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## **7EE5-11: WIND AND SOLAR ENERGY SYSTEM**

**1.Introduction:** Objective, scope and outcome of the course.

**Objective:**

- **To give sufficient knowledge about the promising new and renewable sources of energy**
- **To equip students in working with projects and to take up research work in connected areas.**

**Scope:**

- **India's population is increasing rapidly and is expected to be the world's most populous country by 2020. India is currently facing huge energy demand. Although its energy production has expanded over the years, its population is also increasing at an alarming rate. India has the 5<sup>th</sup> largest power generation portfolio. As the country is facing a huge energy crisis, there is a high need to tap the potential renewable energy resources.**

**Outcome :**

**CO 1: define basic properties of different renewable sources of energy and technologies for their utilisation.**

**CO 2: describe main elements of technical systems designed for utilisation of renewable sources of energy and explain the correlation between different operational parameters**

**CO 3: select engineering approach to problem solving when implementing the projects on renewable sources**

### **2. Physics of Wind Power**

The following historical overview divides the utilisation of the natural resource wind into the generation of mechanical power and the production of electricity (The historical development of wind turbine technology is documented in many publications, for instance see Ancona, 1989; Gipe, 1995; Heymann, 1995; Hill, 1994; Johnson, 1985; Kealey, 1987; Koeppel, 1982; Putnam, 1948; Righter, 1996; Shepherd, 1990, 1994).

**Mechanical power generation** The earliest windmills recorded were vertical axis mills. These windmills can be described as simple drag devices. They have been used in the Afghan highlands to grind grain since the seventh century BC. The first details about horizontal axis windmills are found in historical documents from Persia, Tibet and China at about 1000 AD. This windmill type has a horizontal shaft and blades (or sails) revolving in the vertical plane.

From Persia and the Middle East, the horizontal axis windmill spread across the Mediterranean countries and Central Europe. The first horizontal axis windmill appeared in England around 1150, in France in 1180, in Flanders in 1190, in Germany in 1222 and in Denmark in 1259.

This fast development was most likely influenced by the Crusaders, taking the knowledge about windmills from Persia to many places in Europe. In Europe, windmill performance was constantly improved between the twelfth and nineteenth centuries. By the end of the nineteenth century, the typical European windmill used a rotor of 25 meters in diameter, and the stocks reached up to 30 meters. Windmills were used not only for grinding grain but also for pumping water to drain lakes and marshes. By 1800 about 20 000 modern European windmills were in operation in France alone, and in the Netherlands 90 % of the power used in the industry was based on wind energy. Industrialisation then led to a gradual decline in windmills, but in 1904 wind energy still provided 11 % of the Dutch industrial energy and Germany had more than 18 000 installed units.

When the European windmills slowly started to disappear, windmills were introduced by settlers in North America. Small windmills for pumping water to livestock became very popular. These windmills, also known as American Windmills, operated fully self-regulated, which means they could be left unattended. The self-regulating mechanism pointed the rotor windward during high-speed winds. The European style windmills usually had to be turned out of the wind or the sailing blades had to be rolled up during extreme wind speeds, to avoid damage to the windmill. The popularity of windmills in the USA reached its peak between 1920 and 1930, with about 600 000 units installed. Various types of American Windmills are still used for agricultural purposes all over the world.


Table 1. Historical wind turbines

Turbine and country	Diameter (m)	Swept Area (m <sup>2</sup> )	Power (kW)	Specific Power (kW/m <sup>2</sup> )	Number of blades	Tower height (m)	Date in service
Poul LaCour, Denmark	23	408	18	0.04	4	—	1891
Smith-Putnam, USA	53	2231	1250	0.56	2	34	1941
F. L. Smidth, Denmark	17	237	50	0.21	3	24	1941
F. L. Smidth, Denmark	24	456	70	0.15	3	24	1942
Gedser, Denmark	24	452	200	0.44	3	25	1957
Hütter, Germany	34	908	100	0.11	2	22	1958

Source: Gipe, 1995, page 18 (Reproduced by permission of John Wiley & Sons, Ltd.).

Indian and Global statistics.

The following section will provide a brief overview of the wind energy status around the world at the end of the twentieth century. Furthermore, it will present major wind energy support

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schemes. The overview is divided into grid-connected wind power generation and stand-alone systems.

Table 2. Operational wind power capacity worldwide.

Region	Installed capacity (MW) by end of year						
	1995	1997	1999	2000	2001	2002	2003
Europe	2518	4766	9307	12972	17500	21319	28706
North America	1676	1611	2619	2695	4245	4708	6677
South and Central America	11	38	87	103	135	137	139
Asia and Pacific	626	1149	1403	1795	2330	2606	3034
Middle East and Africa	13	24	39	141	147	149	150

Sources: *Wind Power Monthly*; European Wind Energy Association.


Europe.

Between the end of 1995 and the end of 2003, around 76 % of all new grid-connected wind turbines worldwide were installed in Europe.

Table3. Operational wind power capacity in Europe

Country	Installed capacity (MW) at end of year	
	1995	2003
Germany	1136	14609
Denmark	619	3110
Spain	145	6202
Netherlands	236	912
UK	200	649
Sweden	67	399
Italy	25	904
Greece	28	375
Ireland	7	186
Portugal	13	299
Austria	3	415
Finland	7	51
France	7	239
Norway	4	101
Luxembourg	0	22
Belgium	0	68
Turkey	0	19
Czech Republic	7	10
Poland	1	57
Russia	5	7
Ukraine	1	57
Switzerland	0	5
Latvia	0	24
Hungary	0	3
Estonia	0	3
Cyprus	0	2
Slovakia	0	3
Romania	0	1
Total	2518	28706

Sources: *Wind Power Monthly*; European Wind Energy Association.

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North America.

After the wind power boom in California during the mid-1980s, development slowed down significantly in North America. In the middle of the 1990s the dismantling of old wind farms sometimes exceeded the installations of new wind turbines, which led to a reduction in installed capacity.

Table 4. Operational wind power capacity in North America

Country	Installed capacity (MW) by end of year	
	1995	2003
USA	1655	6350
Canada	21	327
Total	1676	6677

Sources: American Wind Power Association;  
*Wind Power Monthly*.

Table 5. Operational wind power capacity in the USA at end of 2003

State	Installed capacity (MW)
California	2042
Texas	1293
Minnesota	562
Iowa	471
Wyoming	284
Oregon	259
Washington	243
Colorado	223
New Mexico	206
Pennsylvania	129
Oklahoma	176
Kansas	113
North Dakota	66
West Virginia	66
Wisconsin	53
Illinois	50
New York	49
South Dakota	44
Hawaii	18
Nebraska	14
Vermont	6
Ohio	2
Tennessee	2
Alaska	1
Massachusetts	1
Michigan	1
Total	6350

Source: American Wind Energy Association.

### South and Central America

Despite large wind energy resources in many regions of South and Central America, the development of wind energy has been very slow (Table 2.9) because of the lack of a sufficient wind energy policy as well as low electricity prices. Many wind projects in South America have been financially supported by international aid programmes. Argentina, however, introduced a new policy at the end of 1998 that offers financial

support to wind energy generation, but with little success. In Brazil, some regional governments and utilities have started to offer higher feed-in tariffs for wind power. The typical size of existing wind turbines is around 300 kW. Larger wind turbines are difficult to install because of infrastructural limitations for larger equipment (e.g. cranes). Offshore wind projects are not planned, but further small to medium-size (< 100 MW) projects are under development onshore, particularly in Brazil.

Table 6. Operational wind power capacity in South and Central America at end of 2003


Country	Installed capacity (MW)
Costa Rica	71
Argentina	26
Brazil	22
Caribbean	13
Mexico	5
Chile	2
Total	139

*Source: Wind Power Monthly.*

### Asia and Pacific

India achieved an impressive growth in wind turbine installation in the middle of the 1990s, the ‘Indian Boom’. In 1992/93, the Indian government started to offer special incentives for renewable energy investments (e.g. a minimum purchase rate was guaranteed, and a 100 % tax depreciation was allowed in the first year of the project). Furthermore, a ‘power banking’ system was introduced that allows electricity producers to ‘bank’ their power with the utility and avoid being cut off during times of load shedding. Power can be banked for up to one year. In addition, some Indian States have

Table 7. Operational wind power capacity in Asia and Pacific at end of 2003

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Country	Installed capacity (MW)
India	1900
China	468
Japan	401
Australia	196
New Zealand	50
South Korea	8
Taiwan	8
Sri Lanka	3
Total	3034

*Source: Wind Power Monthly.*

introduced further incentives (e.g. investment subsidies). This policy led to a fast development of new installations between 1993 and 1997. Then the development slowed down as a result of uncertainties regarding the future of the incentives but picked up again in the new millennium after a more stable policy towards wind power was provided.


The wind energy development in China is predominately driven by international aid programmes, despite some government programmes to promote wind energy (e.g. the ‘Ride-the-Wind’ programme of the State Planning Commission). In Japan, the development has been dominated by demonstration projects testing different wind turbine technologies. At the end of the 1990s the first commercial wind energy projects started operation on the islands of Hokkaido as well as Okinawa. Interest in wind power is constantly growing in Japan. Also, at the end of the 1990s, the first wind energy projects materialised in New Zealand and Australia. The main driver for wind energy development in Australia is a green certification scheme.

In China and India, the typical wind turbine size is around 300–600 kW; however, some megawatt turbines have also been installed. In Australia, Japan and New Zealand, the 1–1.5 MW range is predominantly used (for installed capacity in countries in Asia and the Pacific, see Table)

#### Middle East and Africa

Wind energy development in Africa is very slow (see also Table). Most projects require financial support from international aid organisations, as there is only limited regional support. Projects are planned in Egypt, where the government agency for the New and Renewable Energy Authority (NREA) would like to build a 600 MW project near the city of Zafarana. Further projects are planned in Morocco as well as in Jordan (25 MW). The typical wind turbine size used in this region is around 300 kW, but there are plans to use 500–600 kW turbines in future projects.

Table 8. Operational wind power capacity in Middle East and Africa at end of 2003

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Country or Region	Installed capacity (MW)
Egypt	69
Morocco	54
Iran	11
Israel	8
Jordan	2
Rest of Africa	3
Total	150

Source: *Wind Power Monthly*.

### Overview of stand-alone generation

Stand-alone systems are generally used to power remote houses or remote technical applications (e.g. for telecommunication systems). The wind turbines used for these purposes can vary from between a few watts and 50 kW. For village or rural electrification systems of up to 300 kW, wind turbines are used in combination with a diesel generator and sometimes a battery system.

### Wind physics

The power of an air mass that flows at speed  $V$  through an area  $A$  can be calculated as follows

$$\text{power in wind} = \frac{1}{2} \rho A V^3 (\text{watts}),$$

where

$$\rho = \text{air density (kg m}^{-3}\text{);}$$

$$V = \text{wind speed (m s}^{-1}\text{).}$$

The power in the wind is proportional to the air density  $\rho$ , the intercepting area  $A$  (e.g. the area of the wind turbine rotor) and the velocity  $V$  to the third power. The air density is a function of air pressure and air temperature, which both are functions of the height above sea level:

$$\rho(z) = \frac{P_0}{RT} \exp\left(\frac{-gz}{RT}\right),$$

where

$$\rho(z) = \text{air density as a function of altitude (kg m}^{-3}\text{);}$$

$$P_0 = \text{standard sea level atmospheric density (1.225 kg m}^{-3}\text{);}$$


$$R = \text{specific gas constant for air (287.05 J kg}^{-1}\text{K}^{-1}\text{);}$$

$$g = \text{gravity constant (9.81 m s}^{-2}\text{);}$$

$$T = \text{temperature (K);}$$

$$z = \text{altitude above sea level (m).}$$



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The power in the wind is the total available energy per unit of time. The power in the wind is converted into the mechanical–rotational energy of the wind turbine rotor, which results in a reduced speed in the air mass. The power in the wind cannot be extracted completely by a wind turbine, as the air mass would be stopped completely in the intercepting rotor area. This would cause a ‘congestion’ of the cross-sectional area for the following air masses. The theoretical optimum for utilising the power in the wind by reducing its velocity was first discovered by Betz, in 1926. According to Betz, the theoretical maximum power that can be extracted from the wind is

$$P_{\text{Betz}} = \frac{1}{2} \rho A V^3 C_{\text{P Betz}} = \frac{1}{2} \rho A V^3 \times 0.59.$$

Hence, even if power extraction without any losses were possible, only 59 % of the wind power could be utilised by a wind turbine (Gasch and Twele, 2002)

**Betz limit.**

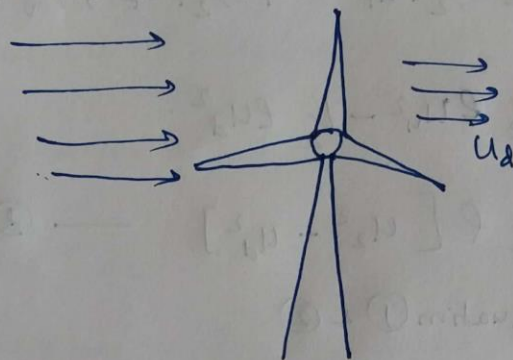


①

Betz Law , Betz Coefficient, Betz limit

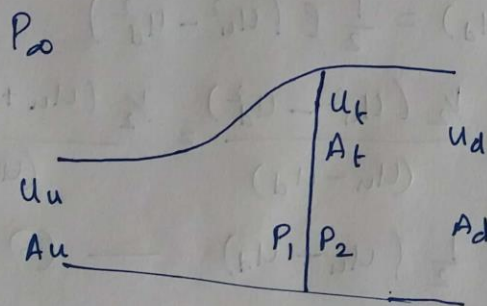
$$\frac{16}{27} = 0.593 = 59.3\% \quad \text{--- (A)}$$

Betz Coefficient      Betz limit .



$$P_{\infty} + \frac{1}{2} \rho u_u^2 = \frac{1}{2} \rho u_d^2 + P_{\infty} - h_t$$

Kinetic energy  
Loss hitting by wind turbine.



$P_{\infty}$  - Atmospheric pressure

$u_u$  - Upper stream Velocity .

$A_u$  - upper stream Area.

$u_t$  - at turbine stream Velocity .

$A_t$  - at turbine stream Area .

$P_1$  - upper stream pressure .

$P_2$  - down stream pressure .

$u_d$  - down stream velocity .

$A_d$  - down stream Area .

$\rho$  - density .

$m$  - mass .

$\Delta V$  = change in velocity

$$u_u A_u = u_t A_t = u_d A_d$$

Due to Continuity principle .

$$F = \dot{m} \times \Delta V$$

$$(P_1 - P_2) A_t = \rho A_u u_u \times (u_u - u_d)$$

$$(P_1 - P_2) A_t = \rho A_t u_t \times (u_u - u_d)$$

$$(P_1 - P_2) = \rho u_t (u_u - u_d) \quad \text{--- (1)}$$



~~Bernoulli's equation.~~ Bernoulli's equation.

$$P_{\infty} + \frac{1}{2} \rho u_n^2 = P_1 + \frac{1}{2} \rho u_t^2$$

$$P_2 + \frac{1}{2} \rho u_t^2 = P_{\infty} + \frac{1}{2} \rho u_d^2$$

Adding both equation.

$$P_{\infty} + \frac{1}{2} \rho u_n^2 + P_2 + \frac{1}{2} \rho u_t^2 = P_1 + \frac{1}{2} \rho u_t^2 + P_{\infty} + \frac{1}{2} \rho u_d^2$$

$$(P_1 - P_2) = \frac{1}{2} \rho u_n^2 - \frac{1}{2} \rho u_d^2$$

$$(P_1 - P_2) = \frac{1}{2} \rho [u_n^2 - u_d^2] \quad \text{--- (2)}$$

By equating equation (1) & (2)

$$\rho u_t (u_n - u_d) = \frac{1}{2} \rho (u_n^2 - u_d^2)$$

$$u_t = \frac{\frac{1}{2} (u_n^2 - u_d^2)}{(u_n - u_d)} = \frac{\frac{1}{2} (u_n + u_d)(u_n - u_d)}{(u_n - u_d)}$$

$$u_t = \frac{1}{2} (u_n + u_d) \quad \text{--- (3)}$$

$$\eta = \frac{P_{out}}{P_{in}}$$

$$P_{in} = \frac{1}{2} \dot{m} \times v^2$$

$$\therefore \dot{m} = \rho A_t u_t$$

$$= \frac{1}{2} \rho A_t u_t \times u_n^2$$

Let us assume  $u_t = u_n$ .

$$P_{in} = \frac{1}{2} \times \rho \times A_t \times u_n^3$$





②

$$P_{\text{output}} = F \times V$$

$$(P_1 - P_2) A_t \times u_t$$

$$\therefore (P_1 - P_2) = \rho u_t (u_u - u_d) \quad \text{--- (1)}$$

Substitute equation (1)

$$P_{\text{output}} = \rho \times u_t (u_u - u_d) A_t \times u_t$$

$$\therefore P_{\text{output}} = \rho \times u_t^2 (u_u - u_d) A_t \times u_t$$

$$u_t = \frac{1}{2} (u_u + u_d) \quad \text{we know that.}$$

$$P_{\text{output}} = \rho \left[ \frac{1}{2} (u_u + u_d) \right]^2 [u_u - u_d] A_t$$

$$P_{\text{output}} = \frac{1}{4} \rho A_t [u_u^2 - u_d^2] [u_u + u_d]$$

$$= \frac{1}{4} \rho A_t u_u^2 \left[ \frac{u_u^2}{u_u^2} - \frac{u_d^2}{u_u^2} \right] u_u \left[ \frac{u_u}{u_u} + \frac{u_d}{u_u} \right]$$

$$P_{\text{out}} = \frac{1}{4} \rho A_t u_u^3 \left[ 1 - \frac{u_d^2}{u_u^2} \right] \left[ 1 + \frac{u_d}{u_u} \right]$$

$$\text{Now } \eta = \frac{P_{\text{out}}}{P_{\text{in}}}$$

$$= \frac{\frac{1}{4} \rho A_t u_u^3 \left[ 1 - \frac{u_d^2}{u_u^2} \right] \left[ 1 + \frac{u_d}{u_u} \right]}{\frac{1}{2} \rho A_t u_u^3}$$

$$= \frac{1}{2} \left[ 1 - \frac{u_d^2}{u_u^2} \right] \left[ 1 + \frac{u_d}{u_u} \right]$$

$$= \frac{1}{2} \left[ 1 - \frac{u_d^2}{u_u^2} \right] \left[ 1 + \frac{u_d}{u_u} \right]$$

$$= \frac{1}{2} [1 - b^2] [1 + b]$$

$$\therefore \frac{u_d}{u_u} = b$$



$$\eta = \frac{1}{2} [1 + b - b^2 - b^3]$$

Polynomial equation  
of order 3.

for maximum efficiency.

$$\frac{d\eta}{db} = 0$$

$$= \frac{1}{2} [1 - 2b - 3b^2] = 0$$

↳ quadratic equation

$$\frac{1}{2} - b - \frac{3}{2} b^2 = 0$$

Then possible roots are.

$$b = -1, \frac{1}{3}$$

$$\eta = \frac{1}{2} \left[ 1 + \frac{1}{3} - \left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^3 \right]$$

$$\eta = \frac{16}{27} = 0.593 \rightarrow \text{Betz Coefficient.}$$

$$\eta = 59.3\% \Rightarrow \text{Betz limit.}$$



## Tip speed ratio.

15-7-20.

$$TSR = \lambda = \frac{\text{rotor Speed (rotor tip Speed)}}{\text{wind Speed.}}$$

$$\lambda = \frac{V}{V} = \frac{\omega r}{V} = \frac{2\pi f \cdot r}{V}$$

$$V = \text{Rotor tip speed [m/sec]}$$

$$V = \text{wind Speed [m/sec]}$$

$$\omega = \text{Angular velocity [rad/sec]}$$

$$r = \text{Rotor Radius [m]}$$

$$f = \text{rotational frequency [Hz], [Sec}^{-1}\text{]}$$

1. No. of blades , 2. Wind turbine type.

Desirable tip ratio 6-8.

Demerits of high TSR.

1. Leading Edge Erosion.
2. Noise (vibration)
3. Rotor efficiency reduction.



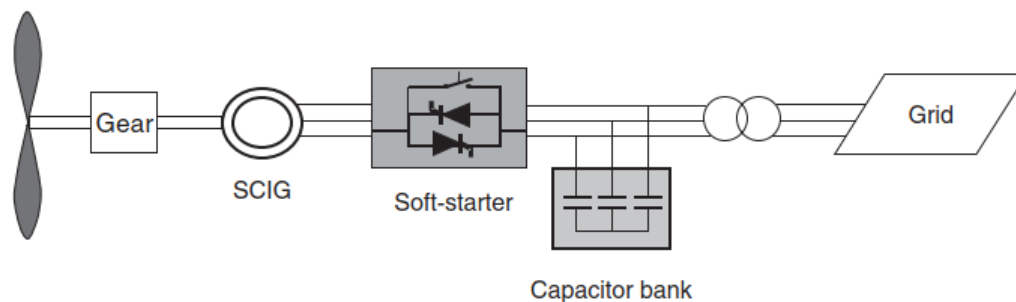
## stall and pitch control

### Type A: fixed speed

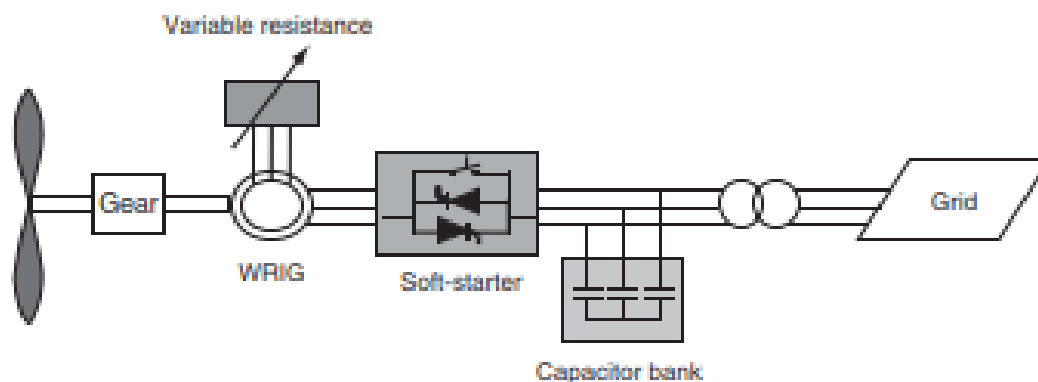
This configuration denotes the fixed-speed wind turbine with an asynchronous squirrel cage induction generator (SCIG) directly connected to the grid via a transformer (see Figure 4.1). Since the SCIG always draws reactive power from the grid, this configuration uses a capacitor bank for reactive power compensation. A smoother grid connection is achieved by using a soft-starter. Regardless of the power control principle in a fixed-speed wind turbine, the wind fluctuations are converted into mechanical fluctuations and consequently into electrical power fluctuations. In the case of a weak grid, these can yield voltage fluctuations at the point of connection. Because of these voltage fluctuations, the fixed-speed wind turbine draws varying amounts of reactive power from the utility grid (unless there is a capacitor bank), which increases both the voltage fluctuations and the line losses. Thus the main drawbacks of this concept are that it does not support any speed control, it requires a stiff grid and its mechanical construction must be able to tolerate high mechanical stress.

All three versions (Type A0, Type A1 and Type A2) of the fixed-speed wind turbine Type A are used in the wind turbine industry, and they can be characterised as follows.

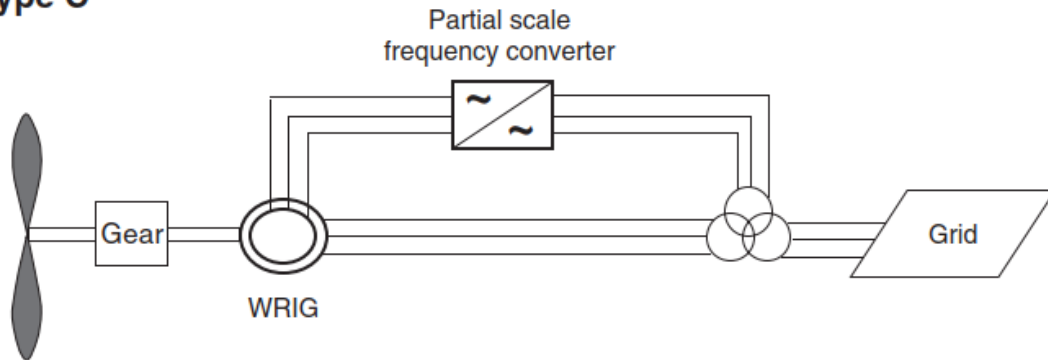
### Type A



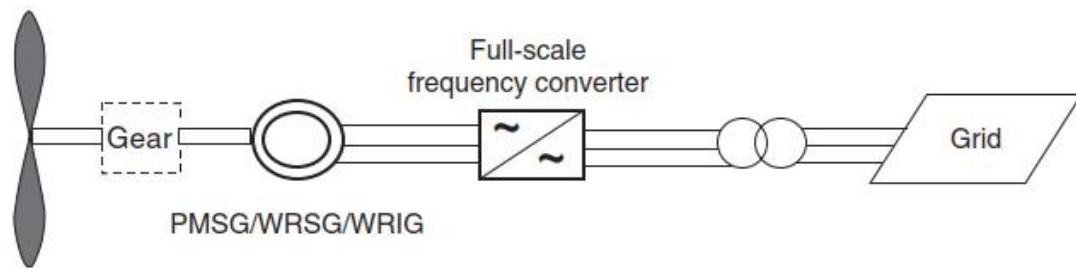
### Type B



### Type C



### Type D




### Wind turbine concepts

Speed control		Power control		
		Stall	Pitch	Active stall
Variable speed	Type A	Type A0	Type A1	Type A2
	Type B	Type B0	Type B1	Type B2
	Type C	Type C0	Type C1	Type C2
Fixed speed	Type D	Type D0	Type D1	Type D2

*Note:* The grey zones indicate combinations that are not in use in the wind turbine industry today.



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#### Type A0: stall control

This is the conventional concept applied by many Danish wind turbine manufacturers during the 1980s and 1990s (i.e. an upwind stall-regulated three-bladed wind turbine concept). It has been very popular because of its relatively low price, its simplicity and its robustness. Stall-controlled wind turbines cannot carry out assisted start-ups, which implies that the power of the turbine cannot be controlled during the connection sequence.

#### Type A1: pitch control

These have also been present on the market. The main advantages of a Type A1 turbine are that it facilitates power controllability, controlled start-up and emergency stopping. Its major drawback is that, at high wind speeds, even small variations in wind speed result in large variations in output power. The pitch mechanism is not fast enough to avoid such power fluctuations. By pitching the blade, slow variations in the wind can be compensated, but this is not possible in the case of gusts.

#### Type A2: active stall control

These have recently become popular. This configuration basically maintains all the power quality characteristics of the stall-regulated system. The improvements lie in a better utilisation of the overall system, as a result the use of active stall control. The flexible coupling of the blades to the hub also facilitates emergency stopping and start-ups. One drawback is the higher price arising from the pitching mechanism and its controller. As illustrated in Figure and Table, the variable speed concept is used by all three configurations, Type B, Type C and Type D. Owing to power limitation considerations, the variable speed concept is used in practice today only together with a fast-pitch mechanism. Variable speed stall or variable speed active stall-controlled wind turbines are not included here as potentially they lack the capability for a fast reduction of power. If the wind turbine is running at maximum speed and there is a strong gust, the aerodynamic torque can get critically high and may result in a runaway situation. Therefore, as illustrated in Table 4.1, Type B0, Type B2, Type C0, Type C2, Type D0 and Type D2 are not used in today's wind turbine industry.

## Wind speed statistics probability distributions

The behavior of wind velocity at a given site can be specified as a probability distribution function,  $f(V)$ . The quantity  $f(V)dV$  represents the fraction of the wind speeds that lie within a range,  $dV$ , about the given velocity,  $V$ . These notes discuss the basics ideas of probability distribution functions with specific application to wind velocity and energy distributions.<sup>1</sup>

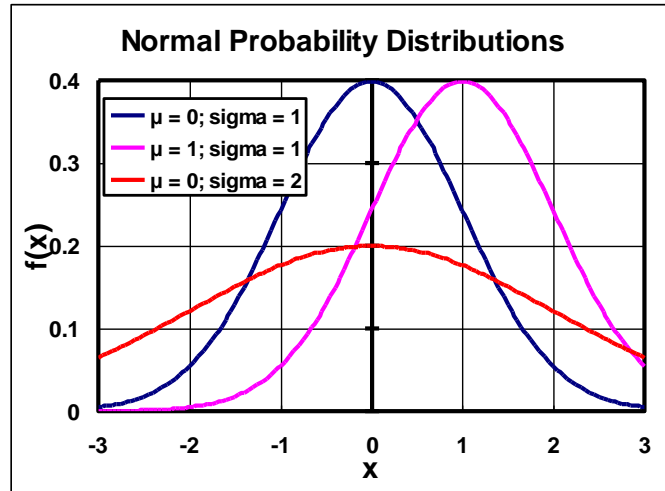
### Probability distribution functions

A probability distribution function (pdf) for a random variable  $x$  is written as  $f(x)$ . The best known pdf is the normal distribution.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} \quad -\infty < x < \infty \quad [1]$$

This distribution has two parameters  $\mu$  and  $\sigma$ . Sketches of this distribution for different values of these parameters are shown in the figure to the right. The distribution is seen to be symmetric about the value of  $\mu$  and the width of the distribution increases as  $\sigma$  increases.

All pdf's are interpreted as follows: the probability that the random variable,  $x$ , lies in a differential range,  $dx$ , about a value  $x^*$  is  $f(x^*)dx$ . Specific statements about the probability that the random



variable,  $x$ , lies in a particular range  $a \leq x \leq b$ , which is denoted by the expression  $P(a \leq x \leq b)$  is obtained by integrating  $f(x)dx$  between the limits of  $a$  and  $b$ .

$$P(a \leq x \leq b) = \int_a^b f(x)dx \quad [2]$$

Probabilities,  $P$ , range from zero (no chance of occurring) to 1 (certain to occur). Because a random variable is certain to lie between its minimum and maximum values, the probability that a pdf lies between its maximum and minimum values,  $P(x_{\min} \leq x \leq x_{\max})$ , must be 1. We can write this as the following equation. The fact that this integral of the pdf must be one is sometimes called the normalization condition or the normalization integral.

$$P(x_{\min} \leq x \leq x_{\max}) = \int_{x_{\min}}^{x_{\max}} f(x)dx = 1 \quad [3]$$

Using the geometric definition of the integral as the area under a curve, we see that the area under the pdf between  $x_{\min}$  and  $x_{\max}$  must be 1. In the normal distributions shown in the figure above, the distribution for  $\sigma = 2$  has a lower peak, but a wider area, compared to the distribution for  $\sigma = 1$ ; both distributions have their integral from  $x_{\min}$  to  $x_{\max}$  equal to 1.

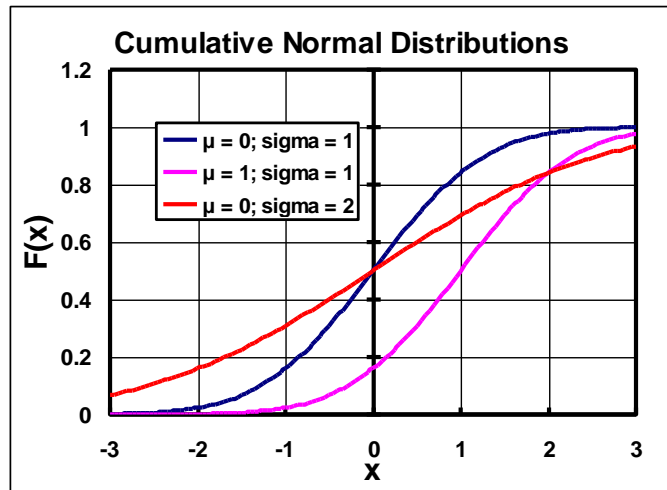
In addition to the pdf, we define the cumulative distribution,  $F(b)$  which is the probability that  $x \leq b$ . This is the probability that  $x$  lies between  $x_{\min}$  and  $b$ . We can use equation [3] to write this cumulative distribution as follows.

$$F(b) = P(x_{\min} \leq x \leq b) = \int_{x_{\min}}^b f(x)dx \quad [4]$$

We can use the usual relationship for the difference of two integrals with the same lower limit to write the probability that  $x$  lies in a certain range in terms of this cumulative distribution.

$$P(a \leq x \leq b) = \int_a^b f(x)dx = \int_{x_{\min}}^b f(x)dx - \int_{x_{\min}}^a f(x)dx = F(b) - F(a) \quad [5]$$

The cumulative normal distribution for the same three sets of parameters  $\mu$  and  $\sigma$  used in the previous plot of the pdf are shown in the figure at the right. From equation [4] we see that at  $b = x_{\max}$ , the integral for  $F(b)$  becomes the same as the normalization condition in equation [3]; thus we must have  $F(x_{\max}) = 1$ . The plots of the cumulative normal distribution shown in the figure at the left show that  $F(x)$  approaches a common value of one as  $x$  becomes large.



Tables, equations, or software for the cumulative distribution are used to find the probability that a random variable for a particular distribution lies in a specified range. The Excel spreadsheet has a normal distribution function,  $\text{NORMDIST}(x, \mu, \sigma, \text{cumulative})$ . In this function  $x$  is the value for which the distribution is desired (e.g.  $a$  or  $b$  in the equations above),  $\mu$  and  $\sigma$  are the parameters in the distribution, and the fourth variable is set to true to give the cumulative distribution. (Setting the fourth variable to false gives the pdf; if this variable is omitted the cumulative distribution is returned.)

Thus the calculation in equation [5] would be obtained by the following Excel formula:  $\text{NDIST}(b, \mu, \sigma, \text{true}) - \text{NDIST}(a, \mu, \sigma, \text{true})$ .


### Mean and variance

For any pdf the mean,  $\mu$ , is defined by the following integral.

$$\mu = \int_{x_{\min}}^{x_{\max}} xf(x)dx \quad [6]$$

The variance,  $\sigma^2$ , is defined as follows.

$$\sigma^2 = \int_{x_{\min}}^{x_{\max}} (x - \mu)^2 f(x)dx \quad [7]$$

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The square root of the variance,  $\sigma$ , is called the standard deviation. The parameters in the normal distribution are given the symbols  $\mu$  and  $\sigma$  because they can be shown to be the mean and standard deviation for the normal distribution.

The derivation in the footnote<sup>2</sup> shows that the following formula can be used to compute the variance.

$$\sigma^2 = \int_{x_{\min}}^{x_{\max}} x^2 f(x) dx - \mu^2 \quad [8]$$

### Functions of a random variable

We would like to be able to compute statistical quantities for functions of a random variable,  $g(x)$ . For example, the energy in the wind flowing into a wind turbine with velocity  $V$  and air density,  $\rho$ , equals the mass flow rate of the wind,  $\rho VA$ , times the kinetic energy in the wind,  $V^2/2$ . Here  $A$  represents the swept area of the wind turbine =  $\pi D^2/4$ , where  $D$  is the diameter of the turbine blades. Thus the wind energy flowing into the turbine is  $\rho V^3 A/2 = \rho V^3 D^2/8$ . In this example,  $V$  is the random variable and the function we would like to examine is the power  $g(V) = \rho V^3 A/2$ . In general, for any pdf,  $f(x)$  and any function  $g(x)$  we can find the mean value of  $g(x)$  as follows.

$$\overline{g(x)} = \int_{x_{\min}}^{x_{\max}} g(x) f(x) dx \quad [9]$$

### Distributions used for wind speed

Two probability distribution functions are commonly used for wind speed. The simpler of the two is the **Rayleigh distribution** which has a single parameter  $c$ .

<sup>2</sup> Start with the definition of the variance and expand the  $(x - \mu)^2$  term in the integrand.

$$\sigma^2 = \int_{x_{\min}}^{x_{\max}} (x - \mu)^2 f(x) dx = \int_{x_{\min}}^{x_{\max}} (x^2 - 2x\mu + \mu^2) f(x) dx = \int_{x_{\min}}^{x_{\max}} x^2 f(x) dx - 2\mu \int_{x_{\min}}^{x_{\max}} x f(x) dx + \mu^2 \int_{x_{\min}}^{x_{\max}} f(x) dx$$

The next to last integral,  $\int_{x_{\min}}^{x_{\max}} x f(x) dx$ , is simply the definition of the mean,  $\mu$ , and the final integral,

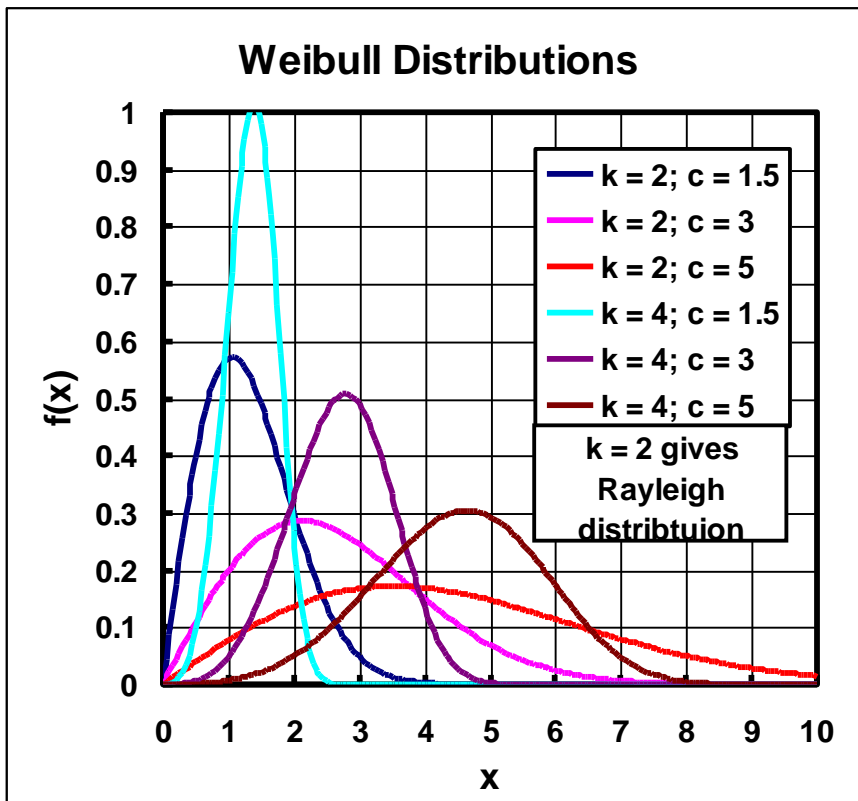
$\int_{x_{\min}}^{x_{\max}} f(x) dx = 1$  since any pdf integrated between  $x_{\min}$  and  $x_{\max} = 1$ . Making these substitutions gives

$$\sigma^2 = \int_{x_{\min}}^{x_{\max}} x^2 f(x) dx - 2\mu \int_{x_{\min}}^{x_{\max}} x f(x) dx + \mu^2 \int_{x_{\min}}^{x_{\max}} f(x) dx = \int_{x_{\min}}^{x_{\max}} x^2 f(x) dx - 2\mu(\mu) + \mu^2(1) = \int_{x_{\min}}^{x_{\max}} x^2 f(x) dx - \mu^2$$

$$f(V) = \frac{2V}{c^2} e^{-(V/c)^2} \quad 0 \leq V < \infty \quad [10]$$

The **Weibull distribution** shown below has two parameters k and c. The Rayleigh distribution is actually a special case of the Weibull distribution with k = 2.

$$f(V) = \frac{k}{c} \left(\frac{V}{c}\right)^{k-1} e^{-(V/c)^k} \quad 0 \leq V < \infty \quad [11]$$



Setting k = 2 in the Weibull distribution gives the Rayleigh distribution. For both distributions,  $V_{\min} = 0$  and  $V_{\max} = \infty$ .

The plot at the left shows the Weibull distribution for various values of the parameters k and c. The plot shows that as the value of c increases for a given value of k the shape of the distribution gets wider. Because of this c is called the **scale parameter**; it has dimensions of velocity. The plot also shows that as k increases from 2 to 4 for a given value of c, the

maximum in the pdf increases. Because of this k is called the **shape parameter**; it is dimensionless.


The following equation for the cumulative Weibull distribution is derived in the appendix.

$$F(b) = \int_{x_{\min}}^b f(V) dV = \int_0^b \frac{k}{c} \left(\frac{V}{c}\right)^{k-1} e^{-(V/c)^k} dV = 1 - e^{-(b/c)^k} \quad \text{Weibull} \quad [12]$$

Setting k = 2 in this result gives the cumulative Rayleigh distribution.

$$F(b) = 1 - e^{-(b/c)^2} \quad \text{Rayleigh} \quad [13]$$

As shown in the appendix, equations [6] and [8] can be used to compute the mean and variance of the Weibull distribution. Once these results are known we can set k = 2 to get the

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mean and variance of the Rayleigh distribution. Those results are shown below. Some results use the gamma function  $\Gamma(z)$ , which is discussed in the Appendix.

$$\mu_{Weibull} = c\Gamma\left(\frac{1}{k} + 1\right) \quad \mu_{Rayleigh} = c\frac{\sqrt{\pi}}{2} \quad [14]$$

$$\sigma_{Weibull}^2 = c^2\left[\Gamma\left(\frac{2}{k} + 1\right) - \Gamma^2\left(\frac{1}{k} + 1\right)\right] \quad \sigma_{Rayleigh}^2 = c^2\left(2 - \frac{\pi}{4}\right) \quad [15]$$

The appendix also gives the following equations for the most probable value of velocity (the one that maximizes the pdf).

$$V_{mp,Weibull} = c\sqrt[k]{\frac{k-1}{k}} \quad V_{mp,Rayleigh} = \frac{c}{\sqrt{2}} \quad [16]$$

For the Rayleigh distribution the single parameter,  $c$ , relates the following three properties:

$$c = V_{mp}\sqrt{2} = \frac{2\mu}{\sqrt{\pi}} = \sigma\sqrt{\frac{4}{8-\pi}} \quad [17]$$


The Rayleigh distribution can be written using  $V_{mp}$  (sometimes using the symbol  $\beta$  for  $V_{mp}$ ) or the mean velocity,  $\mu$ . Substituting the equations in [17] into equation [11] gives the following different forms for the Rayleigh distribution.

$$f(V) = \frac{2V}{c^2}e^{-(V/c)^2} = \frac{V}{V_{mp}^2}e^{-V^2/(2V_{mp}^2)} = \frac{\pi V}{2\mu^2}e^{-\pi V^2/(4\mu^2)} \quad [18]$$

### Computing the Rayleigh distribution $c$ parameter from experimental data

The usual determination of the mean and standard deviation from experimental data for the normal distribution are well known. The minimum-least-squares-error (MLE) estimate of the mean of the normal distribution is the arithmetic mean (the sum of all values divided by the number of values). The formula for the MLE estimate of the variance is also familiar.<sup>3</sup> The parameter  $c$  in the Rayleigh distribution can be evaluated from a set of  $N$  data points on wind velocity,  $V_i$ . When experimental data are used to determine parameters in probability distributions, the computed result is called an estimate of the true parameter. Here we use the symbol  $\hat{c}$  to indicate that the equation below gives us only an estimate of the true distribution parameter,  $c$ .

$$\hat{c} = \sqrt{\frac{1}{2N} \sum_{i=1}^N V_i^2} \quad [19]$$

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To estimate the parameters  $c$  and  $k$  for a Weibull distribution from experimental data, it is first necessary to use an iterative procedure to solve the following equation for the estimator  $\hat{k}$ , typically using an initial guess of  $\hat{k} = 2$ .

$$\hat{k} = \frac{1}{\frac{\sum_{i=1}^N V_i^{\hat{k}} \ln V_i}{\sum_{i=1}^N V_i^{\hat{k}}} - \frac{1}{N} \sum_{i=1}^N \ln V_i} \quad [20]$$

Once the value of  $\hat{k}$  is found, the value of  $\hat{c}$  is found from the equation below.

$$\hat{c} = \left( \frac{1}{2N} \sum_{i=1}^N V_i^{\hat{k}} \right)^{\frac{1}{\hat{k}}} \quad [21]$$

This equation for  $\hat{c}$  is seen to be a generalization of equation [19] from  $k = 2$  for the Rayleigh distribution to the general  $k$  for the Weibull distribution. (If we substitute  $\hat{k} = 2$  in this equation we get equation [19].) There is a Matlab function `wblfit` that can be used to find the estimates  $\hat{k}$  and  $\hat{c}$ . If the wind data are stored in a file called 'C:\Users\All Users\windData.txt', the following Matlab commands (following the `>>` prompt) give the results shown after the `ans =`; the first parameter is  $c$ ; the second is  $k$ .

```
>> V=load('C:\ Users\All Users\windData.txt');
>> wblfit(V)
ans =
8.8393  1.4352
```

It is also possible to use this function to get confidence limits on the estimated parameters. See the Matlab help on the `wblfit` function for directions getting these results.


### Distribution of power in the wind

As noted earlier, the power in the wind is the product of the mass flow rate entering the wind turbine blades,  $\rho VA$ , and the kinetic energy per unit mass in the wind,  $V^2/2$ . (Using the definitions that  $1 \text{ N} = 1 \text{ kg}\cdot\text{m}/\text{s}^2$ ,  $1 \text{ J} = 1 \text{ N}\cdot\text{m}$ , and  $1 \text{ W} = 1 \text{ J}/\text{s}$ , the SI units for this product,  $\rho V^3 A/2$  are  $(\text{kg}/\text{m}^3)(\text{m}/\text{s})^3(\text{m}^2) = \text{kg}\cdot\text{m}^2/\text{s}^3 = \text{N}\cdot\text{m}/\text{s} = \text{J}/\text{s} = \text{W}$ .) The average power in the incoming wind is given by the following application of equation [9].

$$\overline{P(V)} = \int_{V_{\min}}^{V_{\max}} P(V) f(V) dV = \int_0^{\infty} \frac{\rho A V^3}{2} f(V) dV = \frac{\rho A}{2} \int_0^{\infty} V^3 f(V) dV = \frac{\rho A \overline{V^3}}{2} \quad [22]$$

Equations [A-19] and [A-20] give the following results for the mean of the cubed velocity.



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$$\left(\overline{V^3}\right)_{Weibull} = c^3 \Gamma\left(\frac{3}{k} + 1\right) \quad \left(\overline{V^3}\right)_{Rayleigh} = c^3 \frac{3\sqrt{\pi}}{4} \quad [23]$$

Note that there is a difference between the cube of the mean velocity and the mean of the cubed velocity. These two values are related by equation [A-21]

$$\overline{V^3} = \frac{6}{\pi} \overline{V}^3 \quad Rayleigh \quad [24]$$

The fraction of the total power in the wind between a velocity of zero and a velocity b can be obtained by numerical integration of the integral in the following equation from the Appendix.

$$F[0 \leq P(V) \leq P(b)] = \frac{1}{\Gamma\left(\frac{3}{k} + 1\right)} \int_{y=0}^{y=(b/c)^k} y^{\frac{3}{k}} e^{-y} dy \quad [25]$$

Note that the value of this fraction depends only on k the ratio b/c. A table giving the values of  $F[0 \leq P(V) \leq b]$  as a function of k and b/c is given at the end of the appendix.

We can regard the integrand in equation [22] as a distribution function for the distribution of power as a function of velocity. This integrand, multiplied by an arbitrary constant, C, is shown below.

$$g(P) = C \frac{\rho A}{2} V^3 f(V) \quad [26]$$

The constant C is used to make sure that the integral of g(P) over all velocities equals one. Integrating equation over all V from V = 0 to V = ∞ gives.

$$\int_0^{\infty} g(P) dV = \int_0^{\infty} C \frac{\rho A}{2} V^3 f(V) dV = 1 \Rightarrow C = \frac{1}{\frac{\rho A}{2} \int_0^{\infty} V^3 f(V) dV} \quad [27]$$

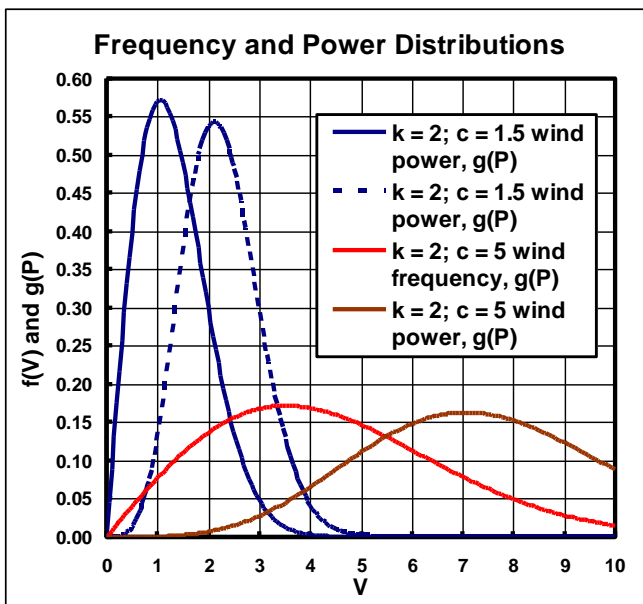
From equations [22] and [23] we see that, for the Weibull distribution,

$$\frac{1}{C} = \frac{\rho A}{2} \int_0^{\infty} V^3 f(V) dV = \frac{\rho A \overline{V^3}}{2} = \frac{\rho A c^3}{2} \Gamma\left(\frac{3}{k} + 1\right) \quad [28]$$

Substituting this result for C into equation [26] gives the final result for the distribution of wind power as a function of velocity for the Weibull distribution.

$$g(P) = \frac{\frac{\rho A}{2} V^3 f(V)}{\frac{\rho A}{2} c^3 \Gamma\left(\frac{3}{k} + 1\right)} = \left(\frac{V}{c}\right)^3 \frac{\frac{k}{c} \left(\frac{V}{c}\right)^{k-1} e^{-(V/c)^k}}{\Gamma\left(\frac{3}{k} + 1\right)} = \frac{\frac{k}{c} \left(\frac{V}{c}\right)^{k+2} e^{-(V/c)^k}}{\Gamma\left(\frac{3}{k} + 1\right)} \quad [29]$$

Because the  $rA/2$  terms cancel out in this equation we see that the distribution for power is really the distribution for  $V^3$ . When speaking of power distributions it is common to really refer to a distribution  $V^3$ .



The distributions of wind power and wind frequency are compared in the figure at the left for two values of the scale parameter  $c$  with the shape parameter  $k = 2$  giving a Rayleigh distribution. These plots show that almost all of the wind power fraction is contained in the higher velocities for each shape parameter. At the point where the velocity frequency is a maximum, the fraction of the available wind power is quite small.

For the Rayleigh distribution with  $c = 2$ ,

$$g(P) = \frac{8V^4}{\sqrt{\pi}c^5} e^{-(V/c)^2}$$


### Wind turbine performance

An actual wind turbine is only operated in a range between a minimum velocity, called the cut-in velocity, and a maximum velocity, called the cut-out velocity. The power coefficient,  $c_p$ , is defined as the fraction of the wind power that is actually captured. (This may be defined either in terms of the wind turbine power to the generator or the generator output power.) If the potential output power of the wind turbine is more than the maximum input power to the generator, the turbine is controlled to produce only the maximum generator power.

With this turbine operating pattern the average power output from the wind turbine, for a given probability distribution of the wind can be found from the modified version of equation [22] shown below.

$$\overline{P(V)} = \int_{V_{cut-in}}^{V_{Pmax}} \frac{c_p \rho A V^3}{2} f(V) dV = \int_{V_{Pmax}}^{V_{cut-out}} P_{max} f(V) dV \quad [30]$$

In this equation  $V_{Pmax}$  is the wind velocity at which the maximum power is delivered by the wind turbine to the generator. This is called the rated wind speed. By the basic definition of the wind power, the wind power that is delivered by the turbine when the wind velocity is  $V_{Pmax}$  is  $c_p \rho A (V_{Pmax})^3 / 2$ . This gives the following definitions of  $V_{Pmax}$ , which depends on the definition of  $c_p$ . In these equations,  $P_{max}$  is the average output power of the generator and  $\eta_{gen}$  is the generator efficiency.

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$$V_{P_{\max}} = \begin{cases} \left( \frac{2P_{\max}}{c_p \rho A} \right)^{1/3} & c_p = \frac{\text{Generator Output}}{\text{Wind Power}} \\ \left( \frac{2P_{\max}}{c_p \rho A \eta_{gen}} \right)^{1/3} & c_p = \frac{\text{Generator Input}}{\text{Wind Power}} \end{cases} \quad [31]$$

For a wind turbine whose generator output has a maximum of 1.5 MW and whose  $c_p$  is based on the generator output,  $V_{P_{\max}}$  would be computed as follows for power coefficient of 0.45, an air density of  $1.2 \text{ kg/m}^3$ , and a rotor diameter of 60 m, which gives an area of  $(60 \text{ m})^2 \pi / 4 = 2827.4 \text{ m}^2$ .

$$V_{P_{\max}} = \left( \frac{2P_{\max}}{c_p \rho A} \right)^{1/3} = \left[ \frac{2 \left( \frac{1.5 \text{ MW}}{10^6 \text{ W}} \right) \frac{1 \text{ kg} \cdot \text{m}^2}{\text{W} \cdot \text{s}^3}}{0.45 \frac{1.2 \text{ kg}}{\text{m}^3} (2827.4 \text{ m}^2)} \right]^{1/3} = \frac{12.53 \text{ m}}{\text{s}} \quad [32]$$

If the  $c_p$  were based on the turbine output, the value of  $V_{P_{\max}}$  computed above would have to be divided by the generator efficiency to the 1/3 power. For a generator efficiency of 95%, this would give a value of  $v_{P_{\max}} = 12.75 \text{ m/s}$ .

Substituting the Weibull distribution into the first integral in equation [30] gives.

$$\begin{aligned} \int_{V_{cut-in}}^{V_{P_{\max}}} \frac{c_p \rho A V^3}{2} \frac{k}{c} \left( \frac{V}{c} \right)^{k-1} e^{-(V/c)^k} dV &= \frac{c_p \rho A}{2} \int_{V=V_{cut-in}}^{V=V_{P_{\max}}} c^3 \left( \frac{V}{c} \right)^3 \frac{k}{c} \left( \frac{V}{c} \right)^{k-1} e^{-(V/c)^k} cd \left( \frac{V}{c} \right) \\ &= \frac{c_p \rho A c^3 k}{2} \int_{V_{cut-in}/c}^{V_{P_{\max}}/c} \left( \frac{V}{c} \right)^{k+2} e^{-(V/c)^k} d \left( \frac{V}{c} \right) \end{aligned} \quad [33]$$

The second integral in equation [30] can be found from the cumulative Weibull distribution from equation [12].

$$\begin{aligned} \int_{V_{P_{\max}}}^{V_{cut-out}} P_{\max} f(V) dV &= P_{\max} \int_{V_{P_{\max}}}^{V_{cut-out}} f(V) dV = \\ P_{\max} \left[ \left( 1 - e^{-(V_{cut-out}/c)^k} \right) - \left( 1 - e^{-(V_{P_{\max}}/c)^k} \right) \right] &= P_{\max} \left( e^{-(V_{P_{\max}}/c)^k} - e^{-(V_{cut-out}/c)^k} \right) \end{aligned} \quad [34]$$

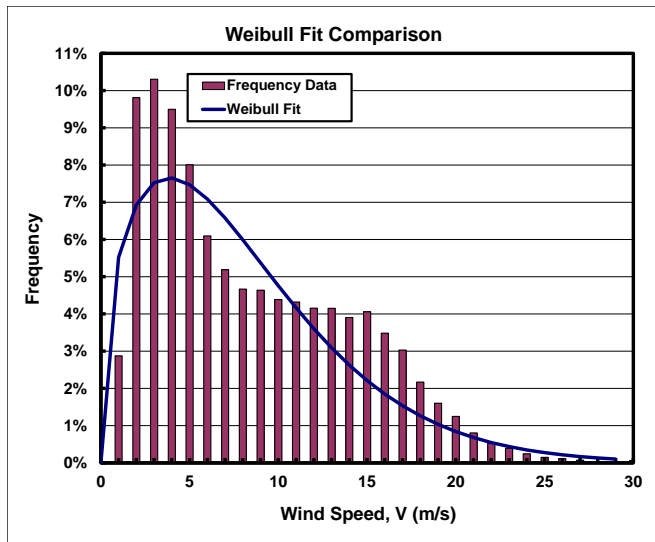
Substituting the results of equation [33] and equation [34] into equation [30] gives the following equation for the average power of the generator from a wind turbine which has the following operating pattern: (1) no operation below a cut-in velocity,  $V_{cut-in}$ , (2) all available power from the turbine delivered to the generator between the cut-in velocity and the velocity which delivers the maximum power to the generator,  $V_{P_{\max}}$ , (3) turbine output power is limited to

maximum generator power between  $V_{P_{max}}$  and a cut-out velocity,  $V_{cut-out}$ , and (4) no operation above the cut-out velocity.

$$\overline{P(V)} = \frac{c_p \rho A c^3 k}{2} \int_{V_{cut-in}/c}^{V_{P_{max}}/c} \left(\frac{V}{c}\right)^{k+2} e^{-(V/c)^k} d\left(\frac{V}{c}\right) + P_{max} \left( e^{-(V_{P_{max}}/c)^k} - e^{-(V_{cut-out}/c)^k} \right) \quad [35]$$

If SI units are used ( $\text{kg/m}^3$  for density,  $\text{m}^2$  for area, and  $\text{m/s}$  for all velocities and the scale factor,  $c$ ) the power will be in watts. This should be the unit used for  $P_{max}$  in the calculations. Reported results can be appropriately scaled to kW or MW. The appropriate averaging time is one year to account for the annual variations in winds. In this case the expected value of the energy generated is simply the product of the average power times the number of hours in a year, 8760 hours in a non-leap year or 8784 hours in a leap year.


### Discrete calculation of wind turbine performance



The calculation outlined above assumes that the wind data are well fitted by a Rayleigh or Weibull distribution. The figure at the left shows actual data that are not well fitted by a Weibull distribution. In such cases, it is necessary to work with a discrete distribution of the data to determine the average wind power. This calculation is best described by an example. The set of discrete wind-speed and frequency data plotted in the figure to the left are shown in the table below. These data show the percent of the wind speed data for given velocity range. For example, the fraction of the wind speed data between speeds of 0

and 1 m/s is 0.028747.

Percent of Wind-Speed Data Between Lower and Upper Velocity Bounds (V in m/s)								
Lower	Upper	Percent	Lower	Upper	Percent	Lower	Upper	Percent
0	1	2.8747 %	10	11	4.3213 %	20	21	0.8028 %
1	2	9.8109 %	11	12	4.1559 %	21	22	0.5310 %
2	3	10.307 %	12	13	4.1527 %	22	23	0.3928 %
3	4	9.4960 %	13	14	3.9050 %	23	24	0.2427 %

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4	5	8.0058 %	14	15	4.0583 %	24	25	0.1476 %
5	6	6.0967 %	15	16	3.4830 %	25	26	0.1102 %
6	7	5.1868 %	16	17	3.0287 %	26	27	0.0716 %
7	8	4.6691 %	17	18	2.1695 %	27	28	0.0310 %
8	9	4.6374 %	18	19	1.6005 %	28	29	0.0114 %
9	10	4.3865 %	19	20	1.2489 %	29		0.0640 %

The Weibull distribution shown in the figure above was computed using equations [20] and [21] to determine the MLE estimators for  $k$  and  $c$ . As such this is a “best fit” between the actual data and the Weibull distribution.


We can define the following variables to use for the calculations with discrete data: the lower limit on velocity for each band,  $V_k$ , and the fraction of the time that the wind velocity occurs in a particular band,  $f_k$ . In the example above,  $k$  ranges from 0 to 29. The lowest band,  $k = 0$ , is bounded by a lower  $V_0 = 0$  m/s and  $V_1 = 1$  m/s; the value of  $f_k$  for this band is  $f_1 = 0.028747$ . We say that a given band, called band  $k$ , extends from velocity  $V_k$  to  $V_{k+1}$  and has the frequency (fraction of time the wind speed in in this velocity range) of  $f_k$ . Within this band the probability of any intermediate wind speed is considered uniform. This corresponds to a uniform probability distribution function (within one band) which is  $f = 1/(V_{k+1} - V_k)$ . For such a probability distribution, the mean velocity within a band is simply the arithmetic average of the band boundaries.

$$\bar{V}_k = \int_{V_k}^{V_{k+1}} \frac{1}{V_{k+1} - V_k} V dV = \frac{1}{V_{k+1} - V_k} \frac{V^2}{2} \Big|_{V_k}^{V_{k+1}} = \frac{1}{V_{k+1} - V_k} \frac{V_{k+1}^2 - V_k^2}{2} = \frac{V_{k+1} + V_k}{2} \quad [36]$$

The mean of the cube of the velocity within a given band is given by the following equation (steps of the integration and final algebra not shown).

$$\overline{V}_k^3 = \int_{V_k}^{V_{k+1}} \frac{1}{V_{k+1} - V_k} V^3 dV = \frac{1}{V_{k+1} - V_k} \frac{V_{k+1}^4 - V_k^4}{4} = \frac{(V_{k+1} + V_k)(V_{k+1}^2 + V_k^2)}{4} \quad [37]$$

We still have the basic concepts for wind turbine operation: (1) for the time that the wind speed is between the cut-in speed and the rated speed the turbine utilizes all the power available in the wind; (2) for the time that the wind speed is between the rated speed and the cut-out speed the turbine operates at its maximum power. For discrete data, we have to use a summation over the discrete distribution instead of the integration over the continuous probability

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distribution function to predict the average operating power. The total time between a lower velocity,  $V_k = V_L$  and an upper velocity,  $V_{k+1} = V_U$  is given by the following equation. Note that the final band is the one in which  $V_U$  is the upper limit so the appropriate index for this velocity is  $k = U - 1$ . This properly counts all the time in the band whose upper limit is  $V_U$ . Since  $V_L$  is the lower limit of the band the proper initial index is  $k = L$

$$\left[ \begin{array}{c} \text{Time between} \\ V_L \text{ and } V_U \end{array} \right] = \sum_{k=L}^{k=U-1} f_k \quad [38]$$

The total “power” over the same range of speeds is given by the following sum using equation [37] for the mean velocity-cubed in a band:

$$\left[ \begin{array}{c} \text{Power between} \\ V_L \text{ and } V_U \end{array} \right] = \sum_{k=L}^{k=U-1} \overline{V_k^3} f_k = \frac{1}{4} \sum_{k=L}^{k=U-1} (V_{k+1} + V_k)(V_{k+1}^2 + V_k^2) f_k \quad [39]$$

We have to consider the general case where any one of the specified wind speeds, generally denoted as  $V_s$ , which may be the cut-in, rated, or cut-out speed, lies within a band between  $V_k$  and  $V_{k+1}$ . The total fraction of time within this band is  $f_k$ . This time fraction for the total band can be divided into two parts: (1) the time at or below  $V_s$  (between  $V_k$  and  $V_s$ ), and (2) the time at or above  $V_s$  (between  $V_s$  and  $V_{k+1}$ ), which are given by the following equations.

$$f_{k-s} = \frac{V_s - V_k}{V_{k+1} - V_k} f_k \quad \text{and} \quad f_{s-(k+1)} = \frac{V_{k+1} - V_s}{V_{k+1} - V_k} f_k \quad [40]$$

The power in these two parts of the band can be found by applying equation [37] to the two velocity ranges within the band.


$$\overline{V_{k-s}^3} = \frac{(V_s + V_k)(V_s^2 + V_k^2)}{4} \quad \text{and} \quad \overline{V_{s-(k+1)}^3} = \frac{(V_{k+1} + V_s)(V_{k+1}^2 + V_s^2)}{4} \quad [41]$$

We can now consider the general case where the cut-in speed lies in the band between  $V_{CI}$  and  $V_{CI+1}$ , the rated wind speed lies in the band between  $V_R$  and  $V_{R+1}$ , and the cut-out wind speed lies in the band between  $V_{CO-1}$ . Below the rated wind speed the total power is given by the following sum; each term in the sum is the product of time in a band (or sub-band) times the power in that band (or sub-band).

$$\left[ \begin{array}{c} \text{Power between} \\ V_{cut-in} \text{ and } V_{Pmax} \end{array} \right] = \frac{(V_{cut-in} + V_{CI+1})(V_{cut-in}^2 + V_{CI+1}^2)}{4} \frac{V_{CI+1} - V_{cut-in}}{V_{CI+1} - V_{CI}} f_{CI} \\ + \sum_{k=CI+1}^{k=R-1} \overline{V_k^3} f_k + \frac{(V_R + V_{Pmax})(V_R^2 + V_{Pmax}^2)}{4} \frac{V_{Pmax} - V_R}{V_{R+1} - V_R} f_R \quad [42]$$

The power between the rated speed and the cut-out speed is given by the following equation, which gives the total time between these two speeds times the cube of the rated speed.

$$\left[ \begin{array}{c} \text{Power between} \\ V_{Pmax} \text{ and } V_{cut-out} \end{array} \right] = V_{Pmax}^3 \left( \frac{V_{R+1} - V_{Pmax}}{V_{R+1} - V_R} f_R + \sum_{k=R+1}^{k=CO-1} f_k + \frac{V_{cut-out} - V_{CO}}{V_{CO+1} - V_{CO}} f_{CO} \right) \quad [43]$$

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Both equations [42] and [43] give the mean of the velocity-cubed. These values must be multiplied by the term  $\rho A c_p / 2$  to give the power. Summing the results of these two equations gives the average expected power over the typical time period for which the data were obtained.

### Derivation of Equations Used in Text

The main part of this appendix contains derivations of formulas for the Weibull and Rayleigh distributions. All the formal derivations are done for the Weibull distribution. The results for the Rayleigh distribution are then found by setting  $k = 2$  in the Weibull distribution results. At the end of the appendix there is a discussion of the gamma function which is used in the derivations. The final item in this appendix is a table giving the fraction of wind power as a function of velocity.

### Cumulative Weibull Distribution

Substituting the Weibull distribution from equation [11] into the definition of the cumulative distribution in equation [4] gives the following result for the cumulative Weibull distribution

$$F(b) = P(V \leq b) = \int_{V_{\min}}^b f(V) dV = \int_0^b \frac{k}{c} \left(\frac{V}{c}\right)^{k-1} e^{-(V/c)^k} dV \quad [A-1]$$

Define a new integration variable,  $y = (V/c)^k$ . The limits of the integral  $V = 0$  and  $V = b$  become  $y = 0$  and  $y = (b/c)^k$ . Differentiating the definition of  $y$  gives  $dy = (kV^{k-1}/c^k)dV$  so that  $dV = c^k/(kV^{k-1})dy$ . Substituting the definition of  $y$ , the new limits and the expression for  $dV$  into equation [A-1] gives

$$\begin{aligned} F(b) &= \int_{V=0}^{V=b} \frac{k}{c} \left(\frac{V}{c}\right)^{k-1} e^{-(V/c)^k} dV = \int_{y=0}^{y=(b/c)^k} \frac{kV^{k-1}}{c^k} e^{-y} \frac{c^k}{kV^{k-1}} dy \\ &= \int_{y=0}^{y=(b/c)^k} e^{-y} dy = -e^{-y} \Big|_{y=0}^{y=(b/c)^k} = 1 - e^{-(b/c)^k} \end{aligned} \quad [A-2]$$


### Moments of Weibull distribution

The computations required for the mean, variance and wind power can be simplified by doing a single integration for the moments of the distribution. The  $n^{\text{th}}$  moment of any probability distribution,  $f(V)$  is defined by the equation below.

$$\overline{V^n} = \int_{V_{\min}}^{V_{\max}} V^n f(V) dV \quad [A-3]$$

For the Weibull distribution, this moment is given by the following integral; the final arrangement of this integral gives a prefactor equal to the exponent  $(V/c)^k$ .



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$$\overline{V^n} = \int_{V_{\min}}^{V_{\max}} V^n f(V) dV = \int_0^{\infty} V^n \frac{k}{c} \left(\frac{V}{c}\right)^{k-1} e^{-(V/c)^k} dV = k \int_0^{\infty} V^{n-1} \left(\frac{V}{c}\right)^k e^{-(V/c)^k} dV \quad [A-4]$$

To evaluate this integral we use the same integration variable  $y = (V/c)^k$  that we used in finding the cumulative distribution. Solving this definition for  $V$  gives  $V = cy^{1/k}$ . Differentiating this equation for  $V$  gives  $dV = (c/k)y^{1/k-1}$ . The limits of the integral  $V = 0$  and  $V = \infty$  become  $y = 0$  and  $y = \infty$ . Substituting the definition of  $y$ , the new limits and the expression for  $dV$  into the integral in equation [A-4] gives a result in the form of a standard integral known as a gamma function,  $\Gamma(x)$ .

$$\begin{aligned} \overline{V^n} &= k \int_{V=0}^{V=\infty} V^{n-1} \left(\frac{V}{c}\right)^k e^{-(V/c)^k} dV = k \int_{y=0}^{y=\infty} V^{n-1} y e^{-y} \frac{c}{k} y^{\frac{1}{k}-1} dy = \\ &= c \int_{y=0}^{y=\infty} \left(cy^{\frac{1}{k}}\right)^{n-1} y^{\frac{1}{k}} e^{-y} dy = c^n \int_{y=0}^{y=\infty} y^{\frac{n}{k}} e^{-y} dy = c^n \Gamma\left(\frac{n}{k} + 1\right) \end{aligned} \quad [A-5]$$

More information about the gamma function, including values of the function for some arguments, a recursion equation that  $\Gamma(x+1) = x\Gamma(x)$ , and the description of Excel and Matlab functions to compute  $\Gamma(x)$ , is provided later in this appendix.

With an equation for the general moment, we can find the mean, variance and mean power by setting  $n = 1, 2,$  and  $3,$  respectively. Once we have the results for the Weibull distribution, we can set  $k = 2$  in those results to obtain the equivalent value for the Rayleigh distribution. Our general result is summarized below.

$$\overline{V^n} \Big|_{Weibull} = c^n \Gamma\left(\frac{n}{k} + 1\right) \quad \overline{V^n} \Big|_{Rayleigh} = c^n \Gamma\left(\frac{n}{2} + 1\right) = c^n \Gamma\left(\frac{n+2}{2}\right) \quad [A-6]$$


### Mean of Weibull and Rayleigh distributions

Substituting the Weibull distribution from equation [11] into the definition of the cumulative distribution in equation [6] gives the following result for the mean of the Weibull distribution,

$$\mu = \int_{V_{\min}}^{V_{\max}} V f(V) dV = \int_0^{\infty} V \frac{k}{c} \left(\frac{V}{c}\right)^{k-1} e^{-(V/c)^k} dV = \overline{V} \quad [A-7]$$

From equation [A-6] for the  $n^{\text{th}}$  moment of a Weibull distribution we can set  $n = 1$  to get the mean as value shown below.

$$\mu_{Weibull} = \int_{V_{\min}}^{V_{\max}} V f(V) dV = \overline{V} = c \Gamma\left(\frac{1}{k} + 1\right) \quad [A-8]$$

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Setting  $k = 2$  gives the mean value for the Weibull distribution.<sup>4</sup>

$$\mu_{Rayleigh} = c\Gamma\left(\frac{1}{2} + 1\right) = c\Gamma\left(\frac{3}{2}\right) = c\frac{\sqrt{\pi}}{2} \quad [A-9]$$

### Variance of Weibull and Rayleigh distributions

Substituting the Weibull distribution from equation [11] into the definition of the cumulative distribution in equation [8] gives the following result for the variance of the Weibull distribution

$$\sigma^2 + \mu^2 = \int_{V_{\min}}^{V_{\max}} V^2 f(V) dV = \int_0^{\infty} V^2 \frac{k}{c} \left(\frac{V}{c}\right)^{k-1} e^{-(V/c)^k} dV = \overline{V^2} \quad [A-10]$$

From equation [A-6] for the  $n^{\text{th}}$  moment of a Weibull distribution we can set  $n = 2$  to get the variance as shown below.

$$\left(\sigma^2 + \mu^2\right)_{Weibull} = c^2 \Gamma\left(\frac{2}{k} + 1\right) \quad [A-11]$$

For the Rayleigh distribution, which is the Weibull distribution with  $k = 2$ , we can use the known value of  $\Gamma(2) = 1$  to obtain.

$$\left(\sigma^2 + \mu^2\right)_{Rayleigh} = c^2 \Gamma\left(\frac{2}{2} + 1\right) = c^2 \Gamma(2) = c^2 \quad [A-12]$$

Using the expression for the mean of a Rayleigh distribution from equation [A-9] gives

$$\left(\sigma^2\right)_{Rayleigh} = \left(\sigma^2 + \mu^2\right)_{Rayleigh} - \left(\mu^2\right)_{Rayleigh} = c^2 - \left(c\frac{\sqrt{\pi}}{2}\right)^2 = c^2\left(1 - \frac{\pi}{4}\right) \quad [A-13]$$


Similarly, using the mean of the Weibull distribution from equation [A-8] gives

$$\sigma_{Weibull}^2 = c^2 \left[ \Gamma\left(\frac{2}{k} + 1\right) - \Gamma^2\left(\frac{1}{k} + 1\right) \right] \quad [A-14]$$

### Maximum of Weibull distribution

The maximum value of a pdf is called the mode or most probable point. Taking the first derivative of the Weibull distribution in equation [11] and setting that derivative to zero gives.

<sup>4</sup> Here we use the result that  $\Gamma(3/2) = (1/2)\Gamma(1/2)$  and  $\Gamma(1/2) = \pi^{1/2}$  to get  $\Gamma(3/2) = \pi^{1/2}/2$ .

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$$\frac{df}{dV} = \frac{d}{dV} \left[ \frac{k}{c} \left( \frac{V}{c} \right)^{k-1} e^{-(V/c)^k} \right] = \frac{k}{c^k} \left[ (k-1)V^{k-2} - V^{k-1} \frac{kV^{k-1}}{c^k} \right] e^{-(V/c)^k} = 0 \quad \text{[A-15]}$$

We can divide by the factors outside the brackets and rearrange the remaining terms to obtain.

$$\left[ (k-1)V^{k-2} - V^{k-1} \frac{kV^{k-1}}{c^k} \right] = \frac{V^{k-2}}{c^k} \left[ (k-1)c^k - kV^k \right] = 0 \quad \text{[A-16]}$$

Solving this equation for V gives the most probable velocity,  $V_{mp}$ .

$$V_{mp} = c \sqrt[k]{\left( \frac{k-1}{k} \right)} \quad \text{Weibull} \quad \text{[A-17]}$$

For the Rayleigh distribution,  $k = 2$ , the most probable velocity is

$$V_{mp} = c \sqrt{2} \left( \frac{2-1}{2} \right) = \frac{c}{\sqrt{2}} \quad \text{Rayleigh} \quad \text{[A-18]}$$

### Wind power integrals

Equation [22] shows that the mean power in the wind is proportional to the mean of the cube of the velocity. From equation [A-6] for the  $n^{\text{th}}$  moment of a Weibull distribution we can set  $n = 3$  to get this mean for a Weibull distribution.

$$\overline{V^3} = c^3 \Gamma \left( \frac{3}{k} + 1 \right) \quad \text{[A-19]}$$


For the Rayleigh distribution we set  $k = 2$  and use the result that  $\Gamma(5/2) = 3\pi^{1/2}/4$  to obtain.

$$\overline{V^3} = c^3 \Gamma \left( \frac{3}{2} + 1 \right) = c^3 \frac{3\sqrt{\pi}}{4} \quad \text{Rayleigh} \quad \text{[A-20]}$$

We can compare this mean of the velocity cubed to the cube of the mean velocity. Cubing both sides of equation [A-9] and dividing the result by equation [A-20] gives the following result for a Rayleigh distribution.

$$\left( \mu_{\text{Rayleigh}} \right)^3 = \overline{V^3} = c^3 \frac{\pi\sqrt{\pi}}{8} \Rightarrow \frac{\overline{V^3}}{\overline{V}^3} = \frac{c^3 \frac{3\sqrt{\pi}}{4}}{c^3 \frac{\pi\sqrt{\pi}}{8}} = \frac{6}{\pi} \Rightarrow \overline{V^3} = \frac{6}{\pi} \overline{V}^3 \quad \text{[A-21]}$$

The fraction of power in the wind between the minimum velocity of zero and a given velocity  $b$  can be written as follows:

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$$F[0 \leq P(V) \leq P(b)] = \frac{1}{V^3} \int_0^b V^3 \frac{k}{c} \left(\frac{V}{c}\right)^{k-1} e^{-(V/c)^k} dV = \frac{k}{V^3} \int_0^b V^2 \left(\frac{V}{c}\right)^k e^{-(V/c)^k} dV \quad [A-22]$$

The usual substitutions  $y = (V/c)^k$ ,  $V = cy^{1/k}$ ,  $dV = (c/k)y^{1/k-1}$  can be made here. The upper limit corresponding to  $V = b$  is  $y = (b/c)^k$ . This gives

$$F[0 \leq P(V) \leq P(b)] = \frac{k}{V^3} \int_{y=0}^{y=(b/c)^k} V^2 y e^{-y} \frac{c}{k} y^{\frac{1}{k}-1} dy = \frac{c^3}{V^3} \int_{y=0}^{y=(b/c)^k} y^{\frac{3}{k}} e^{-y} dy \quad [A-23]$$

Substituting equation [A-19] for the mean cubed velocity gives.

$$F[0 \leq P(V) \leq P(b)] = \frac{c^3}{V^3} \int_{y=0}^{y=(b/c)^k} y^{\frac{3}{k}} e^{-y} dy = \frac{1}{\Gamma\left(\frac{3}{k} + 1\right)} \int_{y=0}^{y=(b/c)^k} y^{\frac{3}{k}} e^{-y} dy \quad [A-24]$$

This integral has no exact form and must be found numerically. The parameter  $c$  and the variable  $v$  do not occur independently; they only occur in the ratio  $v/c$  in the upper limit of the integral. Thus it is possible to compute a table of  $F[0 \leq P(V) \leq P(b)]$  as a function of  $b/c$  for specified values of  $k$ . Such a table is given at the end of this appendix.

### Gamma Functions

The gamma function,  $\Gamma(z)$  is used in various integrals, including probability distribution integrals; it is defined by the following equation.

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt \quad [A-25]$$

We can derive a general recurrence relationship for gamma function values whose argument increases by one using integration by parts with  $u = t^{z-1}$  and  $dv = e^{-t} dt$ .

$$\begin{aligned} \Gamma(z) &= \int_0^{\infty} t^{z-1} e^{-t} dt = (z-1)t^{z-2}(-e^{-t}) \Big|_0^{\infty} - \int_0^{\infty} (z-1)t^{z-2}(-e^{-t}) dt \\ &= 0 - 0 + (z-1) \int_0^{\infty} t^{(z-1)-1} e^{-t} dt = (z-1)\Gamma(z-1) \end{aligned} \quad [A-26]$$

This recurrence relation is commonly written as follows.

$$\Gamma(z+1) = z\Gamma(z) \quad [A-27]$$

For  $z = 1$  the evaluation of the gamma function is simply the integral of  $e^{-z}$ .

$$\Gamma(1) = \int_0^{\infty} t^{1-1} e^{-t} dt = \int_0^{\infty} e^{-t} dt = -e^{-t} \Big|_0^{\infty} = 1 \quad [A-28]$$

Using this result and the recurrence relation that  $\Gamma(z+1) = z\Gamma(z)$  gives  $\Gamma(2) = 1\Gamma(1) = (1)(1) = 1$ ;  $\Gamma(3) = 2\Gamma(2) = (2)(1) = 2$ ;  $\Gamma(4) = 3\Gamma(3) = (3)(2) = 6$ . We see that there is a general relationship for positive integers, n.

$$\Gamma(n) = (n-1)! \quad [A-29]$$

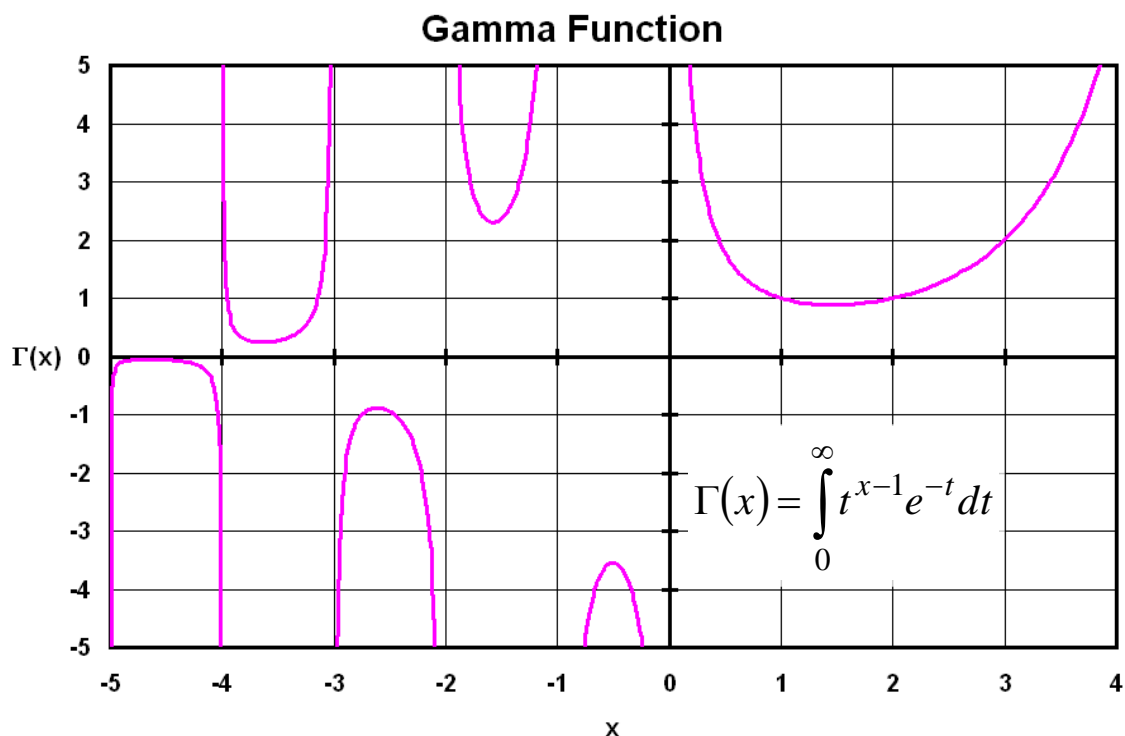
We will state without proof the following result that can be obtained using contour integration for complex variables.

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad [A-30]$$

Using the recurrence relationship from equation [A-10] gives the following results:

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \quad \Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3\sqrt{\pi}}{4} \quad [A-31]$$

Matlab has a function gamma(x) that can be used to compute the gamma function. The Excel spreadsheet has a function GAMMALN(x) than can be used to compute the natural logarithm of  $\Gamma(x)$ . Use the worksheet formula “= exp(gammaln(x))” to compute  $\Gamma(x)$  in Excel. The plot of the gamma function shown below was developed in Excel.



The plot shows some of the values that we have already computed:  $\Gamma(1) = \Gamma(2) = 1$  and  $\Gamma(3) = 2$ . The gamma function is seen to be continuous and positive for positive values of its argument, going to infinity as the (positive) argument approaches zero or infinity. The gamma function is discontinuous at negative integer values. Depending on the direction in which the interer value is approached, the value of  $\Gamma(x)$  approaches plus or minus infinity as the integer value is approached.

An abridged table of gamma functions is shown on the next page. This table gives values for  $\Gamma(x)$  for  $0 < x < 1$ . Recall that the general recursion formula for gamma functions in equation [A-27] allows us to compute  $\Gamma(x+1) = x\Gamma(x)$ . We could use the table value for  $\Gamma(0.16)$  and the this recursion formula to compute the gamma function for  $x = 3.16$  as follows:  $\Gamma(3.16) = 2.16\Gamma(2.16) = (2.16)(1.16)\Gamma(1.16) = (2.16)(1.16)(0.16)\Gamma(0.16) = (2.16)(1.16)(0.16)(5.81127) = 2.32971$ . This calculation procedure can be used with the gamma-function table below for moments of the Weibull distribution which are given in terms of gamma functions.

Abridged Table of Gamma Functions					
x	$\Gamma(x)$	x	$\Gamma(x)$	x	$\Gamma(x)$
0.01	99.4325	0.16	5.8112	0.40	2.218160
0.02	49.4422	0.18	5.1318	0.45	1.968136
0.03	32.7850	0.20	4.5908	0.50	1.772454
0.04	24.4609	0.22	4.1504	0.55	1.616124
0.05	19.4700	0.24	3.7855	0.60	1.489192
0.06	16.1457	0.26	3.4784	0.65	1.384795
0.07	13.7736	0.28	3.2168	0.70	1.298055
0.08	11.9965	0.30	2.9915	0.75	1.225417
0.09	10.6162	0.32	2.7957	0.80	1.164230
0.10	9.51351	0.34	2.6241	0.85	1.112484
0.11	8.61269	0.36	2.4727	0.90	1.068629
0.12	7.86325	0.38	2.3382	0.95	1.031453
0.13	7.23024	For x outside the range (0, 1) you use the relationship that $\Gamma(x+1) = x\Gamma(x)$			
0.14	6.68869				

### Table of Cumulative Wind Power Distribution Fractions

The table of cumulative wind power fractions, shown below, is used to determine the fraction of the total power in the wind that is between two wind speeds  $V_1$  and  $V_2$ . To use this table you first compute  $V_1/c$  and  $V_2/c$  and find the table entries for these values of  $V/c$  at the  $k$  value for your wind speed distribution. If  $F_1$  and  $F_2$  are the cumulative distribution values from the table corresponding to  $V_1/c$  and  $V_2/c$  (with  $F_2 > F_1$ ) then the fraction of the total wind power between these two velocities is given by the difference between  $F_2$  and  $F_1$  multiplied by the total power in the wind given by a combination of equations [22] and [23].

$$\overline{P(V)} = \int_{V_{\min}}^{V_{\max}} P(V)f(V)dV = \frac{\rho AV^3}{2} = \frac{\rho Ac^3}{2} \Gamma\left(\frac{3}{k} + 1\right) \quad [A-32]$$

The average wind power between  $V_1$  and  $V_2$  is then given by the following equation.



$$\overline{P(V_1 \leq V \leq V_2)} = (F_2 - F_1) \frac{\rho A c^3}{2} \Gamma\left(\frac{3}{k} + 1\right) \quad [A-33]$$

The procedure outlined above for computing gamma functions can be used here.

Fraction of Wind Power Between V = 0 and Given V						
V/c	Fraction for Following Values of k					
	k = 1.4	k = 1.6	k = 1.8	k = 2	k = 2.2	k = 2.4
0.00	0	0	0	0	0	0
0.05	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.10	0.000005	0.000005	0.000004	0.000003	0.000002	0.000002
0.15	0.000031	0.000030	0.000027	0.000022	0.000018	0.000014
0.20	0.000108	0.000112	0.000106	0.000094	0.000079	0.000065
0.25	0.000280	0.000305	0.000303	0.000281	0.000250	0.000215
0.30	0.000604	0.000687	0.000709	0.000686	0.000634	0.000567
0.35	0.001151	0.001355	0.001448	0.001448	0.001385	0.001281
0.40	0.002000	0.002425	0.002668	0.002750	0.002708	0.002579
0.45	0.003235	0.004025	0.004545	0.004809	0.004861	0.004753
0.50	0.004948	0.006295	0.007273	0.007877	0.008149	0.008156
0.55	0.007229	0.009380	0.011060	0.012227	0.012916	0.013200
0.60	0.010169	0.013427	0.016118	0.018147	0.019530	0.020338
0.65	0.013856	0.018575	0.022654	0.025925	0.028369	0.030047
0.70	0.018369	0.024959	0.030868	0.035837	0.039800	0.042794
0.75	0.023784	0.032697	0.040936	0.048133	0.054158	0.059017
0.80	0.030166	0.041893	0.053010	0.063024	0.071728	0.079088
0.85	0.037569	0.052630	0.067210	0.080672	0.092725	0.103285
0.90	0.046036	0.064971	0.083615	0.101180	0.117277	0.131769
0.95	0.055600	0.078952	0.102268	0.124585	0.145414	0.164557
1.00	0.066279	0.094589	0.123163	0.150855	0.177061	0.201517
1.05	0.078082	0.111870	0.146256	0.179887	0.212033	0.242361
1.10	0.091003	0.130760	0.171457	0.211508	0.250044	0.286651
1.15	0.105026	0.151202	0.198636	0.245483	0.290712	0.333812
1.20	0.120122	0.173115	0.227629	0.281520	0.333573	0.383162
1.25	0.136255	0.196401	0.258236	0.319279	0.378098	0.433935
1.30	0.153376	0.220943	0.290233	0.358382	0.423715	0.485322
1.35	0.171430	0.246610	0.323375	0.398430	0.469830	0.536504
1.40	0.190353	0.273260	0.357401	0.439009	0.515849	0.586691
1.45	0.210074	0.300740	0.392043	0.479706	0.561197	0.635151
1.50	0.230519	0.328893	0.427029	0.520117	0.605340	0.681244
1.55	0.251609	0.357557	0.462093	0.559861	0.647801	0.724440
1.60	0.273260	0.386569	0.496975	0.598588	0.688167	0.764331
1.65	0.295387	0.415769	0.531431	0.635986	0.726106	0.800640
1.70	0.317906	0.445000	0.565233	0.671782	0.761365	0.833217
1.75	0.340729	0.474112	0.598174	0.705755	0.793773	0.862034
1.80	0.363773	0.502961	0.630071	0.737728	0.823238	0.887166
1.85	0.386953	0.531415	0.660764	0.767575	0.849741	0.908779
1.90	0.410188	0.559351	0.690123	0.795214	0.873330	0.927108
1.95	0.433400	0.586656	0.718039	0.820610	0.894105	0.942437
2.00	0.456514	0.613231	0.744433	0.843764	0.912215	0.955080
2.05	0.479458	0.638988	0.769248	0.864717	0.927840	0.965365
2.10	0.502166	0.663854	0.792453	0.883537	0.941185	0.973617
2.15	0.524576	0.687766	0.814037	0.900318	0.952468	0.980147
2.20	0.546630	0.710674	0.834009	0.915173	0.961913	0.985244
2.25	0.568276	0.732539	0.852395	0.928230	0.969742	0.989168
2.30	0.589468	0.753336	0.869238	0.939627	0.976167	0.992147





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Academic year-  
2020-2021

Fraction of Wind Power Between $V = 0$ and Given $V$						
V/c	Fraction for Following Values of k					
	k = 1.4	k = 1.6	k = 1.8	k = 2	k = 2.2	k = 2.4
2.35	0.610161	0.773048	0.884592	0.949505	0.981389	0.994378
2.40	0.630321	0.791669	0.898521	0.958009	0.985592	0.996026
2.45	0.649913	0.809201	0.911098	0.965281	0.988942	0.997226
2.50	0.668912	0.825655	0.922402	0.971457	0.991586	0.998089
2.55	0.687294	0.841048	0.932515	0.976668	0.993653	0.998700
2.60	0.705043	0.855406	0.941523	0.981036	0.995254	0.999127
2.65	0.722143	0.868757	0.949511	0.984674	0.996482	0.999422
2.70	0.738586	0.881135	0.956564	0.987684	0.997415	0.999622
2.75	0.754365	0.892580	0.962764	0.990159	0.998117	0.999756
2.80	0.769479	0.903130	0.968193	0.992181	0.998640	0.999845
2.85	0.783929	0.912830	0.972925	0.993822	0.999027	0.999903
2.90	0.797719	0.921724	0.977034	0.995146	0.999310	0.999940
2.95	0.810855	0.929856	0.980586	0.996208	0.999515	0.999963
3.00	0.823348	0.937274	0.983646	0.997054	0.999662	0.999978
3.05	0.835208	0.944021	0.986271	0.997724	0.999766	0.999987
3.10	0.846449	0.950144	0.988514	0.998251	0.999840	0.999993
3.15	0.857086	0.955686	0.990423	0.998664	0.999892	0.999996
3.20	0.867136	0.960691	0.992041	0.998985	0.999927	0.999998
3.25	0.876616	0.965198	0.993409	0.999233	0.999952	0.999999
3.30	0.885546	0.969249	0.994559	0.999424	0.999968	1.000000
3.35	0.893944	0.972881	0.995524	0.999570	0.999979	1.000000
3.40	0.901832	0.976130	0.996330	0.999680	0.999987	1.000000
3.45	0.909229	0.979030	0.997000	0.999764	0.999992	1.000000
3.50	0.916156	0.981612	0.997556	0.999826	0.999995	1.000000
3.55	0.922635	0.983906	0.998016	0.999873	0.999997	1.000000
3.60	0.928686	0.985940	0.998394	0.999908	0.999998	1.000000
3.65	0.934330	0.987739	0.998704	0.999933	0.999999	1.000000
3.70	0.939588	0.989328	0.998958	0.999952	1.000000	1.000000
3.75	0.944480	0.990728	0.999165	0.999966	1.000000	1.000000
3.80	0.949025	0.991958	0.999333	0.999976	1.000000	1.000000
3.85	0.953243	0.993037	0.999468	0.999983	1.000000	1.000000
3.90	0.957154	0.993983	0.999578	0.999988	1.000000	1.000000
3.95	0.960774	0.994809	0.999666	0.999992	1.000000	1.000000
4.00	0.964122	0.995529	0.999736	0.999994	1.000000	1.000000
4.05	0.967214	0.996156	0.999793	0.999996	1.000000	1.000000
4.10	0.970067	0.996701	0.999837	0.999997	1.000000	1.000000
4.15	0.972697	0.997173	0.999873	0.999998	1.000000	1.000000
4.20	0.975118	0.997582	0.999901	0.999999	1.000000	1.000000
4.25	0.977344	0.997935	0.999923	0.999999	1.000000	1.000000
4.30	0.979389	0.998240	0.999941	1.000000	1.000000	1.000000
4.35	0.981266	0.998502	0.999954	1.000000	1.000000	1.000000
4.40	0.982986	0.998727	0.999965	1.000000	1.000000	1.000000
4.45	0.984562	0.998920	0.999973	1.000000	1.000000	1.000000
4.50	0.986003	0.999085	0.999979	1.000000	1.000000	1.000000
4.55	0.987321	0.999226	0.999984	1.000000	1.000000	1.000000
4.60	0.988524	0.999347	0.999988	1.000000	1.000000	1.000000
4.65	0.989621	0.999449	0.999991	1.000000	1.000000	1.000000
4.70	0.990621	0.999537	0.999993	1.000000	1.000000	1.000000
4.75	0.991531	0.999611	0.999995	1.000000	1.000000	1.000000
4.80	0.992360	0.999673	0.999996	1.000000	1.000000	1.000000
4.85	0.993112	0.999726	0.999997	1.000000	1.000000	1.000000
4.90	0.993795	0.999771	0.999998	1.000000	1.000000	1.000000
4.95	0.994415	0.999809	0.999998	1.000000	1.000000	1.000000



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**Academic year-  
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Fraction of Wind Power Between $V = 0$ and Given $V$						
$V/c$	Fraction for Following Values of $k$					
	$k = 1.4$	$k = 1.6$	$k = 1.8$	$k = 2$	$k = 2.2$	$k = 2.4$
5.00	0.994977	0.999841	0.999999	1.000000	1.000000	1.000000