Unit-4

Sampling

Sampling - (Statement)

A continuous time sampling may be completely represented in its samples and recovered back if the sampling frequency is $f_s \geq 2f_m$

where

f_s = Sampling frequency

f_m = Max. frequency present in the s/g.

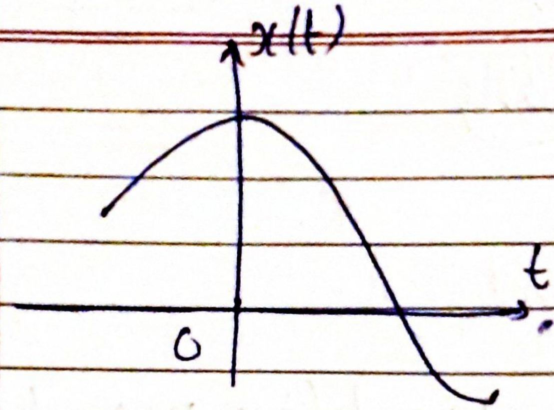
Proof of Sampling Theorem

Let us consider a continuous time signal $x(t)$ whose spectrum is band-limited to f_m Hz. It means that the signal $x(t)$ has no frequency component beyond f_m Hz.

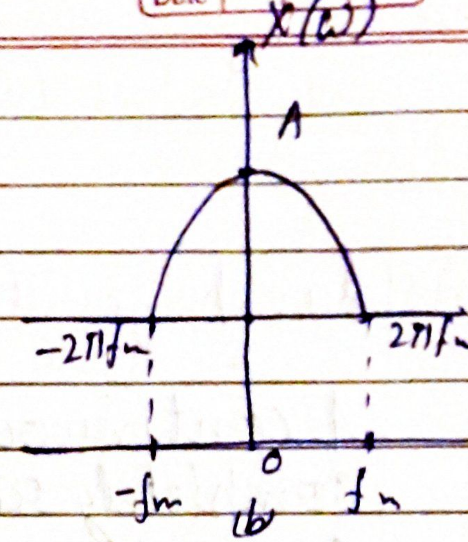
$X(\omega) \rightarrow$ Fourier transform or Frequency Spectrum

Let $x(t)$ be the ~~sma~~ sampling s/g. f_s be the sampling frequency and $\delta_{T_s}(t)$ be the impulse train

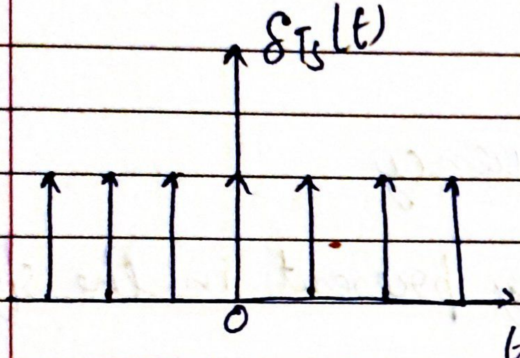
where $T_s = \dfrac{1}{f_s}$ (sampling Time)



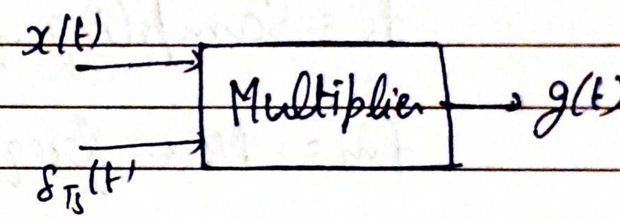
a) A continuous time s/g



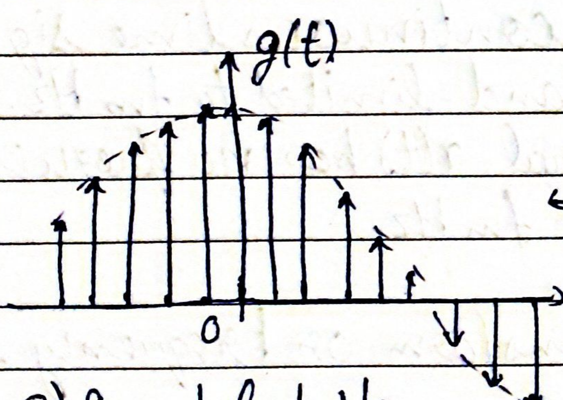
b) Spectrum of CT s/g.



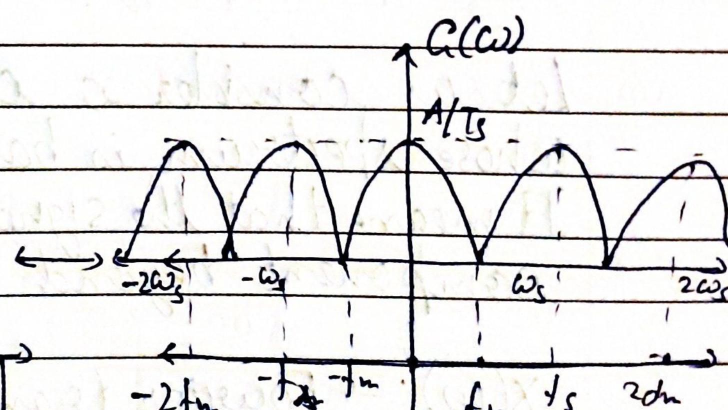
c) Impulse train as sampling function



b) Multiplier



e) Sampled s/g



f) Spectrum of sampled s/g

$$g(t) = x(t)\,\delta_{T_s}(t) \quad \text{—(1)}$$

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Since the impulse train $\delta_{T_s}(t)$ is a periodic s/g of period T_s, the same can be expressed as Fourier series.

$$\delta_{T_s}(t) = \frac{1}{T_s}\left[1 + 2\cos\omega_s t + 2\cos 2\omega_s t + 2\cos 3\omega_s t + \dots\right]$$

where $\omega_s = \frac{2\pi}{T_s} = 2\pi f_s$

Substi: value of δ_{T_s} from above eq in eq (1), we get

$$g(t) = \frac{1}{T_s}\left[x(t) + 2x(t)\cos\omega_s t + 2x(t)\cos 2\omega_s t + 2x(t)\cos 3\omega_s t + \dots\right]$$

Taking Fourier Transformation of g(t)

$G(\omega) \leftrightarrow g(t)$ $\qquad$ $2x(t)\cos\omega_s t \leftrightarrow [X(\omega-\omega_s) + X(\omega+\omega_s)]$

$x(t) \leftrightarrow X(\omega)$ $\qquad$ $2x(t)\cos 2\omega_s t \leftrightarrow X(\omega-2\omega_s) + X(\omega+2\omega_s)]$

$$\therefore G(\omega) = \frac{1}{T_s}\left[X(\omega) + X(\omega-\omega_s) + X(\omega+\omega_s) + X(\omega-2\omega_s) + X(\omega+2\omega_s) + X(\omega-3\omega_s) + X(\omega+3\omega_s) + \dots\right]$$

$$\boxed{G(\omega) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)}$$

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~~Nyqui~~ Nyquist Rate & Nyquist Interval

Nyquist Rate:-

When sampling rate becomes equal to $2f_m$ samples/sec.

It is also called minimum sampling rate.

$$\boxed{f_s = 2f_m}$$

Q1. An analog signal is expressed by equation $x(t) = 3\cos 50\pi t + 10\sin 300\pi t - \cos 100\pi t$. Calculate the Nyquist rate for this s/g.

Sol. $x(t) = 3\cos 50\pi t + 10\sin 300\pi t - \cos 100\pi t$ — (1)

Let $\omega_1, \omega_2, \omega_3$ are the frequencies present.

$$x(t) = 3\cos \omega_1 t + 10\sin \omega_2 t - \cos\omega_3 t \quad -(2)$$

comparing eq (1) & (2), we get

$$\omega_1 t = 50\pi t \Rightarrow \omega_1 = 50\pi$$

$$2\pi f_1 = 50\pi$$

$$\boxed{f_1 = 25\,Hz}$$

lly $\omega_2 t = 300\pi t \Rightarrow \omega_2 = 300\pi$

$$2\pi f_2 = 300\pi$$

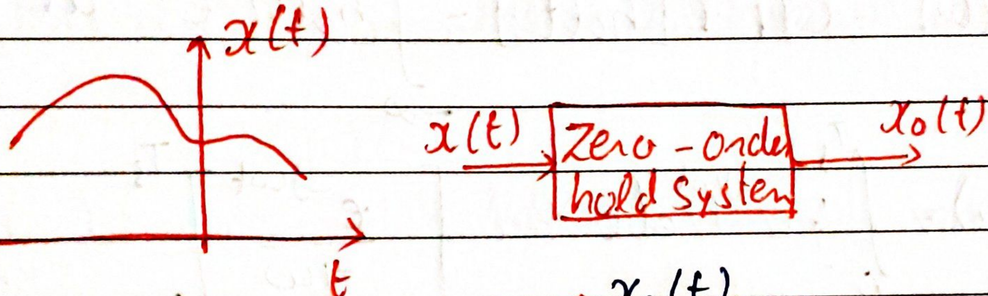
$$\boxed{f_2 = 150\,Hz}$$

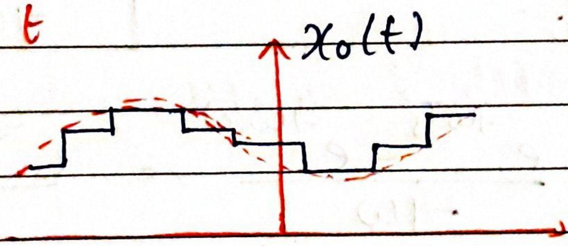
lly $\boxed{f_3 = 50\,Hz}$

Zero-order sampling :

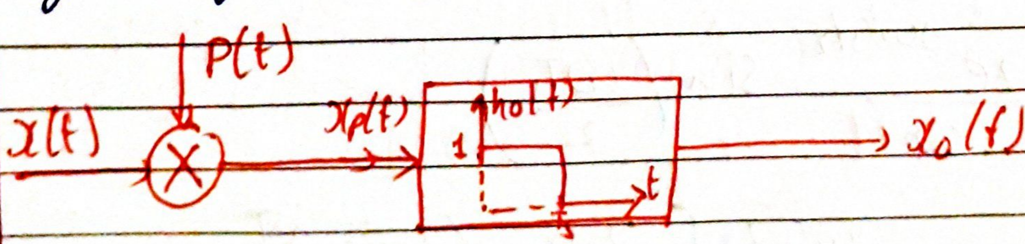
Zero-order hold : A mathematical model of the practical s/g reconstruction done by conventional DAC (Digital to Analog converter.

It describes the effect of converting a discrete-time s/g. to a continuous time s/g. by continuous time s/g by holding each sample value for one sample interval.



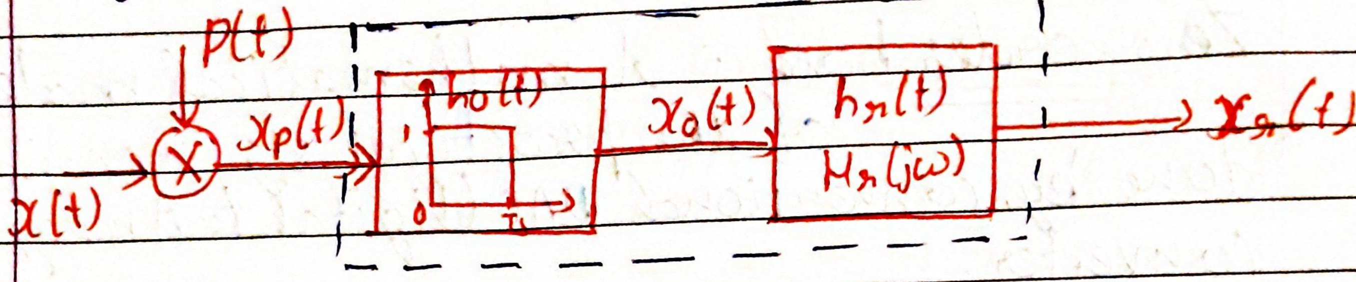


The reconstruction or recovery of a given C.T. s/g $x(t)$ from o/p of a zero-order hold system.



To construct a continuous-time s/g $x(t)$ from the o/p of the zero-order hold $x_0(t)$, we consider processing $x_0(t)$ with C.T. LTI system with impulse response $h_r(t)$.

Let the frequency response ~~to~~ is denoted by $H_r(\omega)$



Freq. response $H_0(\omega)$ may be determined by taking CTFT of the impulse response $h_0(t)$

$$H_0(\omega) = CTFT[h_0(t)] = \int_{-\infty}^{\infty} h_0(t)\, e^{-j\omega t}\, dt$$

$$H_0(\omega) = \int_0^{T_s} 1 \cdot e^{-j\omega t}\, dt = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_0^{T_s}$$

$$= \frac{e^{-j\omega T_s} - e^{-j\omega(0)}}{-j\omega} = \frac{e^{-j\omega T_s} - 1}{-j\omega}$$

$$H_0(\omega) = \frac{2e^{-j\omega(T_s/2)}}{\omega}\left[\frac{e^{j\omega T_s/2} - e^{-j\omega T_s/2}}{2j}\right]$$

$$= \frac{2e^{-j\omega T_s/2}}{\omega} \cdot \sin\left(\frac{\omega T_s}{2}\right)$$

$$= e^{-j\omega T_s/2}\left[\frac{2\sin\left(\frac{\omega T_s}{2}\right)}{\omega}\right]$$

Freq. response $H_r(\omega)$ is determined by channel.

$$H(\omega) = H_0(\omega) \cdot H_r(\omega)$$

$$H_r(\omega) = \frac{H(\omega)}{H_0(\omega)}.$$

Sub value of $H_0(\omega)$

$$H_r(\omega) = \frac{H(\omega)}{H_0(\omega)} = \frac{H(\omega)}{e^{-j\omega T_s/2}\left[\frac{2\sin \omega T_s/2}{\omega}\right]}$$

$$= \frac{e^{j\omega T_s/2} \cdot H(\omega)}{\left[\frac{2\sin \omega T_s/2}{\omega}\right]}$$

Signal Reconstruction :- (Interpolation Formula) or

First order hold :-

The process of reconstructing a continuous-time s/g $x(t)$ from its samples is known as Interpolation.

Interpolation gives either approximate or exact reconstruction or recovery of the continuous time s/g.

Mathematical Analysis:-

Let $x(t)$ be the band limited s/g which can be reconstructed (interpolated) completely from its samples.

Let the sampled s/g is written as

$$g(t) = x(t) \cdot \delta_{T_s}(t)$$

$$g(t) = \frac{1}{T_s}\left[x(t) + 2x(t)\cos\omega_s t + 2x(t)\cos 2\omega_s t + \cdots\right]$$

To recover g $x(t)$ or $X(\omega)$, the sampled s/g must be passed through an ideal low-pass filter of bandwidth of f_m Hz & gain T_s.

$$H(\omega) = T_s \times \text{rect}\left(\frac{\omega}{4\pi f_m}\right)$$

The impulse response $h(t)$ of this filter is the IFT of $H(j\omega)$.

$$h(t) = F^{-1}[H(\omega)]$$

$$= F^{-1}\left[T_s \,\text{rect}\left(\frac{\omega}{4\pi f_m}\right)\right]$$

$$h(t) = 2f_m T_s \,\text{sinc}(2\pi f_m t)$$

Assuming that sampling is done at Nyquist rate, then

$$T_s = \frac{1}{2f_m}$$

So that $2f_m T_s = 1$.

Sub. value of $2f_m T_s$ in eq of $h(t)$

$$h(t) = 1 \cdot \text{sinc}(2\pi f_m t)$$

$$\boxed{h(t) = \text{sinc}(2\pi f_m t)}$$

If $g(t)$ is the filter o/p, then $x(t)$ can be expressed as:

$$x(t) = \sum_{k} x(kT_s)\, h(t - kT_s)$$

$$= \sum_{k} x(kT_s)\, \text{sinc}[2\pi f_m (t - kT_s)]$$

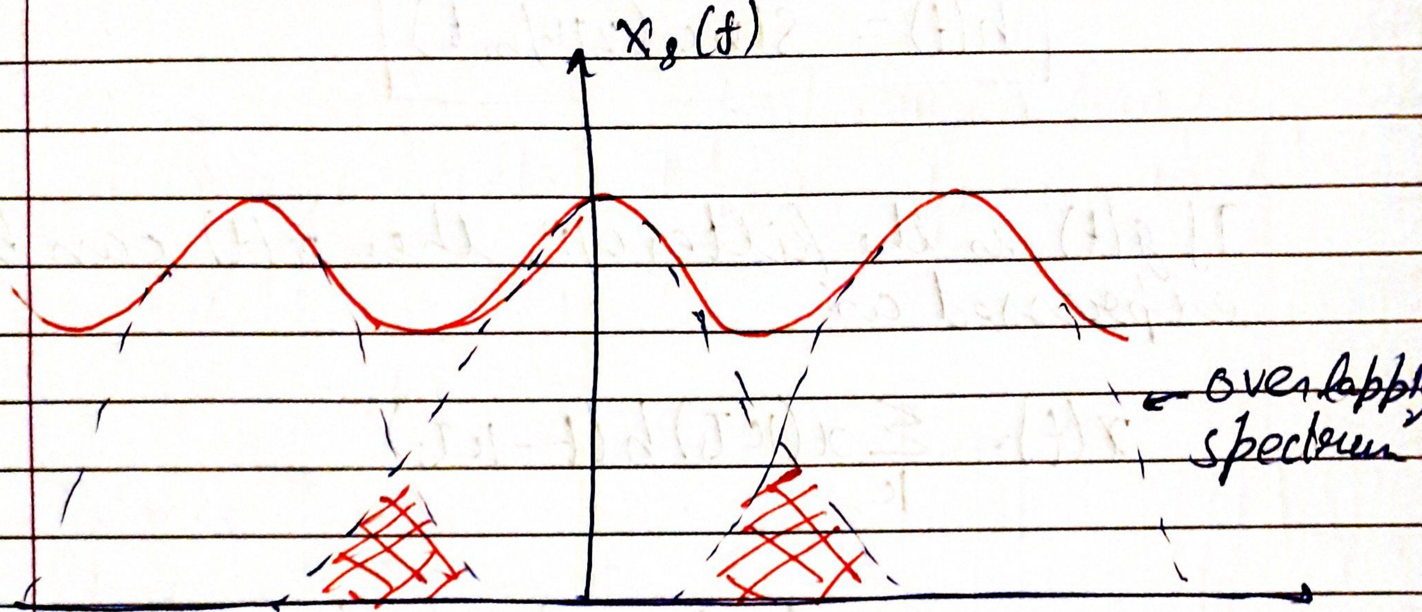
$$\boxed{x(t) = \sum_{k} x(kT_s)\, \text{sinc}(2\pi f_m - k\pi)} \quad \text{— ②}$$

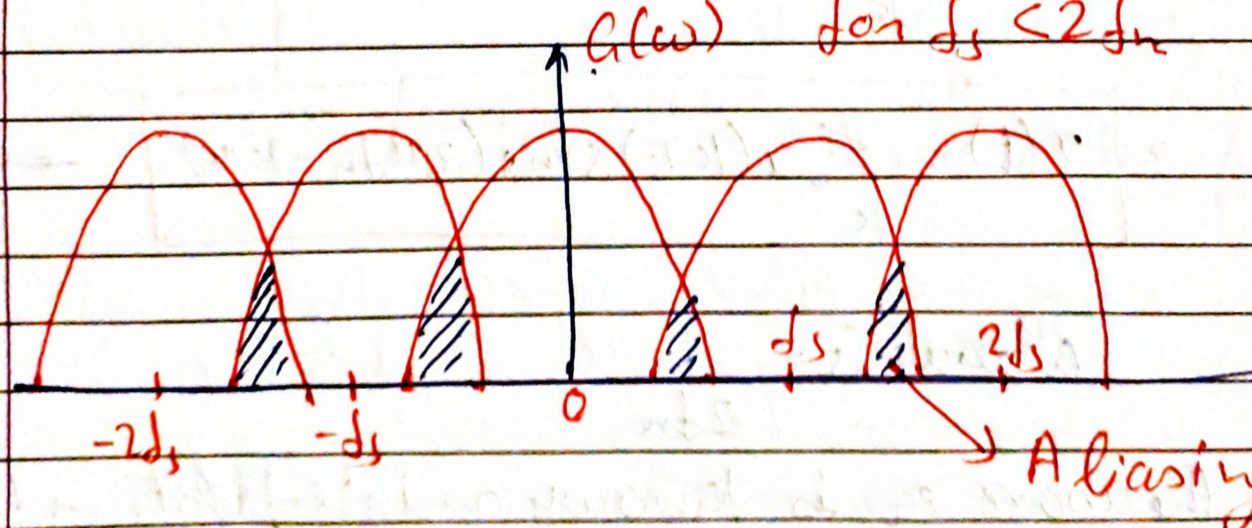
where $T_s = \frac{1}{2f_m}$.

The above eq is known as interpolation formula which provides values of $x(t)$ between samples as a weighted sum of all the sample values.

Aliasing :-

~~The~~ It is an effect that causes different s/g to become indistinguishable (or aliases of one another) when sampled.





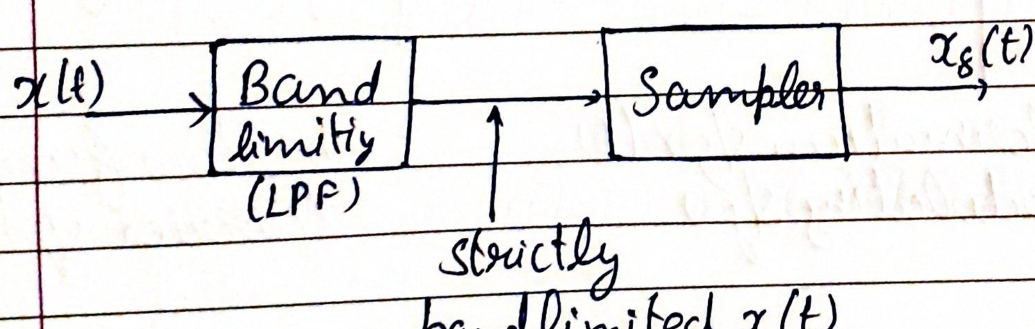
If ~~the sample spacing is T,~~

Due to aliasing some information contained in the original s/g $x(t)$ is lost and cannot be recovered.

* We can eliminate Aliasing by following the below mentioned steps.

1) By using a band limiting LPF & passing $x(t)$ s/g through it before sampling.

The filter is, having cutoff frequency at $f_c = f_m$, due to which it will be strictly band limited. This filter is also called anti aliasing filter or peralias filter.



2) Increase the sampling freq. f_s to a great extent i.e. $f_s >>> 2f_m$. In this case even though $x(t)$ is not strictly band limited, the spectrum will not overlap. A guard band is created betwe the adjacent spectrum.

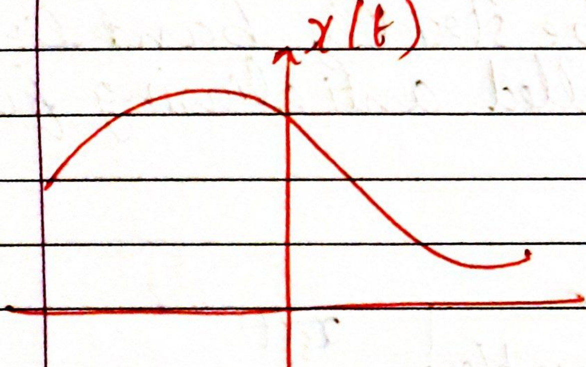
Hence ~~ai~~ aliasing can be prevented by
1) using an antialiasing or prealising filter
2) using the sampling freq. $f_s > 2f_m$.

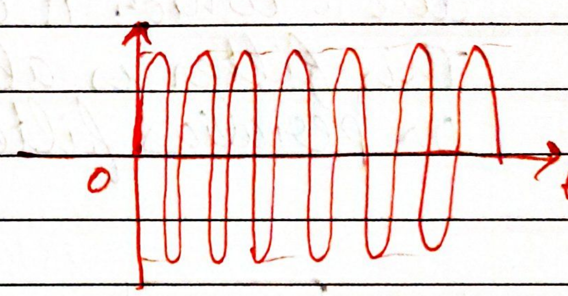
Modulation-:

It is a process by which a low freq. s/g is superimposed on an high freq. s/g which is called carrier s/g.

Or

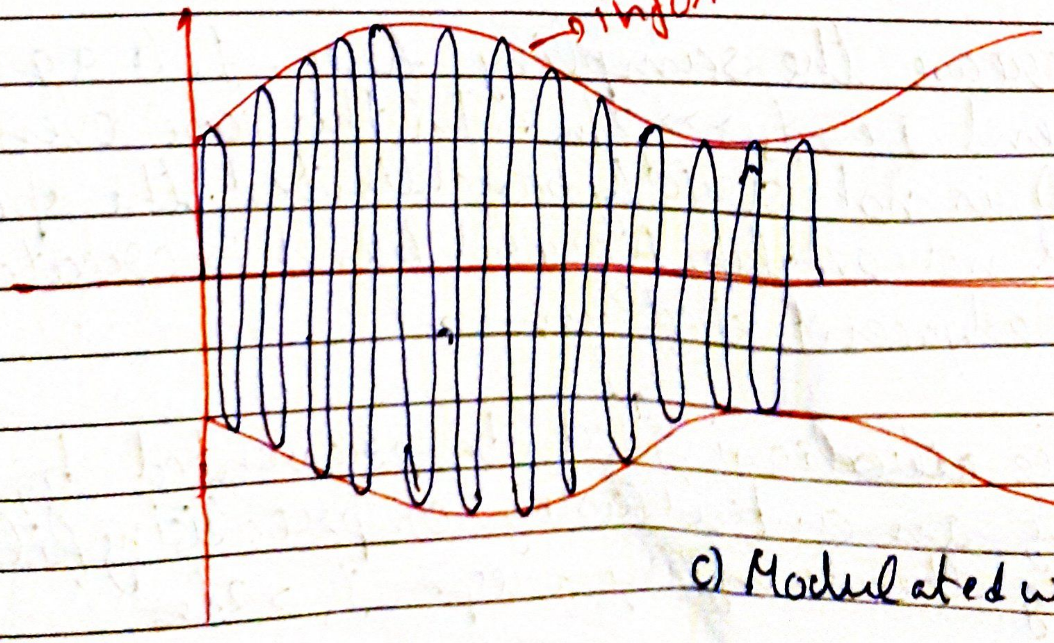
It is a process of superposing a low freq. auido s/g (called modulating s/g) on a high freq. s/g (carrier wave)





a) information s/g x(t) (modulating s/g)

b) Low freq. s/g (carrier s/g)



c) Modulated wave

Types of Modulation :-

1) **Amplitude Modulation** :- ~~where Amplitude~~ It is a modulation technique ~~or process~~ which is most commonly used for transmitting message with a radiowave.

In amplitude modulation (AM) the amplitude of the Carrier wave is varied in proportion to that of message s/g.

2) **Frequency Modulation (FM)** :- is a technique or process of encoding information on a particular s/g by varying the carrier wave freq. in accordance with the freq. of modulating s/g.

3) **Phase Modulation (PM)** :- is a type of modulation in which the phase of a radio carrier wave is varied by an amount proportional to the instantaneous amplitude of the modulating s/g.