JECRC Foundation

JAIPUR ENGINEERING COLLEGE AND RESEARCH CENTRE

JAIPUR ENGINEERING COLLEGE AND RESEARCH CENTRE

Year & Sem – I Year & 1 Sem

Subject –Engineering Mathematics

Unit – V (Multiple Integrals: Double Integrals)

Presented by – (Dr. Tripati Gupta, Associate Professor)

VISION AND MISSION OF INSTITUTE

VISION OF INSTITUTE

To became a renowned centre of outcome based learning and work towards academic professional, cultural and social enrichment of the lives of individuals and communities .

MISSION OF INSTITUTE

- Focus on evaluation of learning, outcomes and motivate students to research aptitude by project based learning.
- Identify based on informed perception of Indian, regional and global needs, the area of focus and provide platform to gain knowledge and solutions.
- Offer opportunities for interaction between academic and industry .
- Develop human potential to its fullest extent so that intellectually capable and imaginatively gifted leaders may emerge.

Engineering Mathematics: Course Outcomes

Students will be able to:

CO1. Understand fundamental concepts of improper integrals, beta and gamma functions and their properties. Evaluation of Multiple Integrals in finding the areas, volume enclosed by several curves after its tracing and its application in proving certain theorems.

CO2. Interpret the concept of a series as the sum of a sequence and use the sequence of partial sums to determine convergence of a series. Understand derivatives of power, trigonometric, exponential, hyperbolic, logarithmic series.

Engineering Mathematics: Course Outcomes

CO3. Recognize odd, even and periodic function and express them in Fourier series using Euler's formulae.

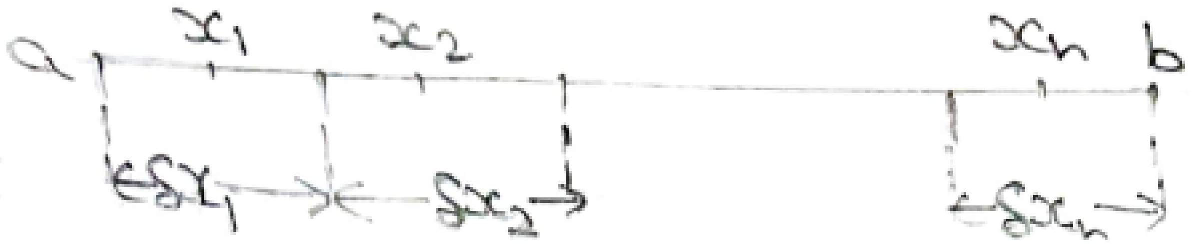
CO4. Understand the concept of limits, continuity and differentiability of functions of several variables. Analytical definition of partial derivative. Maxima and minima of functions of several variables Define gradient, divergence and curl of scalar and vector functions.

CONTENTS (TO BE COVERED)

DOUBLE INTEGRALS AND ITS APPLICATIONS

Double Integral

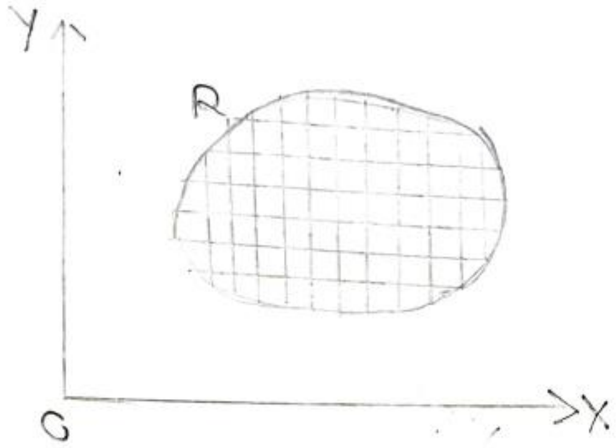
The definite integral $\int_a^b f(x)\,dx$ is defined as the limits of the sum $f(x_1)\delta x_1 + f(x_2)\delta x_2 + \ldots + f(x_n)\delta x_n$ when $n \to \infty$ and each of the lengths $\delta x_1, \delta x_2, \ldots, \delta x_n$ tends to zero. Here $\delta x_1, \delta x_2, \ldots, \delta x_n$ are n subintervals into which the range b-a has been divided and $x_1, x_2, \ldots, x_n$ are values of x lying respectively in the first, second,…, nth sub-interval.



A double integral is its counterpart in two dimensions. Let a single- valued function f(x, y) of two independent variables x, y be defined in a closed region R of the xy-plane. Divide the region R into sub-regions by drawing lines parallel to co-ordinate axes. Number the rectangles which lie entirely inside the region R, from 1 to n. Let (x_r, y_r) be any point inside the rth rectangle whose area is δA_r.

Consider the sum

$f(x_1, y_1)\delta A_1 + f(x_2, y_2)\delta A_1 + \ldots + f(x_n, y_n)\delta A_n$ =
$\sum_{r=1}^{n} f(x_r, y_r)\delta A_r$ (1)



Let the number of these sub-regions increase indefinitely, such that the largest linear dimension of δA_r approaches zero. The limit of the sum (1), if it exists, irrespective of the mode of sub - division , is called the double integral of f(x,y) over the region R and is denoted by $\iint_R f(x, y)\, dA$.

In other words

$$\lim_{\substack{n\to\infty \\ \delta A_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r)\delta A_r = \iint_R f(x, y)\, dA$$

$= \iint_R f(x, y)\, dx\, dy$

Evaluation of double Integrals in cartesian coordinates

The method of evaluating the double integrals depend upon the nature of the curves bounding the region R. let the region R be bounded by the curves x= x_1 ,x=x_2 and y=y_1,y=y_2.

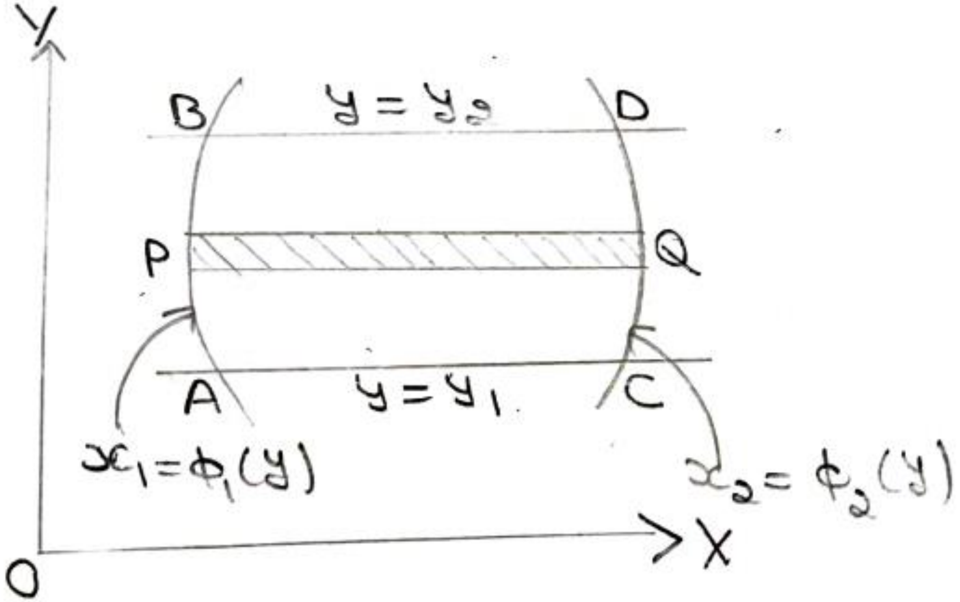
(i)When x_1, x_2 are functions of y and y_1, y_2 are constants

Let AB and CD be the curves $x_1 = \phi_1(y)$ and $x_2 = \phi_2(y)$.

Take a horizontal strip PQ of width δy . Here the double integral is evaluated first w.r.t. x (treating y as a constant). The resulting expression which is a function of y is integrated w.r.t. y between the limits y=y_1 and y=y_2. Thus

$$\iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2} \left\{ \int_{x_1=\phi_1(y)}^{x_2=\phi_2(y)} f(x,y)\,dx \right\} dy$$

The integration being carried from the inner to the outer rectangle. Geometrically, the integral in the inner rectangle indicates that the integration is performed along the horizontal strip PQ (keeping y constant) while the outer rectangle corresponds to the sliding of the strip PQ from AC to BD thus covering the entire region ABCD of integration.

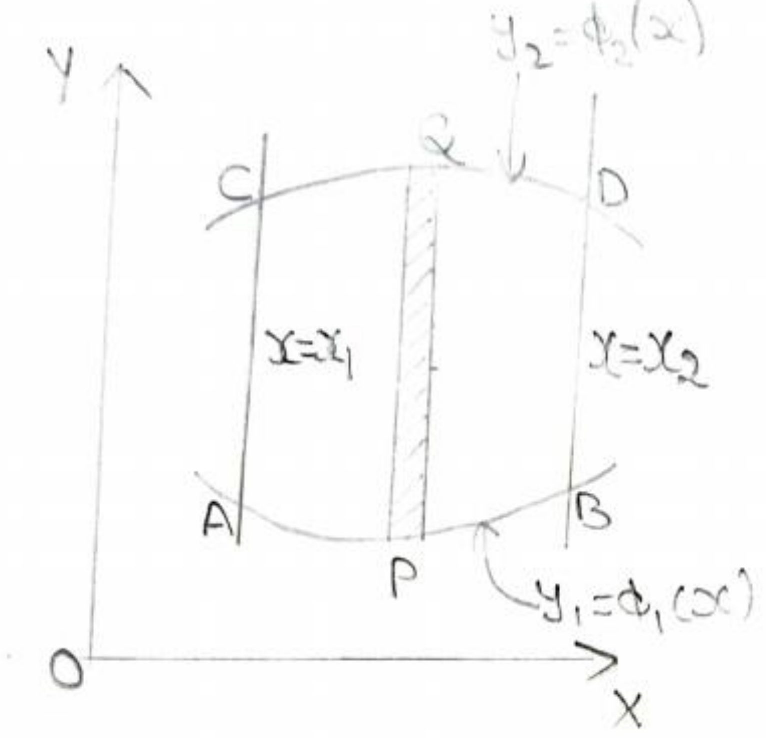


(ii) When y_1, y_2 are functions of x and x_1, x_2 are constants

Let AB and CD be the curves $y_1 = \phi_1(x)$ and $y_2 = \phi_2(x)$. Take a vertical strip PQ of width δx . Here the double integral is evaluated first w.r.t. y(treating x as a constant). The resulting expression which is a function of x is integrated w.r.t. x between the limits $x=x_1$ and $x=x_2$. Thus

$$\iint_R f(x,y)\,dx\,dy = \int_{x_1}^{x_2} \left\{ \int_{y_1=\phi_1(x)}^{y_2=\phi_2(x)} f(x,y)\,dy \right\} dx$$

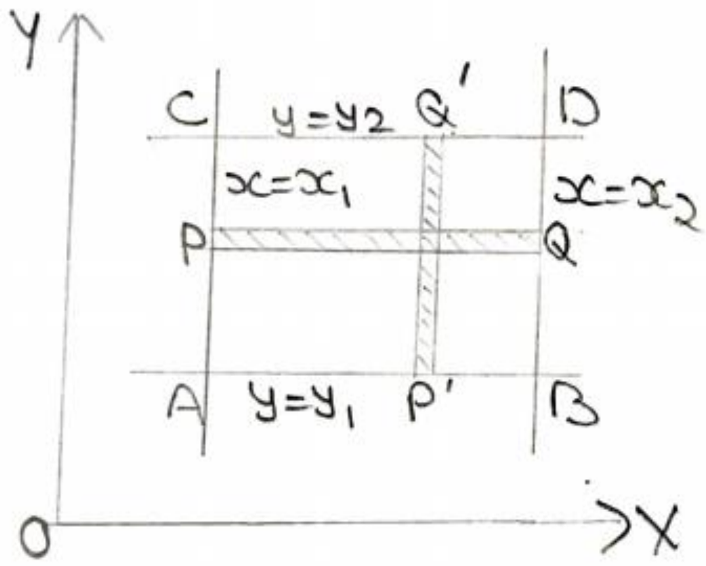
The integration being carried from the inner to the outer rectangle. Geometrically, the integral in the inner rectangle indicates that the integration is performed along the vertical strip PQ(keeping x constant) while the outer rectangle corresponds to the sliding of the strip PQ from AC to BD thus covering the entire region ABCD of integration.



(iii)When y_1, y_2, x_1, x_2 are constants

Here the region of integration R is the rectangle ABCD. It is immaterial whether we integrate first along the horizontal strip PQ and then slide it from AB to CD; or we integrate first along the vertical strip $P'Q'$ and then slide from AC to BD. Thus, the order of integration is immaterial, provided the limits of integration are changed accordingly.

$$\iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2}\left\{\int_{x_1}^{x_2} f(x,y)\,dx\right\} dy = \int_{x_1}^{x_2}\left\{\int_{y_1}^{y_2} f(x,y)\,dy\right\} dx$$

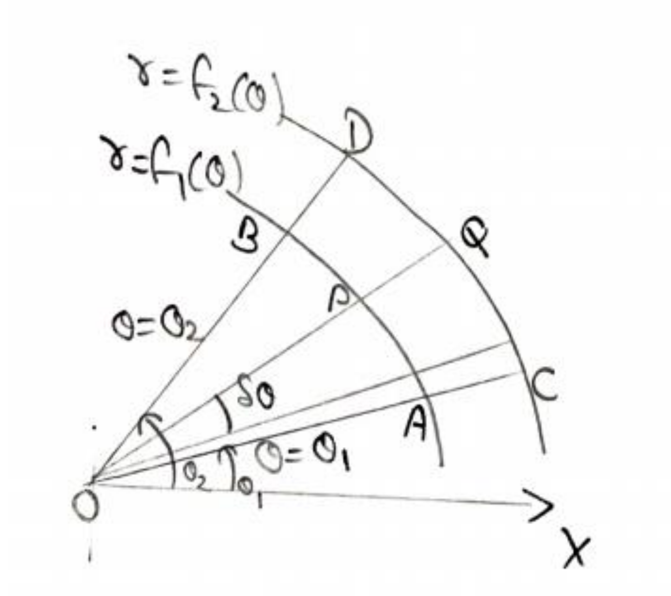


Evaluation of Double Integrals in Polar Coordinates

Let us consider the double integral in polar coordinates as

$$I = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r,\theta)\, dr\ d\theta$$

To evaluate I, integrate R.H.S. first w.r.t. r between the limits r_1 , r_2 treating θ as constant and then the resulting expression is integrated w.r.t. θ within the limits θ_1, θ_2 i.e.



$$I = \int_{\theta_1}^{\theta_2} \left\{ \int_{r_1}^{r_2} f(r,\theta)\, dr \right\} d\theta$$

Question: Evaluate $\iint_R y\,dx\,dy$, where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$.

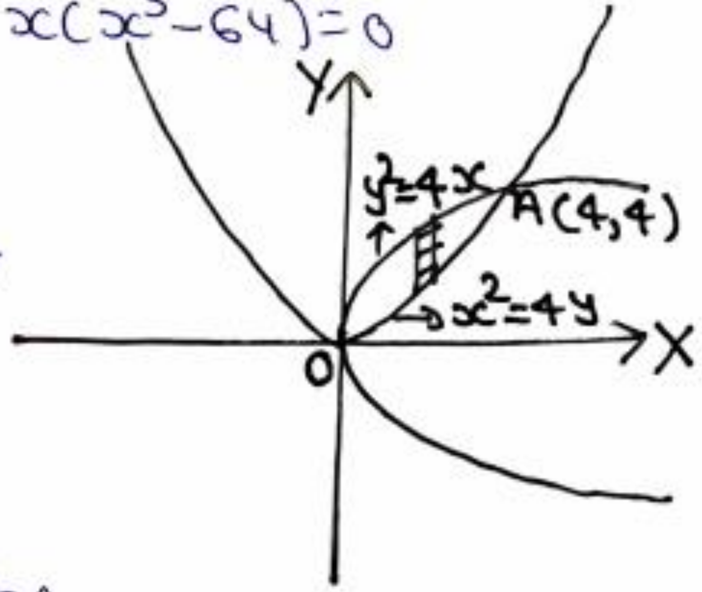
Solution: Solving $y^2 = 4x$ and $x^2 = 4y$, we have

$$\left(\frac{x^2}{4}\right)^2 = 4x \Rightarrow x(x^3 - 64) = 0$$

$$\Rightarrow x = 0, 4$$

when $x = 4$, $y = 4$

$\therefore$ Coordinates of A are $(4, 4)$.



The region R can be expressed as $0 \le x \le 4$, $\frac{x^2}{4} \le y \le 2\sqrt{x}$.

$$\therefore \iint_R y\,dx\,dy = \int_{x=0}^{4} \int_{y=x^2/4}^{2\sqrt{x}} y\,dy\,dx$$

$$= \int_0^4 \frac{1}{2}\left[y^2\right]_{x^2/4}^{2\sqrt{x}}$$

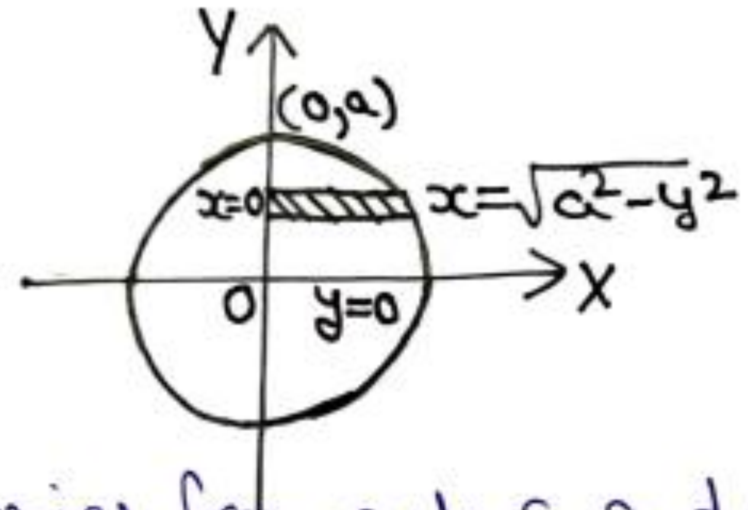
$$= \frac{1}{2}\int_0^4 \left(4x - \frac{x^4}{16}\right)dx$$

$$= \frac{1}{2}\left[2x^2 - \frac{x^5}{80}\right]_0^4$$

$$= \frac{1}{2}\left[32 - \frac{1024}{80}\right] = \frac{48}{5}$$

Question: Evaluate $\iint_R xy\,dy\,dx$, where the region R in positive quadrant of the circle $x^2+y^2=a^2$

Solution: Given $I=\iint_R xy\,dy\,dx$



Here y varies from 0 to a and x varies from 0 to $\sqrt{a^2-y^2}$

$$\therefore \quad I = \int_{y=0}^{a} y \left[\int_{x=0}^{\sqrt{a^2-y^2}} x \, dx \right] dy$$

$$= \int_0^a y \left[\frac{x^2}{2} \right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \frac{1}{2} \int_0^a y (a^2 - y^2) \, dy$$

$$= \frac{1}{2} \left[\frac{a^2 y^2}{2} - \frac{y^4}{4} \right]_0^a$$

$$= \frac{1}{2} \left[\frac{a^4}{2} - \frac{a^4}{4} \right]$$

$$= \frac{1}{2} \left(\frac{a^4}{4} \right) = \frac{a^4}{8}$$

Question: Evaluate: $\int_0^1 \left\{ \int_0^1 \frac{x-y}{(x+y)^3} dy \right\} dx$

Solution: $I = \int_0^1 \left\{ \int_0^1 \frac{x-y}{(x+y)^3} dy \right\} dx$

$$= \int_0^1 \left\{ \int_0^1 \frac{-(x+y)+2x}{(x+y)^3} dy \right\} dx$$

$$= -\int_0^1 \left\{ \int_0^1 \frac{dy}{(x+y)^2} - 2x \int_0^1 \frac{dy}{(x+y)^3} \right\} dx$$

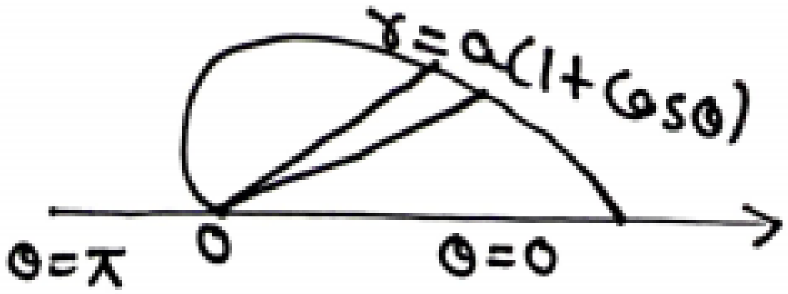
$$= -\int_0^1 \left[-\frac{1}{x+y} + \frac{x}{(x+y)^2} \right]_0^1 dx$$

$$= -\int_0^1 \left[-\frac{1}{x+1} + \frac{x}{(x+1)^2} + \frac{1}{x} - \frac{1}{x} \right] dx$$

$$= \int_0^1 \frac{1}{(x+1)^2} dx = \left(-\frac{1}{x+1} \right)_0^1 = -\frac{1}{2} + 1 = \frac{1}{2}$$

Question: Evaluate $\iint r\sin\theta \, dr \, d\theta$ over the area of the cardioid $r = a(1+\cos\theta)$ above the initial line.

Solution: The region of integration R is



covered by radial strips whose

ends are $r=0$ and $r=a(1+\cos\theta)$, the strips starting from $\theta=0$ and ending at $\theta=\pi$.

$$\therefore \iint_R r\sin\theta\, dr\, d\theta = \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\sin\theta\, dr\, d\theta$$

$$= \int_0^{\pi}\sin\theta\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_0^{\pi}\sin\theta \cdot a^2(1+\cos\theta)^2\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2}\right)^2 d\theta$$

$$= 4a^2\int_0^{\pi}\sin\frac{\theta}{2}\cos^5\frac{\theta}{2}\, d\theta$$

$$= 4a^2\int_0^{\pi/2} 2\sin\phi\cos^5\phi\, d\phi$$

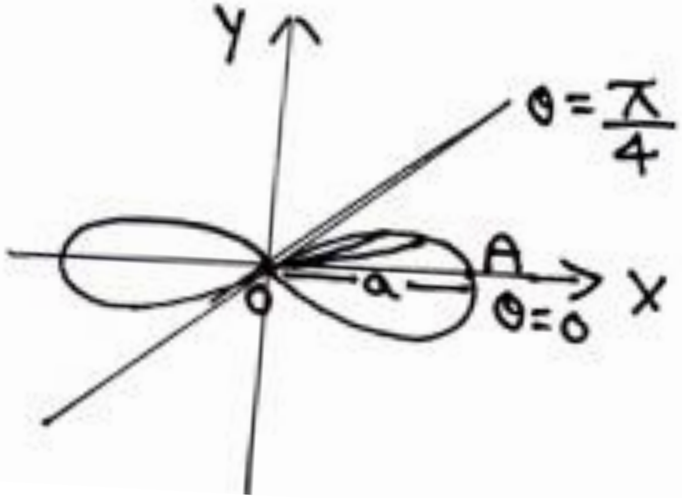
$$\left(\text{Put } \frac{\theta}{2}=\phi\ ;\ d\theta = 2d\phi\right)$$

$$= -8a^2\int_0^{\pi/2}\cos^5\phi\,(-\sin\phi)\, d\phi$$

$$= -8a^2\left[\frac{\cos^6\phi}{6}\right]_0^{\pi/2} = \frac{-4a^2}{3}(0-1) = \frac{4a^2}{3}$$

Question: Evaluate $\iint \frac{r\,d\theta\,dr}{\sqrt{a^2+r^2}}$ over one loop of the leminiscate $r^2 = a^2\cos 2\theta$

Solution:



from figure, for upper half of one loop

$r = 0,\ r = a\sqrt{\cos 2\theta}$ and $\theta = 0,\ \theta = \frac{\pi}{4}$

$$I = 2\int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} \frac{r\,d\theta\,dr}{\sqrt{(a^2+r^2)}}$$

$$= 2\int_0^{\pi/4}\left[\sqrt{a^2+r^2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= 2\int_0^{\pi/4}\left[a\sqrt{1+\cos 2\theta} - a\right] d\theta$$

$$= 2\sqrt{2}\,a\int_0^{\pi/4}\cos\theta\,d\theta - 2\int_0^{\pi/4} a\,d\theta$$

$$= 2a\left[\sqrt{2}\sin\theta - \theta\right]_0^{\pi/4}$$

$$= 2a\left[\sqrt{2}\,\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right] = 2a\left(1 - \frac{\pi}{4}\right)$$

$$I = \frac{a}{2}(4 - \pi)$$

Change of variables: Cartesian to Polar Form

Let us consider the integral in cartesian form as

$I = \iint_R \ f(x, y)\, dx\, dy$
Now to transform the above integral in polar coordinates put $x \ = \ r\ cos\theta$, $y \ = \ r\ sin\theta$ and replace the elementary area dx dy by the corresponding area in polar coordinates i.e. $r\ d\theta\ dr$.
Thus, we get

$$\iint_{\mathrm{R}} \ \mathrm{f(x,y)\ dx\ dy} \ = \iint_{\mathrm{R}} \ \mathrm{f(r\ cos}\theta, \mathrm{r\ sin}\theta)\ \mathrm{r\ d}\theta \mathrm{dr}$$

Question: Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy$

by changing it to polar Coordinates.

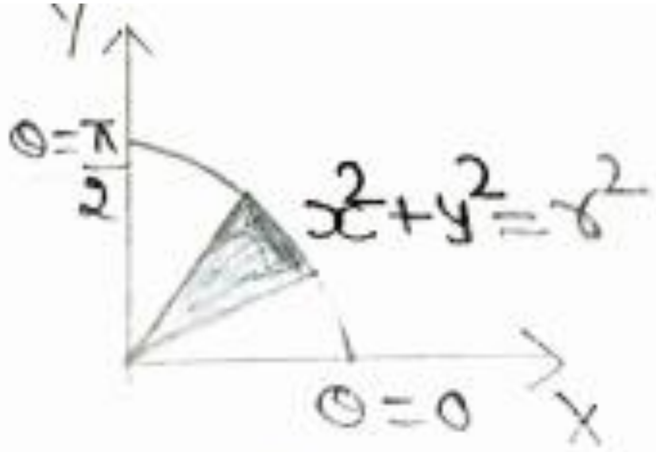
Solution: To change the integral into polar form put $x = r\cos\theta$, $y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$ in the given integral, we get

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy = \iint_R e^{-r^2} r\,dr\,d\theta$$

$$[\because x^2+y^2=r^2]$$

Since given that $x=0$, $x=\infty$; $y=0$, $y=\infty$ so the region of integration is the first quadrant thus we have the limits of θ and r as

$\theta = 0$ to $\frac{\pi}{2}$; $r = 0$ to ∞



$$\therefore \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy$$

$$= \int_0^{\pi/2} \int_0^\infty e^{-r^2} r\,d\theta\,dr$$

$$= \int_0^{\pi/2} \left[\int_1^0 \frac{du}{-2} \right] d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} [u]_0^1 \, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} d\theta = \frac{1}{2} (\theta)_0^{\pi/2}$$

$$= \pi/4$$

put $e^{-r^2} = u$

$r e^{-r^2} dr = \frac{du}{-2}$

and $r = 0 \Rightarrow u = 1$;

$r = \infty \Rightarrow u = 0$

Question: Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$ by changing to polar Co-ordinates.

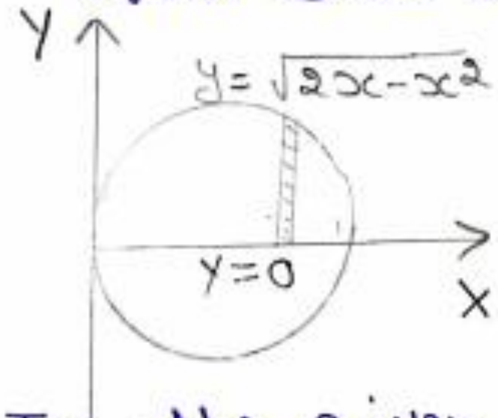
Solution: In the given integral, y varies from 0 to $\sqrt{2x-x^2}$ and x varies from 0 to 2.

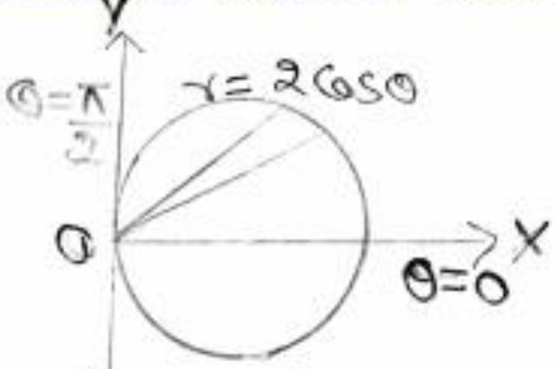
$$y = \sqrt{2x-x^2} \Rightarrow y^2 = 2x-x^2 \Rightarrow x^2+y^2 = 2x$$

In polar Co-ordinates, we have $r^2 = 2r\cos\theta$

$$\Rightarrow r = 2\cos\theta$$

$\therefore$ For the region of integration, r varies from 0 to $2\cos\theta$ and, θ varies from 0 to $\frac{\pi}{2}$.





In the given integral, replacing x by $r\cos\theta$, y by $r\sin\theta$, $dy\,dx$ by $r\,dr\,d\theta$, we have

$$I = \int_0^{\pi/2}\int_0^{2\cos\theta} \frac{r\cos\theta\, r\,dr\,d\theta}{r}$$

$$= \int_0^{\pi/2}\int_0^{2\cos\theta} r\cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2} \cos\theta\left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta = \int_0^{\pi/2} 2\cos^3\theta\,d\theta = \frac{4}{3}$$

Change of Order of Integration

Sometimes we encounter problem in solving a given double integral. Such integrals can easily be solved by changing the order of integration. Let the given integral be

$$I = \int_{x=a}^{x=b} \int_{y=f_1(x)}^{f_2(x)} f(x, y)\, dx\, dy$$

Here the region of integration is bounded by the lines x=a, x=b , curves $y=f_1(x)$ and $y= f_2(x)$.

First, sketch the region of integration on XY- plane and divide it into vertical strips.

Now for changing the order of integration, divide the region into horizontal strips and obtain the new limits of x and y by moving the horizontal strip from top to bottom in the region of integration. Since in the given integral the limits for x are constants while for y are variables so after changing the order of integration, the limits of y must become constants and the limits for x must be variable i.e. we obtain

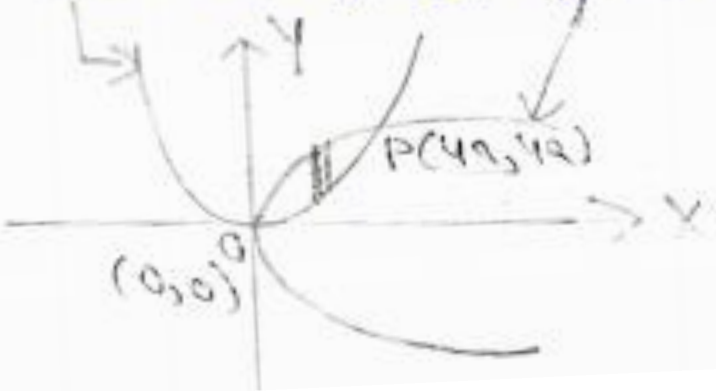
$$I = \int_{y=c}^{y=d} \int_{x=\phi_1(y)}^{\phi_2(y)} f(x, y)\, dy\, dx$$

Question: Change the order of integration of the integral $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} f(x,y)\,dx\,dy$

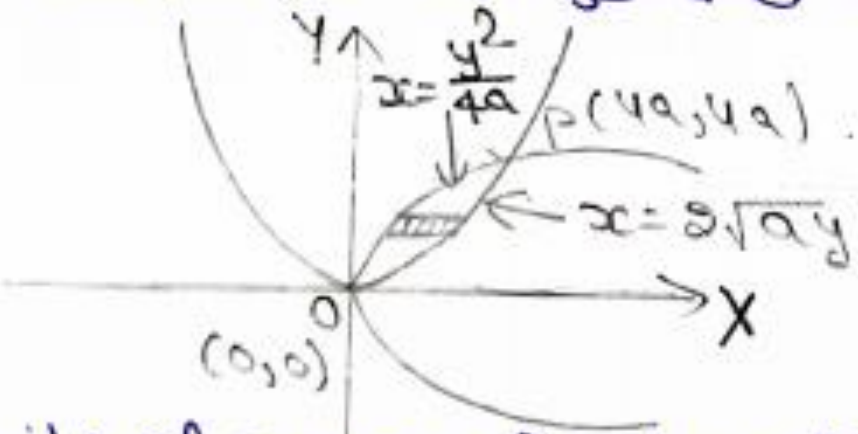
Solution: Let $I = \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} f(x,y)\,dx\,dy$

Here x varies from $x=0$ to $x=4a$ and y varies from $y=\frac{x^2}{4a}$ to $y=2\sqrt{ax}$ $\Rightarrow$

$x^2=4ay$ to $y^2=4ax$



For changing the order of integration, we divide the region of integration by means of horizontal strips



The limits of x are $\frac{y^2}{4a}$ to $2\sqrt{ay}$ and the corresponding limits of y are 0 to $4a$.

$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} f(x,y)\,dx\,dy = \int_{y=0}^{4a}\int_{x=y^2/4a}^{2\sqrt{ay}} f(x,y)\,dy\,dx$$

Question: Change the order of integration of the integral $\int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\,dx\,dy$

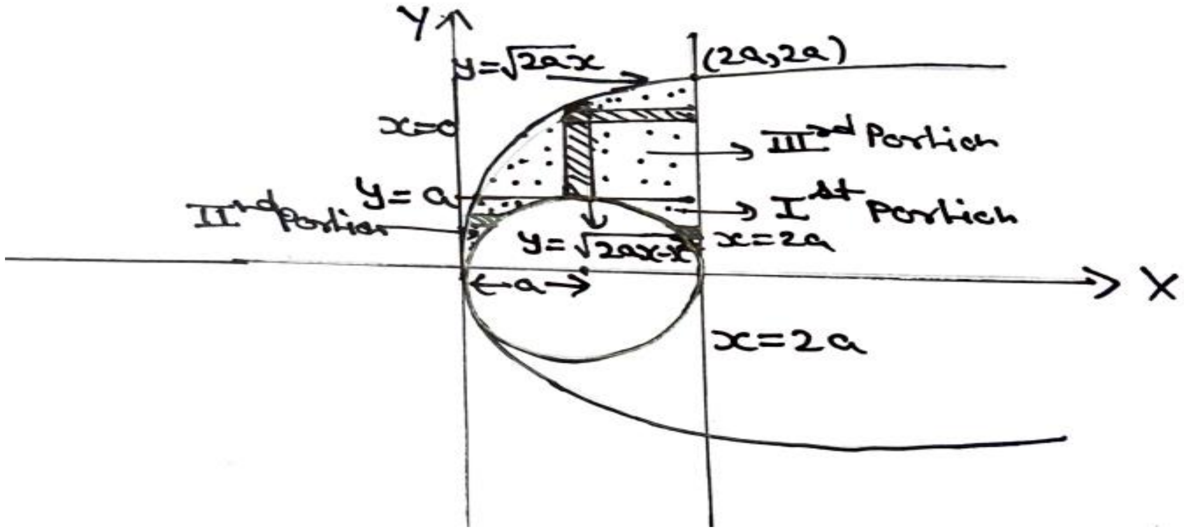
Solution: Given $I=\int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\,dx\,dy$

Here $x=0$ and $x=2a$; $y=\sqrt{2ax-x^2}$ and $y=\sqrt{2ax}$

or $y^2=2ax-x^2 \Rightarrow x^2+y^2-2ax$ & $y^2=2ax$

$\Rightarrow (x-a)^2+(y-0)^2=a^2$ & $y^2=2ax$

So, the region of integration is shown shaded in the following figure:



Now, taking horizontal strips in the shaded region, the area divided by three strips

(i) Horizontal strips starting from the circle and ending on line $x=2a$. So in this case the limits are,

$x=a+\sqrt{a^2-y^2}$ to $x=2a$ and $y=0$ to $y=a$

(ii) Horizontal strips starting from parabola and ending on the circle. So, in this case limits are, $x = \frac{y^2}{2a}$ to $x = a - \sqrt{a^2 - y^2}$ and $y = 0$ to $y = a$.

(iii) Horizontal strips starting from the parabola and ending on line $x = 2a$. So, in this case the limits are, $x = \frac{y^2}{2a}$ to $x = 2a$ and $y = a$ to $y = 2a$

So, the required integral is

$$I = \int_{y=0}^{a} \int_{x=y^2/2a}^{a-\sqrt{a^2-y^2}} f(x,y)\, dy\, dx$$

$$+ \int_{y=0}^{a} \int_{x=a+\sqrt{a^2-y^2}}^{2a} f(x,y)\, dy\, dx$$

$$+ \int_{y=a}^{2a} \int_{x=y^2/2a}^{2a} f(x,y)\, dy\, dx.$$

Area by Double Integration: Cartesian Coordinates

(a) Cartesian co-ordinates: The area A of the region bounded by the curves $y=f_1(x)$, $y= f_2(x)$ and the lines $x=a$, $x=b$ is given by

$$A = \int_{a}^{b} \int_{y=f_1(x)}^{f_2(x)} dy\, dx$$

The area A of the region bounded by the curves $x=f_1(y)$, $x= f_2(y)$ and the lines $y =c$, $y = d$ is given by

$$A = \int_{c}^{d} \int_{x=f_1(y)}^{f_2(y)} dy\, dx$$

(a) Polar Co-ordinates: The area A of the region bounded by the curves $r=f_1(\theta)$, $r= f_2(\theta)$ and lines $\theta = \alpha, \theta = \beta$ is given by

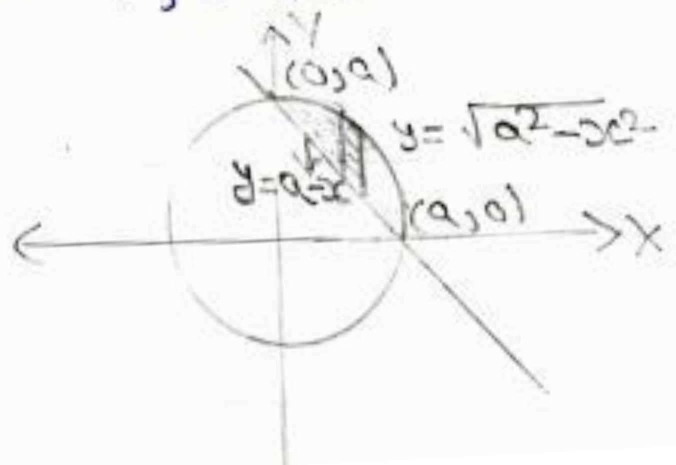
$$A = \int_{\alpha}^{\beta} \int_{x=f_1(\theta)}^{f_2(\theta)} r\ dr\, d\theta$$

Question: Find by double integration the area of the region enclosed by the curve $x^2+y^2=a^2$ and $x+y=a$ (in the first quadrant)

Solution: For points of intersection solving the given curves, we get

$$x^2+(a-x)^2=a^2 \Rightarrow 2x^2-2ax=0$$

$\Rightarrow x=0, a \Rightarrow$ points of intersection are $(0,a)$; $(a,0)$.



$$\therefore \quad A = \int_{x=0}^{a}\int_{y=a-x}^{\sqrt{a^2-x^2}} dx\, dy$$

$$= \int_0^a \left[\sqrt{a^2-x^2} - (a-x)\right] dx$$

$$= \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) - \left(ax - \frac{x^2}{2}\right)\right]_0^a$$

$$= \frac{a^2}{2}\left(\frac{\pi}{2}\right) - \left(a^2 - \frac{a^2}{2}\right) = \frac{a^2}{4}(\pi - 2)$$

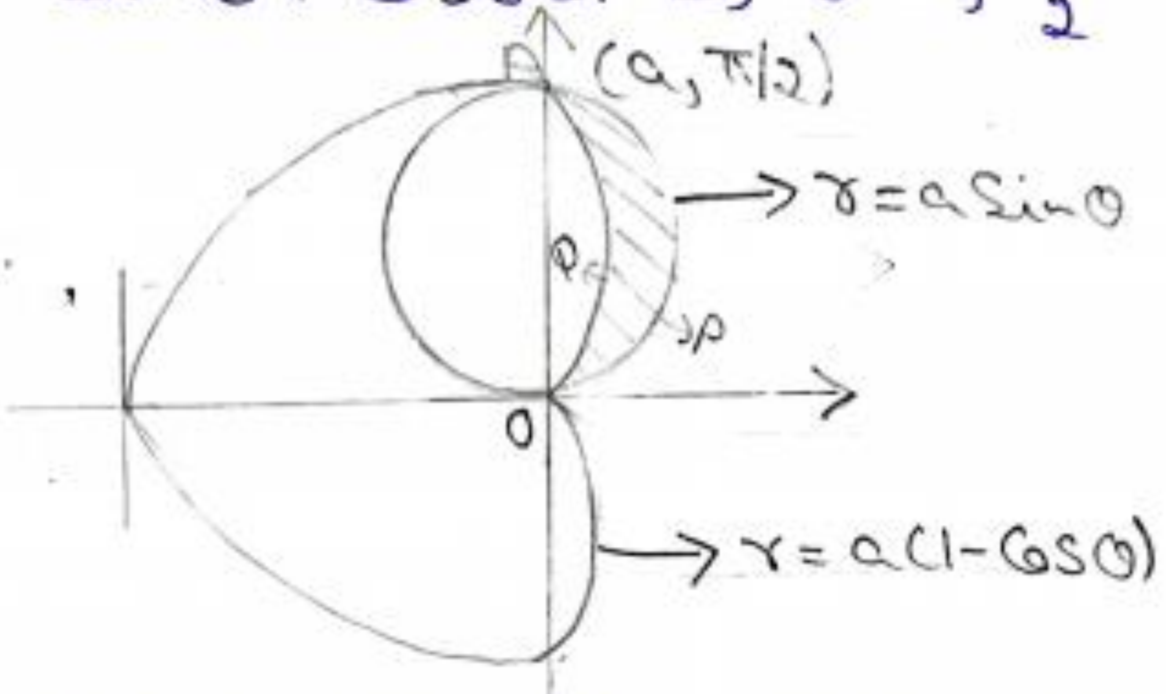
Question: Find by double integration the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1-\cos\theta)$.

Solution: The curve $r = a\sin\theta$ is a circle with centre at the point for which $r = \frac{a}{2}$, $\theta = \frac{\pi}{2}$ (i.e. $[0, \frac{a}{2}]$ in cartesian form) and radius $\frac{a}{2}$.

For the points of intersection of the two given curves, put the value $r = a\sin\theta$ in the equation of the cardioid $r = a(1-\cos\theta)$, we get

$a\sin\theta = a(1-\cos\theta)$

$\Rightarrow \quad \sin\theta + \cos\theta = 1 \Rightarrow \theta = 0, \frac{\pi}{2}$



when $\theta = 0$, $r = 0$; when $\theta = \frac{\pi}{2}$, $r = a$

The required area in the area of the region OPAQO. Taking a radial strip in this region, then the limits for r are from $a(1-\cos\theta)$ to $r = a\sin\theta$ and θ varies from 0 to $\frac{\pi}{2}$.

$\therefore$ Required area $= \iint_R dx\,dy$

$$= \int_{\theta=0}^{\pi/2} \int_{r=a(1-\cos\theta)}^{a\sin\theta} r\,d\theta\,dr$$

$$= \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{1}{2}a^2 \int_0^{\pi/2} \left[\sin^2\theta - (1-\cos\theta)^2\right] d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi/2} (2\cos\theta - 1 - \cos 2\theta)\,d\theta$$

$$= \frac{a^2}{2} \left[2\sin\theta - \theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{a^2}{2}\left[2 - \frac{\pi}{2}\right]$$

$$= a^2\left(1 - \frac{\pi}{4}\right)$$

Volume by Double Integration

Let the problem be to evaluate the volume inside the cylinder $\phi(x, y) = 0$ which is bounded by the surface z=f(x, y) and the XOY plane. Now divide the region R, bounded by $\phi(x, y) = 0$ in XOY plane, into small rectangles by drawing lines parallel to X and Y-axes.

Let the area of the rth rectangle be $dx_r\, dy_r$ then the volume of the prism with this rectangle as base and of height z_r is $z_r\, dx_r\, dy_r$.

Now the required volume is the sum of volumes of all such elementary prisms, i.e.

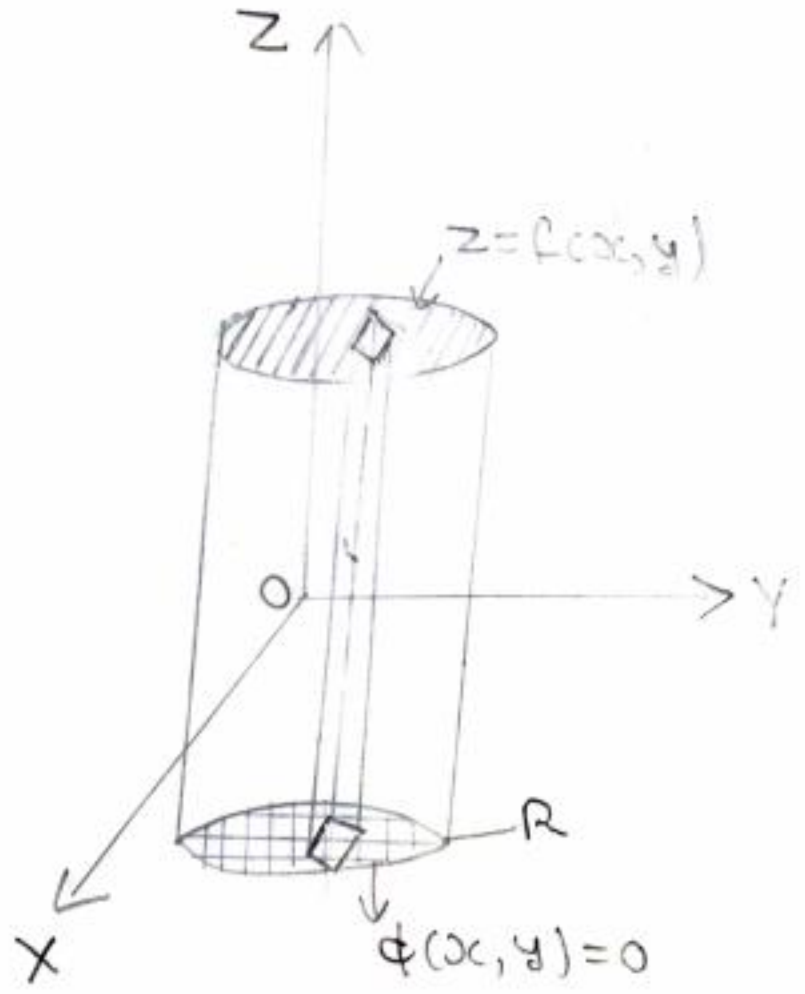
Required volume

$$\lim_{\substack{n\to \\ dx_r\, dy_r \to 0}} \sum_{r=1}^{n} z_r\, dx_r\, dy_r = \iint_R z\, dx\, dy$$

Or $V=\iint_R f(x,y)dx\, dy$

To obtain the formula for volume in polar coordinates put $x = r\cos\theta$, $y = r\sin\theta$ and replace dx dy by $r\, d\theta\, dr$ in above equation, we get

$$V = \iint_{\mathrm{R}} \mathrm{f}(\mathrm{r}\cos\theta, \mathrm{r}\sin\theta)\ \mathrm{r}\ \mathrm{d}\theta\mathrm{dr}$$



Question: Find the volume in the first octant bounded by the parabolic cylinders $z = 9 - x^2$, $x = 3 - y^2$.

Solution: Volume in the first octant

$$V = \iint_R z \, dx\,dy$$

Here $z = 9 - x^2$ and $x = 3 - y^2$. Now, in first octant x varies from 0 to 3 ($\because x = 3 - y^2 = 3$ at $y = 0$) and y varies from 0 to $\sqrt{3-x}$ ($\because x = 3 - y^2$ $\Rightarrow y^2 = 3 - x \Rightarrow y = \sqrt{3-x}$; in first octant).

$\therefore$ Required volume $V = \int_{x=0}^{3} \int_{y=0}^{\sqrt{3-x}} (9 - x^2)\, dx\,dy$

$$= \int_0^3 (9-x^2)\,[y]_0^{\sqrt{3-x}} = \int_0^3 (9-x^2)\sqrt{3-x}\,dx$$

Put $x = 3\sin^2\theta \Rightarrow dx = 6\sin\theta\cos\theta\,d\theta$

$$V = \int_0^{\pi/2} (9-9\sin^4\theta)\sqrt{3-3\sin^2\theta}\cdot 6\sin\theta\cos\theta\,d\theta$$

$$= 54\sqrt{3}\int_0^{\pi/2} [\sin\theta\cos^2\theta - \sin^5\theta\cos^2\theta]\,d\theta$$

$$= 54\sqrt{3}\left[\frac{\Gamma\frac{2}{2}\,\Gamma 3/2}{2\,\Gamma 5/2} - \frac{\Gamma 3\,\Gamma 3/2}{2\,\Gamma 9/2}\right]$$

$$= 54\sqrt{3}\left[\frac{1}{3} - \frac{8}{105}\right] = 54\sqrt{3}\times\frac{9}{35}$$

$$\Rightarrow V = \frac{486\sqrt{3}}{35}$$

Question: Find the volume common to the sphere $x^2+y^2+z^2=a^2$ and cylinder $x^2+y^2=ay$.

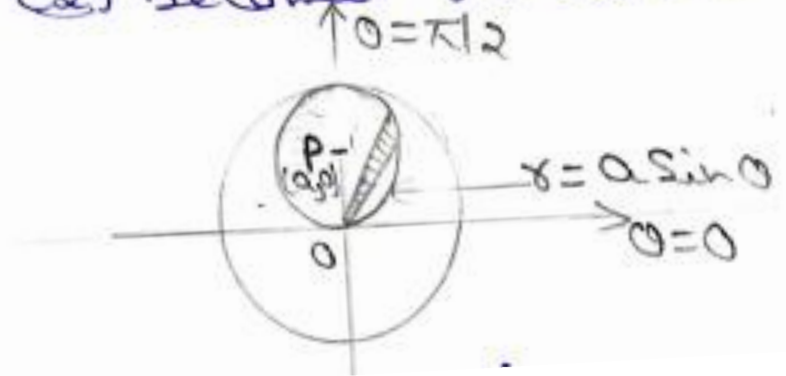
Solution: Given $x^2+y^2+z^2=a^2$ — (1) and $x^2+y^2=ay$ — (2)

Considering the section in positive quadrant of xy-plane and taking z to be positive (i.e. volume above the xy-plane)

Changing to polar coordinates, (1) becomes

$$r^2+z^2=a^2 \Rightarrow z^2=a^2-r^2 \Rightarrow z=\sqrt{a^2-r^2}$$

and (2) becomes $r^2=ar\sin\theta \Rightarrow r=a\sin\theta$



$\therefore$ Required volume $V = \iint_R z\, dx\, dy$

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sin\theta} \sqrt{a^2-r^2}\; r\, d\theta\, dr$$

$$= 4\int_0^{\pi/2}\left[\int_a^{a\cos\theta} t(-t)\, dt\right] d\theta \qquad \text{Put } a^2-r^2=t^2,\; r\,dr = -t\,dt$$

$$= -\frac{4}{3}\int_{\theta=0}^{\pi/2}\left[t^3\right]_a^{a\cos\theta} = -\frac{4}{3}\int_0^{\pi/2}\left[a^3\cos^3\theta - a^3\right] d\theta$$

$$= \frac{4}{3}a^3\left[\int_{\theta=0}^{\pi/2} d\theta - \int_0^{\pi/2}\cos^3\theta\, d\theta\right]$$

$$= \frac{4}{3}a^3\left[(\theta)_0^{\pi/2} - \frac{\Gamma(1/2)\Gamma 2}{2\,\Gamma 5/2}\right] = \frac{4}{3}a^3\left(\frac{\pi}{2} - \frac{2}{3}\right)$$

CALCULATION OF MASS

(a) For a plane lamina, let the surface density at the point P(x,y) be $\rho = f(x, y)$. Then elementary mass at P $=\rho\ \delta x\ \delta y$

Therefore total mass of lamina $=\iint \rho\ dx\ dy$

In polar co-ordinates, taking $\rho = \varphi\ (r, \theta)$ at the point P(r,θ)

Total mass of lamina $= \iint \rho\ r\ d\theta\ dr$

(b) For a solid, let the density at the point P(x, y, z) be $\rho = f(x, y, z)$

Then total mass of the solid $= \iiint \rho\ dx\ dy\ dz$ with suitable limits of integration.

Question: Find the mass of the tetrahedron bounded by the co-ordinates planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, the variable density μxyz.

Solution : Elementary mass at P= $\mu xyz\ \delta x\ \delta y\ \delta z$

Therefore the whole mass= $\iiint \mu xyz\ dx\ dy\ dz$

The limits for z are from 0 to $z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$. The limits for y are 0 to y = b(1-x/a) and the limits for x are from 0 to a.

Therefore required mass $=\int_0^a \int_0^{\mathrm{b(1-x/a)}} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} \mu xyz\ dz\ dy\ dx$

$$=\mu \int_0^a \int_0^{\mathrm{b(1-x/a)}} x\ y.\frac{c^2}{2}\left(1-\frac{x}{a}-\frac{y}{b}\right) dy\ dx$$

$$=\frac{\mu c^2}{2}\int_0^a x\left[\frac{b^2}{2}\left(1-\frac{x}{a}\right)^4-\frac{2b^2}{3}\left(1-\frac{x}{a}\right)^4+\frac{b^2}{4}\left(1-\frac{x}{a}\right)^4\right] dx$$

$$=\frac{\mu\, b^2 c^2}{24}\int_0^a x\left[\left(1-\frac{x}{a}\right)^4\right] dx$$

$$=\frac{\mu\, a^2\, b^2 c^2}{720}$$

CENTRE OF GRAVITY (C.G.)

(a) To find the C.G.$(\bar{x}, \bar{y})$ of a plane lamina, take the element of mass as $\rho\ \delta x\ \delta y$ at the point P (x, y).

Then $\bar{x} = \dfrac{\iint x\ \rho\ dx\ dy}{\iint \rho\ dx\ dy}$, $\bar{y} = \dfrac{\iint y\ \rho\ dx\ dy}{\iint \rho\ dx\ dy}$,

While using polar co-ordinates, take the elementary mass as $\rho\ r\delta\theta\ \delta r$ at the point P(r,θ) so that x= r cosθ , y= r sinθ , therefore

$$\bar{x} = \frac{\iint r\ cos\theta\ \rho\ r\ d\theta\ dr}{\iint \rho\ rd\theta\ dr},\ \ \bar{y} = \frac{\iint r\ sin\theta\ \rho\ r\ d\theta\ dr}{\iint \rho\ rd\theta\ dr}$$

(b) To find the C.G .$(\bar{x}, \bar{y}, \bar{z})$ of a solid, take the element of mass $\rho\ \delta x\ \delta y\ \ \delta z$ enclosing the point P (x, y, z).

Then $\bar{x} = \dfrac{\iint\int\ \ x\ \rho\ dx\ dy\ dz}{\iint\int\ \ \rho\ dx\ dy\ \mathrm{dz}}$, $\bar{y} = \dfrac{\iint\int\ \ y\ \rho\ dx\ dydz}{\iint\int\ \ \rho\ dx\ dy\ \mathrm{dz}}$,$\bar{z} = \dfrac{\iint\int\ \ z\rho\ dx\ dy\ dz}{\iint\int\ \ \rho\ dx\ dy\ \mathrm{dz}}$

Question : Find by double integration , the centre of gravity of the area of the cardioid $r = a\,(1+\cos\theta)$.

Solution: The given cardioid is symmetrical about the initial line hence its C.G. lies on OX, i.e. y

$$\bar{x} = \frac{\iint r\,cos\theta\ \rho\ r\ d\theta\ dr}{\int \rho\ r d\theta\ dr}$$

$$= \frac{\int_{-\pi}^{\pi}\int_{0}^{a\,(1+\cos\theta)} cos\theta\ r^2\ dr\ d\theta}{\int_{-\pi}^{\pi}\int_{0}^{a\,(1+\cos\theta)} r\ dr\ d\theta}$$

$$= \frac{2a\ \int_{-\pi}^{\pi} cos\theta\,(1+cos\theta)^3\ d\theta}{3\ \int_{-\pi}^{\pi}\,(1+cos\theta)^2\ d\theta}$$

$$= \frac{2a}{3}\frac{2\int_0^{\pi}(3cos^2\theta + cos^4\theta)d\theta}{2\int_0^{\pi}(1 + cos^2\theta)d\theta}$$

$$= \frac{2a}{3}\frac{2\int_0^{\pi/2}(3cos^2\theta + cos^4\theta)d\theta}{2\int_0^{\pi/2}(1 + cos^2\theta)d\theta}$$

$$= \frac{2a}{3}\frac{3.{}^1/_2 \cdot {}^{\pi}/_2 + {}^3/_4 \cdot {}^1/_2 \cdot {}^{\pi}/_2}{{}^{\pi}/_2 + {}^1/_2 \cdot {}^{\pi}/_2}$$

$$= \frac{5a}{6}$$

$$hence\ the\ C.G. of\ the\ cardioid\ is\ at\ (\frac{5a}{6}, 0)$$

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Thank you!