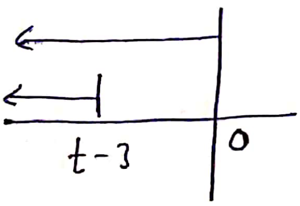
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{2\tau} u(-\tau)\, u(t-\tau-3)\, d\tau$$

Now $u(-\tau) = 1 \qquad -\tau \geq 0$ or $\tau \leq 0$

and $u(t-\tau-3) = 1 \qquad t-\tau-3 \geq 0$ or $\tau \leq (t-3)$

Case (i) $t-3 < 0$, $t < 3$

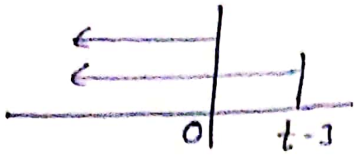


So both signal are one when $\tau \leq (t-3)$

So $$y(t) = \int_{-\infty}^{t-3} e^{2\tau}\, d\tau = \left(\frac{e^{2\tau}}{2}\right)_{-\infty}^{t-3}$$

$$= \frac{1}{2}\left[e^{2(t-3)} - 0\right] = \frac{1}{2} e^{2(t-3)}$$

Case (ii) $t-3 \geq 0$, $t \geq 3$



$$y(t) = \int_{-\infty}^{0} e^{2\tau}\, d\tau = \left(\frac{e^{2\tau}}{2}\right)_{-\infty}^{0}$$

$$= \frac{1}{2}[1-0] = \frac{1}{2} \quad \text{Ans.}$$

So
$$y(t) = \begin{cases} \frac{1}{2} e^{2(t-3)} & t < 3 \\ \frac{1}{2} & t \geq 3 \end{cases} \quad \text{Ans.}$$

Q. Find $y(n)$ if $x(n) = \alpha^n u(n)$
and $h(n) = u(n)$

Sol:
$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} \alpha^k u(k)\, u(n-k)$$

Now $u(k) = 1$ $\quad k \geq 0$

and $u(n-k) = 1$ $\quad n-k \geq 0$ or $k \leq n$

So
$$y(n) = \sum_{k=0}^{n} \alpha^k \qquad \boxed{n \geq 0}$$

$$= 1 + \alpha + \alpha^2 + \cdots \alpha^n$$

This is a Finite GP with

$a = 1$ $\quad r = \alpha$

So

$$y(n) = \frac{(1)\left[1-\alpha^{n+1}\right]}{1-\alpha} \qquad n \geq 0 \qquad \text{Ans.}$$

Q. $x(n) = 2^n u(-n)$

$h(n) = u(n)$

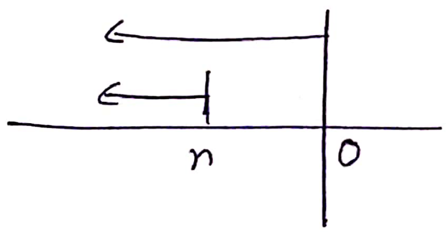
Sol: $$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} 2^k u(-k)\, u(n-k)$$

Now $u(-k) = 1 \qquad -k \geq 0, \quad k \leq 0$

$u(n-k) = 1 \qquad n-k \geq 0, \quad k \leq n$

Case (i) $n < 0$



$$y(n) = \sum_{k=-\infty}^{n} 2^k$$

$$= \ldots\ldots + 2^{n-2} + 2^{n-1} + 2^n$$

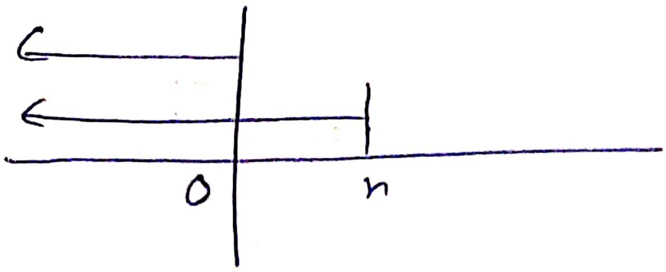
$$= 2^n \left[1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right]$$

$$= 2^n \left[\frac{1}{1-1/2}\right]$$

$$= 2^n [2]$$

$$y(n) = 2^{n+1}$$

Case (ii) $n \geq 0$



$$y(n) = \sum_{k=-\infty}^{0} 2^{k} = \cdots \cdots + 2^{-2} + 2^{-1} + 1$$

$$= 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots$$

$$= \frac{1}{1 - 1/2} = 2$$

$$y(n) = \begin{cases} 2^{n+1} & n < 0 \\ 2 & n \geq 0 \end{cases}$$

Ans.

Q. Find $y(t)$ if

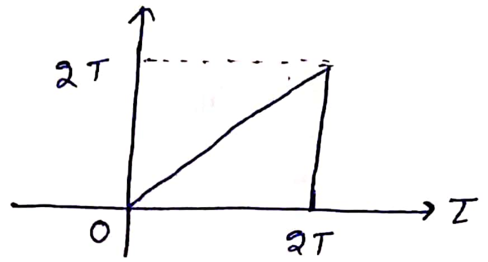
$$x(t) = \begin{cases} 1 & 0 < t < T \\ 0 & \text{otherwise} \end{cases} \qquad h(t) = \begin{cases} t & 0 < t < 2T \\ 0 & \text{otherwise} \end{cases}$$

Sol:
$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

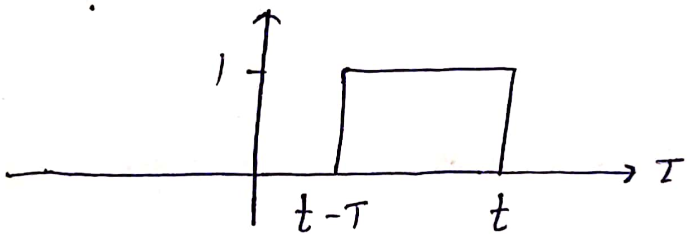
$$h(\tau) = \begin{cases} \tau & 0 < \tau < 2T \\ 0 & \text{otherwise} \end{cases}$$

$$x(t-\tau) = \begin{cases} 1 & 0 < (t-\tau) < T \ , \quad t-T < \tau < t \\ 0 & \text{otherwise} \end{cases}$$

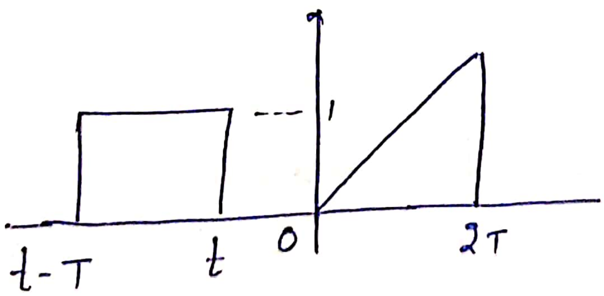
$h(\tau) =$



$x(t-\tau) =$



Case (i) $t < 0$

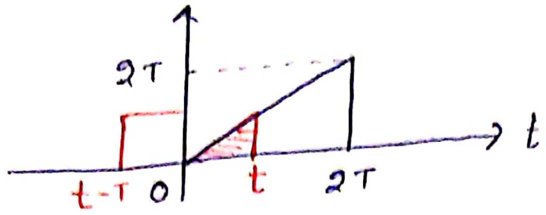


No overlapping occur so

$$y(t) = 0$$

Case (ii) $t \geq 0$ and $t - T < 0$

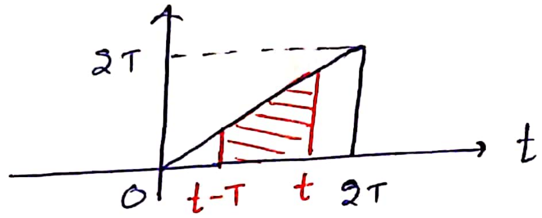
i.e $0 \leq t < T$



$$y(t) = \int_0^t \tau \, d\tau = \left(\frac{\tau^2}{2}\right)_0^t = \frac{1}{2} t^2$$

Case (iii) $t - T \geq 0$ and $t < 2T$

i.e $T \leq t < 2T$



$$y(t) = \int_{t-T}^{t} \tau \, d\tau = \left(\frac{\tau^2}{2}\right)_{t-T}^{t}$$

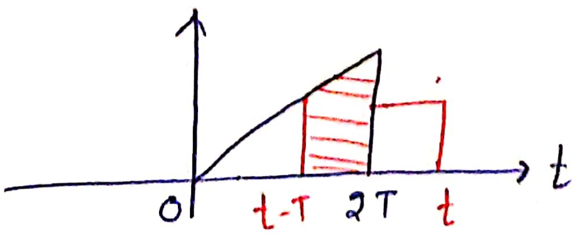
$$= \frac{1}{2}\left[t^2 - (t-T)^2\right]$$

$$= \frac{1}{2}\left[t^2 - t^2 - T^2 + 2tT\right]$$

$$y(t) = Tt - \frac{1}{2} T^2$$

Case (iv) $t \geq 2T$ and $t - T < 2T$

i.e. $2T \leq t < 3T$



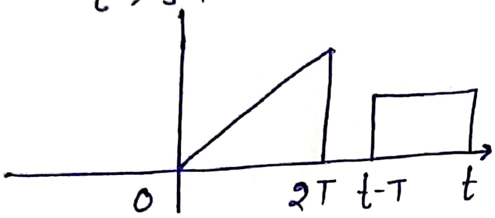
$$y(t) = \int_{\tau = t-T}^{2T} \tau \, d\tau = \left[\frac{\tau^2}{2}\right]_{t-T}^{2T}$$

$$= \frac{1}{2}\left[4T^2 - (t^2 + T^2 - 2tT)\right]$$

$$= \frac{1}{2}\left[-t^2 + 3T^2 + 2tT\right]$$

$$= -\frac{1}{2}t^2 + \frac{3}{2}T^2 + tT$$

Case (v) $t > 3T$



No overlapping occur.

So $y(t) = 0$

So
$$y(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2}t^2 & 0 < t < T \\ Tt - \frac{1}{2}T^2 & T < t < 2T \\ -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2 & 2T < t < 3T \\ 0 & t > 3T \end{cases}$$

Ans.

Q: Find $y(n)$ if

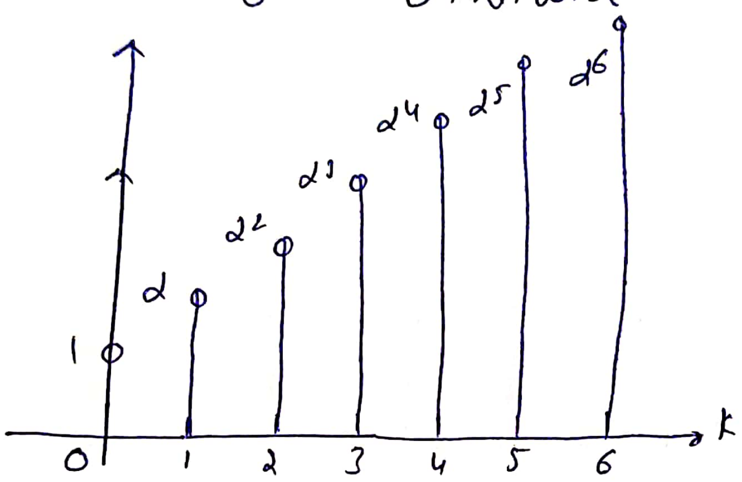
$$x(n) = \begin{cases} 1 & 0 \le n \le 4 \\ 0 & \text{otherwise} \end{cases} \qquad h(n) = \begin{cases} \alpha^n & 0 \le n \le 6 \\ 0 & \text{otherwise.} \end{cases}$$

Sol: $$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

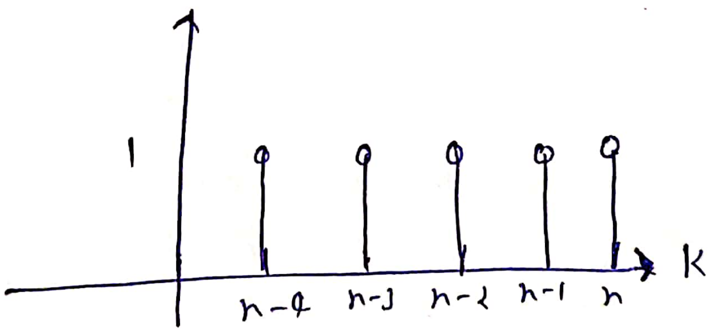
Now $$h(k) = \begin{cases} \alpha^k & 0 \le k \le 6 \\ 0 & \text{otherwise.} \end{cases}$$

$$x(n-k) = \begin{cases} 1 & 0 \le (n-k) \le 4, \quad n-4 \le k \le n \\ 0 & \text{otherwise} \end{cases}$$

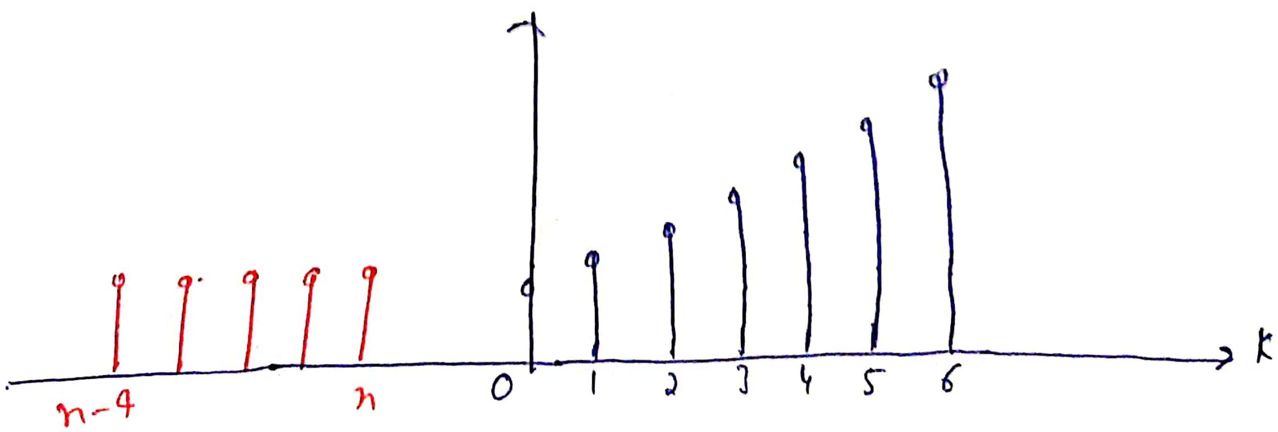
$h(k) =$



$x(n-k) \Rightarrow$



Case (i) $n < 0$

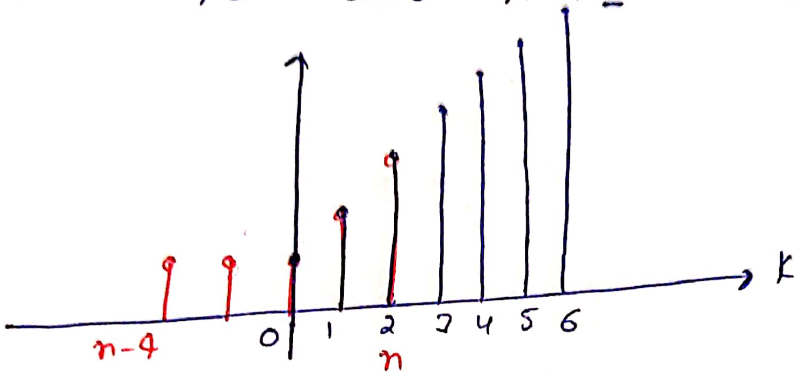


No overlapping occur so

$y(n) = 0$

~~Case~~ ~~Case~~

Case (2) $n \geq 0$ and $n-4 \leq 0$ So $0 \leq n \leq 4$

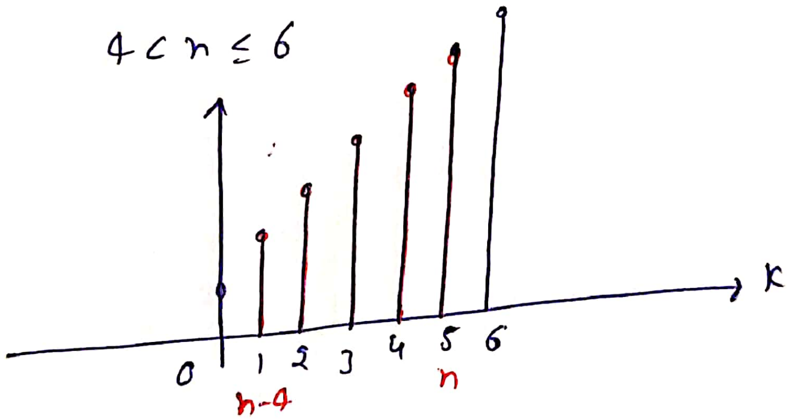


$$y(n) = \sum_{k=0}^{n} \alpha^k$$

$$= 1 + \alpha + \alpha^2 + \cdots \alpha^n = \frac{1-\alpha^{n+1}}{1-\alpha}$$

case (3) $n-4 > 0$, and $n \leq 6$

i.e $4 < n \leq 6$



$$y(n) = \sum_{k=n-4}^{n} \alpha^k$$

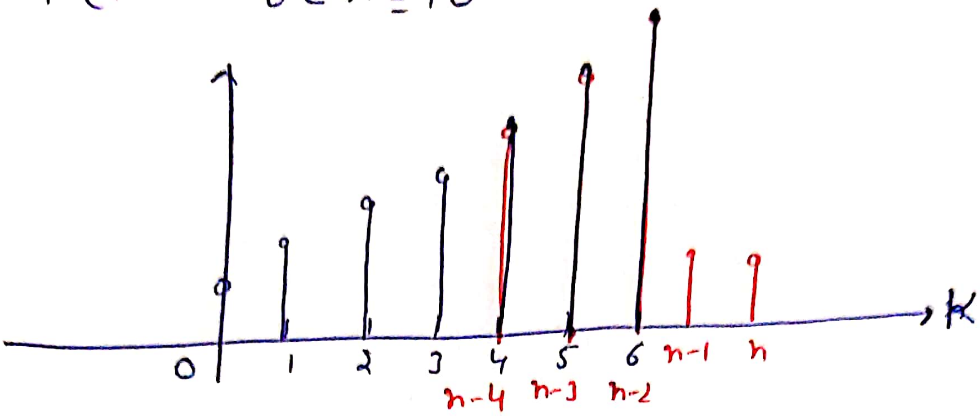
$$= \alpha^{n-4} + \alpha^{n-3} + \alpha^{n-2} + \cdots \alpha^n$$

$$= \alpha^{n-4}\left[1 + \alpha + \alpha^2 + \cdots \alpha^4\right]$$

$$= \alpha^{n-4}\left[\frac{1-\alpha^5}{1-\alpha}\right] = \frac{\alpha^{n-4} - \alpha^{n+1}}{1-\alpha}$$

Case (iv) $\quad n > 6$ and $n-4 \le 6$

i.e. $\quad 6 < n \le 10$



$$y(n) = \sum_{k=n-4}^{6} \alpha^k$$

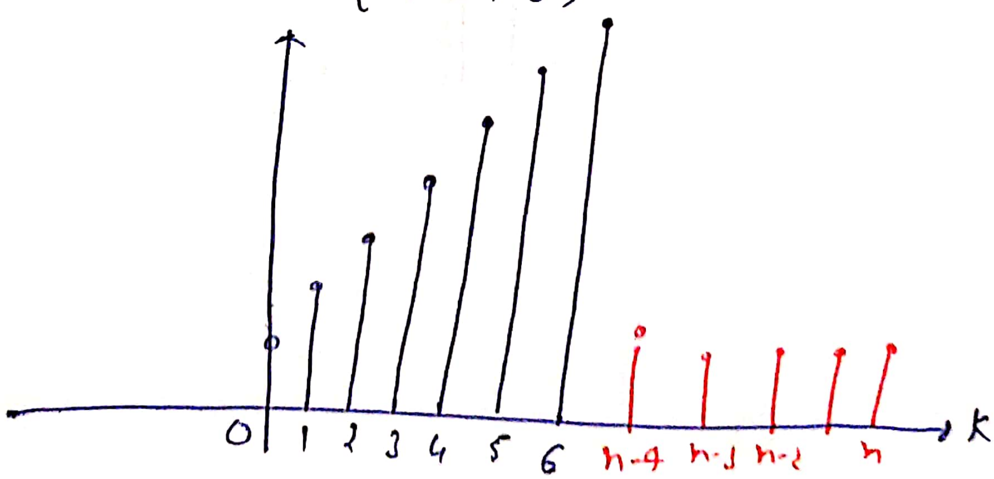
$$= \alpha^{n-4} + \alpha^{n-3} + \cdots \alpha^6$$

$$= \alpha^{n-4}\left[1 + \alpha + \alpha^2 + \cdots \alpha^{-n+10}\right]$$

$$= \alpha^{n-4}\left[\frac{1-\alpha^{-n+11}}{1-\alpha}\right]$$

$$= \frac{\alpha^{n-4} - \alpha^7}{1-\alpha}$$

Case (v) $\quad n > 10 \quad (n-4 > 6)$



No overlapping occur

$y(n) = 0$

$$y(n) = \begin{cases} 0 & n < 0 \\ \dfrac{1-\alpha^{n+1}}{1-\alpha} & 0 \le n \le 4 \\ \dfrac{\alpha^{n-4} - \alpha^{n+1}}{1-\alpha} & 4 < n \le 6 \\ \dfrac{\alpha^{n-4} - \alpha^{7}}{1-\alpha} & 6 < n \le 10 \\ 0 & n \; n > 10 \end{cases}$$

Ans.

Q. Find Convolution of

$x(n) = \{\underset{\uparrow}{0}, 1, 2\}$ $\quad h(n) = \{1, \underset{\uparrow}{2}, 3\}$

Sol: $\quad y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$

Now $h(k) = \{1, \underset{\uparrow}{2}, 3\}$

So $\quad y(n) = \sum_{k=-1}^{1} h(k)\, x(n-k)$

$$= h(-1)\,x(n+1) + h(0)\,x(n) + h(1)\,x(n-1)$$

$$y(n) = x(n+1) + 2x(n) + 3x(n-1)$$

put $n=0$ $\quad y(0) = x(1) + 2x(0) + 3x(-1)$

$$= 1 + 2(0) + 0 = 1$$

$n=1$ $\qquad y(1) = x(2) + 2x(1) + 3x(0)$

$\qquad\qquad = 2 + 2(1) + 3(0) = 4$

$n=2$ $\qquad y(2) = x(3) + 2x(2) + 3x(1)$

$\qquad\qquad = 0 + 2(2) + 3(1) = 7$

$n=3$ $\qquad y(3) = x(4) + 2x(3) + 3x(2)$

$\qquad\qquad = 0 + 0 + 3(2) = 6$

$n=4$ $\qquad y(4) = x(5) + 2x(4) + 3x(3) = 0$

$\vdots$

Now put Negative Value

$n=-1$ $\qquad y(-1) = x(0) + 2x(-1) + 3x(-2)$

$\qquad y(-1) = 0 + 0 + 0 = 0$

$n=-2$ $\qquad y(-2) = x(-1) + 2x(-2) + 3x(-3)$

$\qquad\qquad = 0$

$\vdots$

So $\qquad y(n) = \{ \underset{\uparrow}{1}, 4, 7, 6 \}$

Q. Find $y(n)$ if

$x(n) = \{4 \ \underset{\uparrow}{0} \ 2\}$ $\quad h(n) = \{4, \underset{\uparrow}{5}\}$

Sol: $y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$

$= \sum_{k=-1}^{1} x(k)\, h(n-k)$

$= x(-1)\, h(n+1) + x(0)\, h(n) + x(1)\, h(n-1)$

$= 4\, h(n+1) + 0 + 2\, h(n-1)$

$y(n) = 4h(n+1) + 2h(n-1)$

$n=0$ $\quad y(0) = 4h(1) + 2h(-1) = 0 + 2(+4) = +8$

$n=1$ $\quad y(1) = 4h(2) + 2h(0) = 0 + 2(5) = 10$

$n=2$ $\quad y(2) = 4h(3) + 2h(1) = 0 + 0 = 0$

$\vdots$

$n=-1$ $\quad y(-1) = 4h(0) + 2h(-2) = 4(5) + 0 = 20$

$n=-2$ $\quad y(-2) = 4h(-1) + 2h(-3) = 4(4) + 0 = 16$

$n=-3$ $\quad y(-3) = 4h(-2) + 2h(-4) = 0 + 0 = 0$

So $\quad y(n) = \{16 \ \ 20 \ \ \underset{\uparrow}{+8} \ \ 10\}$ Ans.

NOTE:- No. of Terms equal to

$M + N - 1$

$3 + 2 - 1 = 4$

Q. The inpulse Response of LTI system is given by $h(n) = \left(\frac{1}{2}\right)^n u(n)$. let $y(n)$ be the output of system with input $x(n) = 2\delta(n) + \delta(n-3)$.

Find $y(1)$ and $y(4)$ RTU 2018

Sol:

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

or

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u(k) \left\{2\delta(n-k) + \delta(n-k-3)\right\}$$

$$y(n) = 2\sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u(k)\, \delta(n-k) + \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u(k)\, \delta(n-k-3)$$

Now $\delta(n-k) = 1$ at $k = n$

0 otherwise

and $\delta(n-k-3) = 1$ at $k = n-3$

0 otherwise

$$\boxed{y(n) = 2\left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{2}\right)^{n-3} u(n-3)}$$

Now $y(1) = 2\left(\frac{1}{2}\right)^1 u(1) + \left(\frac{1}{2}\right)^{-2} u(-2) = 1$ Ans.

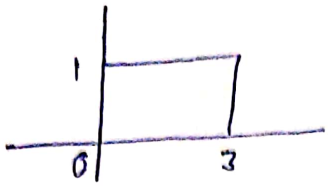
and $y(4) = 2\left(\frac{1}{2}\right)^4 u(4) + \left(\frac{1}{2}\right)^1 u(1)$

$$= \frac{1}{8} + \frac{1}{2} = \frac{1+4}{8} = 5/8 \text{ Ans.}$$

Q. if $x(t) = u(t) - u(t-3)$, $h(t) = u(t) - u(t-2)$

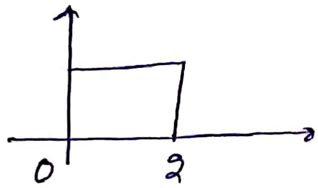
Find $y(t) = x(t) * h(t)$ RTU 2018 (15 marks)

Sol: given $x(t) = u(t) - u(t-3)$



So $x(t) = \begin{cases} 1 & 0 < t < 3 \\ 0 & \text{otherwise} \end{cases}$

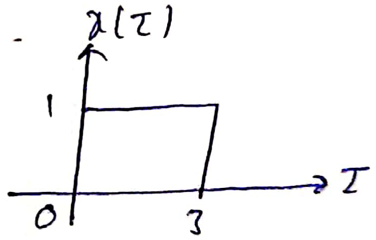
and $h(t) = u(t) - u(t-2)$



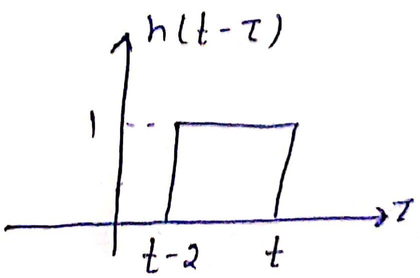
$h(t) = \begin{cases} 1 & 0 < t < 2 \\ 0 & \text{otherwise} \end{cases}$

Now $y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

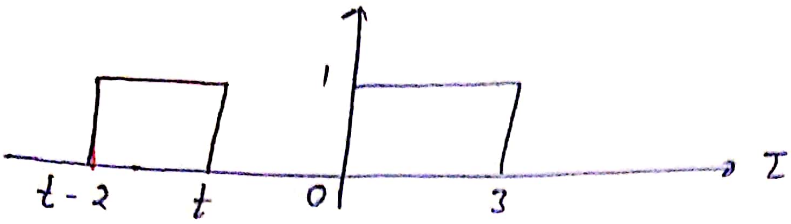
$x(\tau) = \begin{cases} 1 & 0 < \tau < 3 \\ 0 & \text{otherwise} \end{cases}$



$h(t-\tau) = \begin{cases} 1 & 0 < (t-\tau) < 2, \quad t-2 < \tau < t \\ 0 & \text{otherwise} \end{cases}$



Case (i) $t < 0$

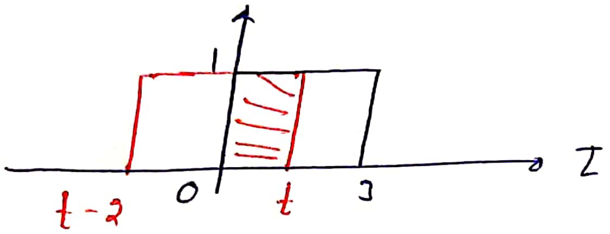


No overlapping occur

So $y(t) = 0$

Case (ii) $t \geq 0$ and $t-2 < 0$

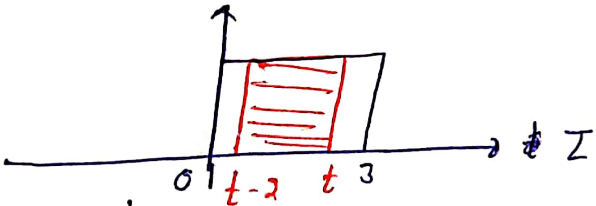
i.e $0 \leq t < 2$



$$y(t) = \int_0^t (1)(1)\,d\tau = (\tau)_0^t = t$$

Case (iii) $t-2 \geq 0$ and $t < 3$

i.e $2 \leq t < 3$

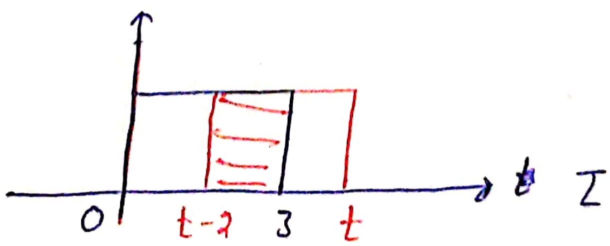


$$y(t) = \int_{t-2}^{t} (1)(1)\,d\tau = (\tau)_{t-2}^{t} = (t - t + 2)$$

$$= 2$$

Case (iv) $t \geq 3$ and ~~$t-2<0$~~ $t-2 < 3$

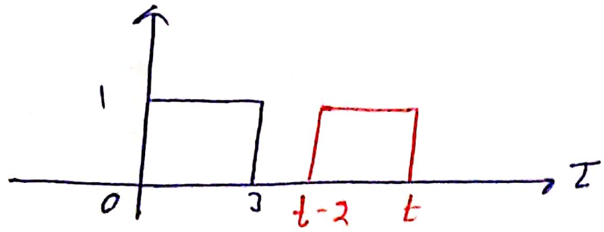
i.e $3 \leq t < 5$



$$y(t) = \int_{t-2}^{3} (1)(1)\,d\tau = (\tau)_{t-2}^{3} \Rightarrow (3 - t + 2)$$

$$= -t + 5$$

case (v) $t > 5$ $(t-2 > 3)$



No overlapping $y(t) = 0$

$$y(t) = \begin{cases} 0 & t < 0 \\ t & 0 \le t < 2 \\ 2 & 2 \le t < 3 \\ -t+5 & 3 \le t < 5 \\ 0 & t > 5 \end{cases}$$

Practice and Assignment Questions :-

Q.1 Find the convolution of the two continous time signals.

$f(t) = e^{-t^2}$ and $g(t) = 3t^2$

Q.2 Compute and plot $y(n) = x(n) * h(n)$

$$x(n) = \begin{cases} 1 & 3 \le n \le 8 \\ 0 & \text{otherwise} \end{cases}$$

$$h(n) = \begin{cases} 1 & 4 \le n \le 15 \\ 0 & \text{otherwise} \end{cases}$$

Q.3 Find $y(n)$ if

$x(n) = \left(\frac{1}{3}\right)^{-n} u(-n-1)$ and $h(n) = u(n-1)$

Q.4 Determine and sketch the convolution of the following two signals.

$$x(t) = \begin{cases} t+1 & 0 \le t \le 1 \\ 2-t & 1 < t \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

and $h(t) = \delta(t+2) + 2\delta(t+1)$

Properties of Convolution integral

(i) **Commutative Property:-**

$$x(t) * h(t) = h(t) * x(t)$$

$$x(t) \longrightarrow \boxed{h(t)} \xrightarrow{y(t)} \;\equiv\; \xrightarrow{h(t)} \boxed{x(t)} \longrightarrow y(t)$$

(ii) **Associative Property:-**

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$$
$$= [x(t) * h_2(t)] * h_1(t)$$

(iii) **Distributive property:-**

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$

(iv) **Shift Property:-**

if $x(t) * h(t) = y(t)$ then $x(t) * h(t-t_0) = x(t-t_0) * h(t)$
$$= y(t-t_0)$$

and $x(t-t_1) * h(t-t_2) = y(t-t_1-t_2)$

(v) **Convolution with an impulse:-**

The Convolution of a signal $x(t)$ with a unit impulse fn Results in the signal $x(t)$ itself

$$x(t) * \delta(t) = x(t)$$

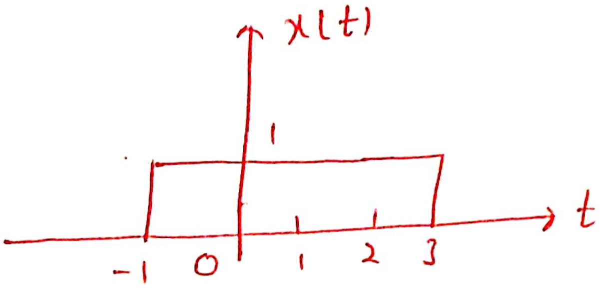
or $x(t) * \delta(t-t_0) = x(t-t_0)$

Width Property:-

The width of the non zero extent (the interval of time b/w the first and last nonzero values) of the continous convolution of two f/n equals the sum of the widts of the non zero extents of the two functions.

In other words, if the duration (widts) of $x(t)$ and $h(t)$ are finite, given by W_x and W_h, respectively, Then the duration (width) of $x(t) * h(t)$ is $W_x + W_h$.

Q. For given $x(t)$ and $h(t)$. What will be the width of the non zero extent of the convolution.



Sol: Right edge of $x(t) = RE_x = 3$

Right edge of $h(t) = RE_h = 4$

left edge of $x(t) = LE_x = -1$

left edge of $h(t) = LE_h = 0$

Now width of $x(t)$ $\quad W_x = RE_x - LE_x = 4$

and width of $h(t)$ $\quad W_h = RE_h - LE_h = 4$

Now $y(t) = x(t) * h(t)$

~~Rigth~~ Right Edge of $y(t) \Rightarrow RE_y = RE_x + RE_h = 7$

left Edge of $y(t) = LE_y = LE_x + LE_h = -1$

so Duratio of $y(t) = W_x + W_h = 4 + 4 = 8$

(7) Differentiation Property:-

if $x(t) * h(t) = y(t)$ Then $\left(\frac{d}{dt}x(t)\right) * h(t)$

$$= x(t) * \left(\frac{d}{dt}h(t)\right)$$

$$= \frac{d}{dt}y(t)$$

(8) Time - Scaling property:-

if $x(t) * h(t) = y(t)$

Then $x(at) * h(at) = \frac{1}{|a|}y(at)$

Q. Show that

(a) the Convolution of an odd and an even f/n is an odd function.

(b) the Convolution of two odd f/n is an even f/n.

(c) the Convolution of two Even f/n is an even f/n.

Sol: by Scaling Property

$$x(at) * h(at) = \frac{1}{|a|}y(at)$$

Put $a = -1$

$$x(-t) * h(-t) = y(-t) \quad \text{--(i)}$$

(a) let $x(t)$ is odd so $x(-t) = -x(t)$

and $h(t)$ is Even so $h(-t) = h(t)$

So $-x(t) * h(t) = y(-t)$

$\Rightarrow$ $x(t) * h(t) = -y(-t) = y(t)$

so $y(-t) = -y(t)$

odd f/n.

(b) $x(t)$ and $h(t)$ both are odd

So $x(-t) = -x(t)$

and $h(-t) = -h(t)$

So Eq. (i)

$$-x(t) * (-h(t)) = y(-t)$$

$$\Rightarrow \quad x(t) * h(t) = y(-t) = y(t)$$

So $y(-t) = y(t)$

Even fn.

(c) Now let $x(t)$ and $h(t)$ both are even

So $x(-t) = x(t)$

and $h(-t) = h(t)$

So Eq. (i)

$$x(t) * h(t) = y(-t) = y(t)$$

So $y(-t) = y(t)$

Even fn.

Relationship b/w LTI System Properties and the impulse response:-

(a) LTI Systems with and without memory:-

A LTI System is memoryless if and only if

$$h(t) = k\,\delta(t) \quad \text{for Continous time}$$

and $$h(n) = k\,\delta(n) \quad \text{for discrete time}$$

it means for memory less LTI System its impulse Response is defined only at $t=0$ or $n=0$.

(b) Causality for LTI Systems:-

A Continous time LTI System is causal if

$$h(t) = 0 \quad \text{for } t<0$$

and for discrete time system

$$h(n) = 0 \quad \text{for } n<0$$

(c) Stability for LTI Systems:-

If the impulse Response of a System is absolutely summable or integrable, Then System is stable.

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty \quad \text{or} \quad \int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

(a) Unit step Response of an LTI System:-

$$y(n) = x(n) * h(n)$$

$$= \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

~~Now $x(n) = \delta(n)$~~

Now $x(n) = u(n)$

$$y(n) = u(n) * h(n) = \sum_{k=-\infty}^{\infty} h(k)\, u(n-k)$$

Now
$$u(n-k) = \begin{cases} 1 & k \le n \\ 0 & k > n \end{cases}$$

$$\boxed{\text{So } y(n) = \sum_{k=-\infty}^{n} h(k)}$$

Similarly for Continous time System

$$y(t) = x(t) * h(t)$$

$$= \int_{\tau=-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Now $x(t) = u(t)$

$$y(t) = u(t) * h(t)$$

$$= \int_{\tau=-\infty}^{\infty} h(\tau)\, u(t-\tau)\, d\tau$$

$$u(t-\tau) = \begin{cases} 1 & \tau \le t \\ 0 & \tau > t \end{cases}$$

$$\boxed{y(t) = \int_{\tau=-\infty}^{t} h(\tau)\, d\tau}$$

Q. For given impulse responses $h(t)$ or $h(n)$. Determine whether each system is causal and/or stable.

(a) $h(t) = e^{-4t} u(t-2)$

(b) $h(t) = e^{-6t} u(3-t)$

(c) $h(t) = e^{-6|t|}$

(d) $h(n) = \left(\frac{1}{8}\right)^n u(n)$

(e) $h(n) = (0.8)^n u(n+2)$

Sol: (a) $h(t) = e^{-4t} u(t-2)$

Causal:- $u(t-2) = 1 \quad t \geq 2$

$0 \quad t < 2$

So $h(t) = e^{-4t} \quad t \geq 2$

$0 \quad t < 2$

it means $h(t) = 0$ for $t < 0$

So system is causal.

Now Stability $\int_{-\infty}^{\infty} |h(\tau)| d\tau$

$$\Rightarrow \int_{-\infty}^{\infty} |e^{-4\tau} u(\tau-2)| d\tau$$

$$= \int_{\tau=2}^{\infty} e^{-4\tau} d\tau = \left(\frac{e^{-4\tau}}{-4}\right)_2^{\infty}$$

$$= -\frac{1}{4}\left[0 - e^{-8}\right] = \frac{e^{-8}}{4} < \infty$$

So system is stable.

(b) $h(t) = e^{-6t} u(3-t)$

Causality:- given $h(t) = e^{-6t} u(3-t)$

Now $u(3-t) = 1 \quad t \leq 3$

$\qquad\qquad\quad 0 \quad t > 3$

So $h(t) = e^{-6t} \quad t \leq 3$

$\qquad\qquad\quad 0 \quad t > 3$

So $h(t) \neq 0$ for $t < 0$

So System is Non Causal.

Stability:- $\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$

$$\Rightarrow \cancel{\int_{-\infty}^{3} e^{-6t}} \int_{-\infty}^{3} e^{-6\tau} d\tau$$

$$= \left(\frac{e^{-6\tau}}{-6}\right)_{-\infty}^{3} = -\frac{1}{6}\left[e^{-18} - e^{\infty}\right]$$

$$= \infty$$

Unstable.

(c) $h(t) = e^{-6|t|}$

Causality $h(t) = e^{-6t} \quad t \geq 0$

$\qquad\qquad\quad e^{6t} \quad t < 0$

So $h(t) \neq 0$ for $t < 0$

So System is Non Causal.

Stability:- $\int_{-\infty}^{\infty} |h(\tau)| d\tau$

$$\Rightarrow \int_{-\infty}^{0} e^{6\tau} d\tau + \int_{0}^{\infty} e^{-6\tau} d\tau$$

$$= \left(\frac{e^{6\tau}}{6}\right)_{-\infty}^{0} + \left(\frac{e^{-6\tau}}{-6}\right)_{0}^{\infty}$$

$$= \frac{1}{6}[1-0] - \frac{1}{6}[0-1] = \frac{1}{3} < \infty$$

So System is stable.

(d) $h(n) = \left(\frac{1}{5}\right)^n u(n)$

Causality

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$h(n) = \begin{cases} (1/5)^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

So $h(n) = 0 \quad n < 0$

So System is causal.

Stability:- $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$

$$\Rightarrow \sum_{k=-\infty}^{-1} (0) + \sum_{k=0}^{\infty} \left(\frac{1}{5}\right)^k$$

$$= 1 + 1/5 + (1/5)^2 + \dots$$

$$= \frac{1}{1 - 1/5} = \frac{5}{4} < \infty$$

System is stable.

(e) $h(n) = (0.8)^n u(n+2)$

Sol: Causality:-

$$h(n) = (0.8)^n u(n+2)$$

$$u(n+2) = \begin{cases} 1 & n \geq -2 \\ 0 & n < -2 \end{cases}$$

So $h(n) \neq 0$ for $n < 0$

System is Non Causal.

Stability:- $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$

$$\Rightarrow \sum_{k=-\infty}^{-3} (0) + \sum_{k=-2}^{\infty} (0.8)^k$$

$$= (0.8)^{-2} + (0.8)^{-1} + (1) + (0.8)^1 + \cdots$$

$$= \left(\frac{1}{0.8}\right)^2 + \frac{1}{0.8} + (1) + (0.8) + \cdots$$

$$= \left(\frac{1}{0.8}\right)^2 \left[1 + 0.8 + (0.8)^2 + \cdots\right]$$

$$= \left(\frac{1}{0.8}\right)^2 \left[\frac{1}{1-0.8}\right]$$

$$= \left(\frac{5}{4}\right)^2 \left[\frac{1}{0.2}\right]$$

$$= \left(\frac{5}{4}\right)^2 (5) = \frac{125}{16} < \infty$$

Stable.

Q. Evaluate the step Response of following System.

(a) $h(n) = \left(-\frac{1}{2}\right)^n u(n)$

Sol:

$$y(n) = \sum_{k=-\infty}^{n} h(k)$$

$$= \sum_{k=-\infty}^{n} \left(-\frac{1}{2}\right)^k u(k)$$

$$u(k) = \begin{matrix} 1 & k \geq 0 \\ 0 & k < 0 \end{matrix}$$

$$y(n) = \sum_{k=0}^{n} \left(-1/2\right)^k$$

$$= 1 + (-1/2) + (-1/2)^2 + \cdots (-1/2)^n$$

$$= \frac{1-(-1/2)^{n+1}}{1-(-1/2)} = \frac{2}{3} + \frac{1}{3}(-1/2)^n \quad \text{Ans.}$$

(b) $h(t) = \frac{1}{4}\left[u(t) - u(t-4)\right]$

Sol:

$$y(t) = \int_{\tau=-\infty}^{t} h(\tau)\, d\tau$$

$$= \frac{1}{4}\left[\int_{\tau=-\infty}^{t} u(\tau)\, d\tau - \int_{\tau=-\infty}^{t} u(\tau-4)\, d\tau\right]$$

$$= \frac{1}{4}\left[\int_{\tau=0}^{t} d\tau - \int_{\tau=4}^{t} d\tau\right]$$

$$= \frac{1}{4}\left[(\tau)_0^t - (\tau)_4^t\right] = \frac{1}{4}\left[(t-0) - (t-4)\right]$$