NUMERICAL METHODS

Definition :- The technique or Method of estimating unknown values from a given set of observation is Known as Interpolation. Mathematically, let

x :-	a	$a+h$	$a+2h$,	---,	$a+nh$
$y=f(x)$:-	$f(a)$	$f(a+h)$	$f(a+2h)$,	---;	$f(a+nh)$

Then the Method of finding $f(m)$ for $x=m$ where $a \le m \le a+nh$, is known as Interpolation and where m lies out side this range, It is Known as Extrapolation, eg.

x :-	0	5	10	15	20
$y=f(x)$:-	7	11	19	23	27

Then the method of finding $f(3)$, $f(7)$, $f(17)$ with The help of The given data is Known as Interpolation and that of finding $f(23)$ or $f(0.5)$ is Known as Extrapolation.

The study of interpolation is based on The "CALCULUS OF FINITE DIFFERENCES" We begin by deriving many Important

Interpolation Formulae by Means of The Following (i) forward differences
(ii) Backward differences
(3) Central differences

These deals with The changes in the value of The function y (dependent variable) due to change in x (Independent Variable) by equal intervals.

FORWARD DIFFERENCES

x	$y = f(x)$	1st diff	2nd diff	3rd diff	4th diff	5th diff.
x_0	y_0					
		Δy_0				
$x_1 = x_0 + h$	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
$x_2 = x_0 + 2h$	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
$x_3 = x_0 + 3h$	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
$x_4 = x_0 + 4h$	y_4		$\Delta^2 y_3$			
		Δy_4				
$x_5 = x_0 + 5h$	y_5					

Subscript remains constant along each forward diagonal.

From The table, The First forward differences of $y = f(x)$ are defined as

$$\Delta y_n = y_{n+1} - y_n, \quad n = 0, 1, 2, 3, \ldots$$

In particular

$$\Delta y_0 = y_1 - y_0, \quad \Delta y_1 = y_2 - y_1, \quad \Delta y_2 = y_3 - y_2 \text{ etc.}$$

The second forward differences are defined as $\Delta^2 y_n = \Delta y_{n+1} - \Delta y_n$

$n = 0, 1, 2, 3, \ldots$

which gives $\Delta^2 y_0 = \Delta y_1 - \Delta y_0$, $\Delta^2 y_1 = \Delta y_2 - \Delta y_1$ etc

$\Delta f(x) \rightarrow$ known as Forward operator

In a similar manner, we can also define the higher order differences.

Thus $\Delta^3 y_n = \Delta^2 y_{n+1} - \Delta^2 y_n$.

$\Delta^4 y_n = \Delta^3 y_{n+1} - \Delta^3 y_n$ etc.

$n = 0, 1, 2, 3, \ldots$

NEWTON - GREGORY FORWARD INTERPOLATION FORMLA:

Let the function $y = f(x)$ has the set of $(n+1)$ values of y that is $y_0, y_1, y_2, \ldots y_n$ corresponding to the values of x as $x_0, x_1, x_2, \ldots; x_n$ such that x is equi-spaced. i.e $x_i = x_0 + ih$. $(i = 0, 1, 2, 3, \ldots)$. Let $y_n(x)$ be a polynomial of nth degree such that $y_n(x_i) = y_i = f(x_i)$ and $y_n(x)$ is assumed as.

$$y_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}) \quad \text{--- (i)}$$

where $a_0, a_1, a_2, \ldots, a_n$ are $(n+1)$ constants which are obtained by putting

$x = x_0, x_1, \ldots, x_n$ sucessively in eqⁿ (1)

Then equation (1) becomes

$$y_n(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{\lfloor 2}\Delta^2 y_0 \quad \text{--(2)}$$

$$+ \frac{u(u-1)(u-2)}{\lfloor 3}\Delta^3 y_0 + \cdots + \frac{u(u-1)(u-2)\cdots\{u-(n-1)\}}{\lfloor n}\Delta^n y_0$$

where $x = x_0 + uh$

or $u = \dfrac{x - x_0}{h}$. $\quad h \rightarrow$ class Interval.

(2) is known as Newton's forward Interpolation formula for equal Intervals.

NOTE :-

(i) u should be taken between 0 to 1 or between 0 to -1

(ii) This formula is used mainly for Interpolating the values of $y = f(x)$ near the top of a given set of tabulated values.

BACKWARD DIFFERENCES

Below we give a table which gives backward differences at different points.

TABLE.

x	y	1st diff	2nd diff	3rd diff	4th diff
x_0	y_0				
		∇y_1			
$x_0+h=x_1$	y_1		$\nabla^2 y_2$		
		∇y_2		$\nabla^3 y_3$	
$x_2 = x_0+2h$	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$
		∇y_3		$\nabla^3 y_4$	
$x_3 = x_0+3h$	y_3		$\nabla^2 y_4$		
		∇y_4			
$x_4 = x_0+4h$.	y_4				

Note:- The Subscripts remains constant along each backward diagonal of the table.

As shown in table. here we use the line of differences starting from the bottom and going backward along a diagonal. These are called as backward differences and $\nabla \rightarrow$ is known as backward operator.

From The Table above.

The first backward difference ∇y of a function $y = f(x)$ can be defined as

$$\nabla y_n = y_n - y_{n-1} \qquad n = 1, 2, 3, \ldots$$

In Particular,

$$\nabla y_1 = y_1 - y_0, \quad \nabla y_2 = y_2 - y_1 \text{ etc.}$$

The second backward difference may be defined as

$$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}, \quad n = 1, 2, 3, \ldots$$

which gives

$$\nabla^2 y_1 = \nabla y_1 - \nabla y_0, \quad \nabla^2 y_2 = \nabla y_2 - \nabla y_1$$

NEWTON-GREGORY BACKWARD FORMULA

Let The function $y = f(x)$ has the set of $(n+1)$ values of y That is $y_0, y_1, y_2, \ldots, y_n$. Corresponding to the values of x as $x_0, x_1, x_2, \ldots, x_n$ such that x is equispaced i.e $x_i = x_0 + ih$ $(i = 0, 1, 2, 3, \ldots)$.

Let $y_n(x)$ be a polynomial of nth degree such That $\quad y_n(x_i) = y_i = f(x_i)$

and $y_n(x)$ is assumed as.

$$y_n(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + \cdots + a_n(x - x_n)(x - x_{n-1}) \cdots (x - x_1) \quad \text{--- (1)}$$

where $a_0, a_1, a_2, \ldots, a_n$ are $(n+1)$ constants which are obtained by putting $x = x_n, x_{n-1}, \ldots, x_0$ successively. in eqn (1).

Then above eqn (i) becomes.

$$y_n(x) = y_n + u \nabla y_n + \frac{u(u+1)}{\lfloor 2} \nabla^2 y_n + \cdots + \frac{u(u+1)\cdots(u+n-1)}{\lfloor n} \nabla^n y_n \quad \text{--- (2)}$$

where $x = x_0 + uh$ i.e $u = \dfrac{x - x_0}{h}$.

$0 < u < 1$.

(2) is called the Newton's Backward Interpolation formula with equal Intervals.

CENTRAL DIFFERENCES

We derived Newton's forward and backward interpolation formulae which are applicable for Interpolation near the beginning and end of tabulated values. Now we shall develop central difference formulae which are best suited for Interpolation near the middle of the table.

Central diff. Table

x	y	1st Diff	2nd diff	3rd diff	4th diff
$x_{-2} = x_0 - 2h$	y_{-2}				
		$\Delta y_{-2} (= \delta y_{-\frac{3}{2}})$			
$x_{-1} = x_0 - h$	y_{-1}		$\Delta^2 y_{-2} (= \delta^2 y_{-1})$		
		$\Delta y_{-1} (= \delta y_{-\frac{1}{2}})$		$\Delta^3 y_{-2} (= \delta^3 y_{-\frac{1}{2}})$	
x_0	y_0		$\Delta^2 y_{-1} (= \delta^2 y_0)$		$\Delta^4 y_{-2} (= \delta^4 y_0)$
		$\Delta y_0 (= \delta y_{1/2})$		$\Delta^3 y_{-1} (= \delta^3 y_{\frac{1}{2}})$	
$x_1 = x_0 + h$	y_1		$\Delta^2 y_0 (= \delta^2 y_1)$		
		$\Delta y_1 (= \delta y_{3/2})$			
$x_2 = x_0 + 2h$	y_2				

GAUSS Backward Path

STIRLING'S PATH

Gauss Forward Path

Since from the Table δ is known as central operator

$$\delta y_{n+\frac{1}{2}} = y_{n+1} - y_n$$

$$\delta^2 y_n = \delta y_{n+\frac{1}{2}} - \delta y_{n-\frac{1}{2}}$$

$$\delta^3 y_{n+\frac{1}{2}} = \delta^2 y_{n+1} - \delta^2 y_n$$

$$\delta^4 y_n = \delta^3 y_{n+\frac{1}{2}} - \delta^3 y_{n-\frac{1}{2}}$$

— — — — — —

Thus we see that $\Delta y_0 = y_1 - y_0$ and $\nabla y_1 = y_1 - y_0$, also $\delta y_{1/2} = y_1 - y_0$.

$$\therefore \quad y_1 - y_0 = \Delta y_0 = \nabla y_1 = \delta y_{1/2}$$

GAUSS'S FORWARD INTERPOLATION FORMULA

The Newton' forward Interpolation formula is

$$y_n(x_0+uh) = y_0 + u\Delta y_0 + \frac{u(u-1)}{\lfloor 2} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{\lfloor 3} \Delta^3 y_0 + \cdots \quad \text{---(1)}$$

we have from the table above

$$\Delta^3 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$$

$$\text{or} \quad \Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$

Similarly we determine the values of $\Delta^2 y_0$, $\Delta^3 y_0$, $\Delta^4 y_0$ from Table and substituting in (1), we get

$$y_n(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{\lfloor 2}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{\lfloor 3}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{\lfloor 4}\Delta^4 y_{-2} + \cdots \quad (0<u<1) \quad (2)$$

where $x = x_0 + uh \Rightarrow u = \frac{x - x_0}{h}$

which is called the Gauss's forward Interpolation formula

NOTE

In the central difference notation, This formula will be

$$y_n(x_0+uh) = y_0 + u\,\delta y_{1/2} + \frac{u(u-1)}{\lfloor 2}\delta^2 y_0 + \frac{(u+1)u(u-1)}{\lfloor 3}\delta^3 y_{1/2} + \frac{(u+1)u(u-1)(u-2)}{\lfloor 4}\delta^4 y_0 + \cdots \quad (3)$$

here students are advised to use formula given by (3) For GAUSS Forward Interpolation.

GAUSS'S BACKWARD INTERPOLATION FORMULA

The Newton's forward Interpolation formula is

$$y_n(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{\lfloor 2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{\lfloor 3}\Delta^3 y_0 + \cdots \quad (1)$$

we have From The table

$$\Delta y_0 - \Delta y_{-1} = \Delta^2 y_{-1}$$

$$\text{or } \Delta y_0 = \Delta^2 y_{-1} + \Delta y_{-1}$$

Similarly determine $\Delta^2 y_0, \Delta^3 y_0$ from the table and substituting in (1), we get

$$y_n(x) = y_0 + u\,\Delta y_{-1} + \frac{(u+1)u}{\underline{|2}}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{\underline{|3}}\Delta^3 y_{-2} \quad \text{---(2)}$$

$$+ \frac{(u+2)(u+1)u(u-1)}{\underline{|4}}\Delta^4 y_{-2} + \cdots$$

This is called Gauss backward Interpolation formula.

$$u = \frac{x - x_0}{h} \qquad \text{i.e. } x = x_0 + uh.$$

$-1 < u < 0$ i.e It is interpolate the values of y for a negative values of u

In the central difference notation, this formula will be

$$y_n(x) = y_0 + u\,\delta y_{-1/2} + \frac{(u+1)u}{\underline{|2}}\delta^2 y_0$$

$$+ \frac{(u+1)u(u-1)}{\underline{|3}}\delta^3 y_{-1/2}$$

$$+ \frac{(u+2)(u+1)u(u-1)}{\underline{|4}}\delta^4 y_0 + \cdots \quad \text{---(3)}$$

(As per the path mentioned in the table above)

(3) is called Gauss's backward Interpolation FORMULA.

students are advised to use Formula (3) above

STIRLING'S FORMULA

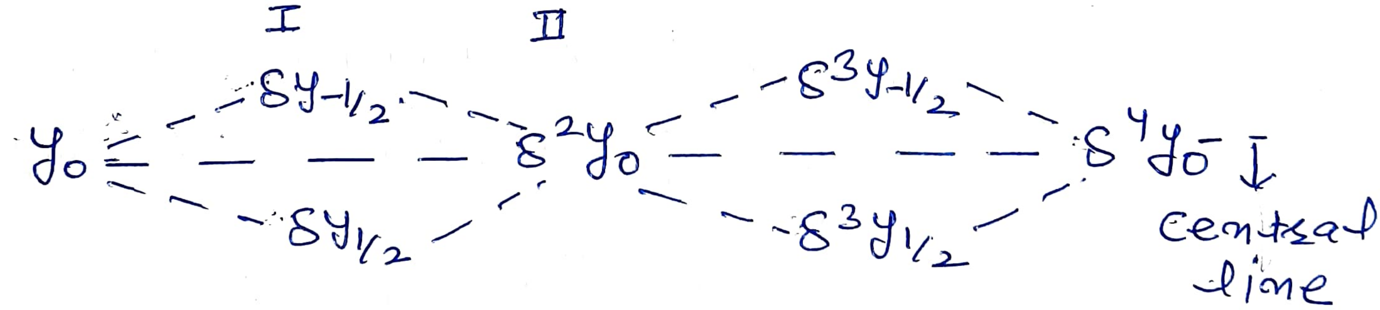
It is obtained by taking the Mean of Gauss's forward and backward Interpolation Formula as given:

$$y_n(x_0+uh) = y_0 + u(\mu\delta y_0) + \frac{u^2}{\lfloor 2}\delta^2 y_0 + \frac{u(u^2-1^2)}{\lfloor 3}(\mu\delta^3 y_0) + \frac{u^2(u^2-1^2)}{\lfloor 4}\delta^4 y_0 + \cdots \quad (1)$$

where $\mu\delta y_0 = \frac{1}{2}(\delta y_{1/2} + \delta y_{-1/2})$

$$\mu\delta^3 y_0 = \frac{1}{2}(\delta^3 y_{1/2} + \delta^3 y_{-1/2})$$

This formula involves means of the odd differences just above and below the central line and even differences on this line as shown below:



LAGRANGE'S INTERPOLATION FORMULA FOR UNEQUAL INTERVALS:

Let $y(x)$ be continuous and differentiable $(n+1)$ times in the Interval (a, b). Now to find a polynomial $y_n(x)$ of degree n, let $f(x_0), f(x_1), f(x_2), \ldots f(x_n)$ be the $(n+1)$ values of the function $f(x)$ corresponding to $x_0, x_1, \ldots x_n$ which are not equally spaced. Then the polynomial $y_n(x)$ at any arbitrary point 'x' may be written as

$$y_n(x) = f(x) = A_0 (x-x_1)(x-x_2)\cdots(x-x_n)$$
$$+ A_1 (x-x_0)(x-x_2)\cdots(x-x_n) + \cdots \quad \cdots (1)$$
$$+ A_n (x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

where $A_0, A_1, \ldots A_n$ are the $(n+1)$ constants

putting $x = x_0, x_1, x_2, \ldots x_n$ in (1) we have

$$f(x_0) = A_0 (x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)$$

$$\Rightarrow A_0 = \frac{f(x_0) = y_0}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}$$

proceeding in the same way, we get the values of $A_1, A_2, \ldots, A_n$

Substituting these values in (1), we get

$$y = f(x) = y_n(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} y_0$$

$$+ \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)} y_2 + \cdots$$

$$+ \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} y_n \quad \text{--} (2)$$

where

x :-	x_0	x_1	x_2	x_3	$\cdots x_n$
y :-	y_0	y_1	y_2	y_3	$\cdots y_n$

For example.

x :-	x_0	x_1	x_2
y :-	y_0	y_1	y_2

Then y for unknown value of x using (2) is given by

$$y_n(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \cdot y_1$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

(2) is known as Lagrange's Interpolation polynomial or formula.

INVERSE INTERPOLATION.

The dechnique of determining unknown value of argument (x) for a given entry (y) lying between two tabulated values of y's is known as Inverse Interpolation. here in next Article we are going to discuss Lagrange's method (only included in our syllabus)

LAGRANGE'S METHOD FOR INVERSE INTERPOLATION :

If we interchange the role of x and y in Lagrange's Interpolation formula, we get

$$x = f(y) = Y_n(y) = \frac{(y-y_1)(y-y_2)---(y-y_n)}{(y_0-y_1)(y_0-y_2)--(y_0-y_n)} x_0$$

$$+ \frac{(y-y_0)(y-y_2)---(y-y_n)}{(y_1-y_0)(y_1-y_2)---(y_1-y_n)} x_1 + ----$$

$$+ \frac{(y-y_0)(y-y_1)-----(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)---(y_n-y_{n-1})} x_n$$

This gives the values of x when y is known.