Subject

Unit-2

LTI

LTI: A system which has both linearity and time-invariance properties.

Why LTI system is req:-

① Most practical and physical process around us can be modelled in the form of LTI system.

Characterization of LTI system:-

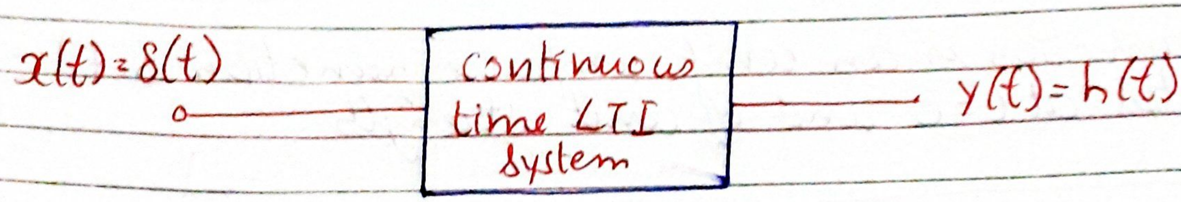
Superposition theorem can be applied to find the response y(t) to a given i/p x(t).

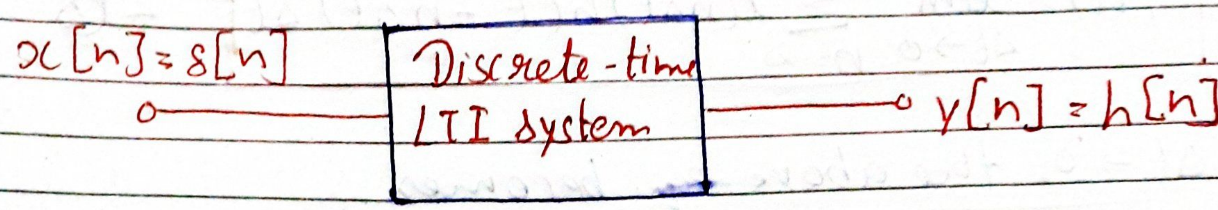
Steps to followed to find the response of a LTI system

- Resolve the i/p function x(t) in terms of simpler or basic function for which response can be easily evaluated.

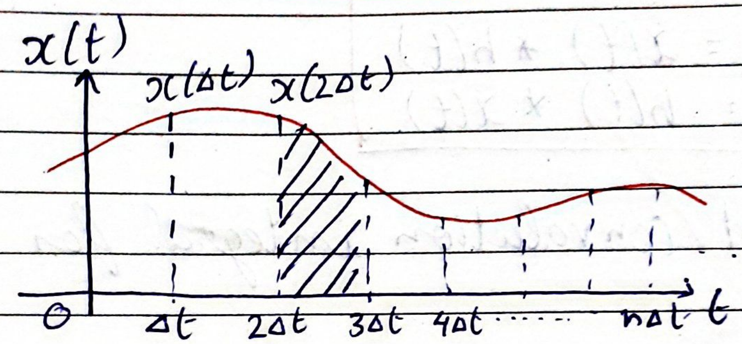
- Determine individually the response of LTI system for the simpler i/p impulse function.

Using superposition theorem, find the sum of individual responses which will become the overall response y(t) of function x(t)





CT unit impulse response and convolution Integral



If limit $\Delta t \to 0$, n^{th} element area can be considered as a rectangle of width Δt & height $[x(n\Delta t)]$

The function $x(t)$ is continuous sum of such impulse function. Mathematically.

$$x(t) = \lim_{\Delta t \to 0} \sum_{n=-\infty}^{\infty} x(n\Delta t)(\Delta t) \cdot \delta(t - n\Delta t) \quad — (1)$$

Let $h(t)$ be the unit impulse response of a LTI System.

i.e $h(t)$ is the ~~est~~ response of LTI system when

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i/p function is an unit impulse function $\delta(t)$ located at $t=0$ and of unit strength.

$$\boxed{y(t) = \lim_{\Delta t \to 0} \sum_{n=-\infty}^{\infty} x(n\Delta t)\, h(t - n\Delta t)\, \Delta t} \quad - (2)$$

if $\Delta t \to 0$, the above eq. becomes.

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau.}$$

$$\boxed{\begin{aligned} y(t) &= x(t) * h(t) \\ y(t) &= h(t) * x(t) \end{aligned}}$$

This is called convolution integral for CT. LTI System.

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Numerical

1) Obtain the convolution of two continuous time functions given below :-

$\boxed{x(t) = e^{-t^2}}$ $\boxed{h(t) = 3t^2}$ for all values of t

Sol:-

$$y(t) = x(t) * h(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \quad -(1)$$

Now

$x(\tau) = e^{-\tau^2}$

$h(t-\tau) = 3(t-\tau)^2$

Sub. values in eq (1)

$$= \int_{\tau=-\infty}^{\infty} e^{-\tau^2} \cdot [3(t-\tau)^2]\, d\tau$$

$$= \int_{\tau=-\infty}^{\infty} e^{-\tau^2} [3t^2 - 6t\tau + 3\tau^2]\, d\tau$$

$$= 3t^2 \int_{-\infty}^{\infty} e^{-\tau^2} d\tau - 6t \int_{-\infty}^{\infty} \tau e^{-\tau^2} d\tau + 3\int_{-\infty}^{\infty} \tau^2 e^{-\tau^2} d\tau$$

$$= 3t^2 \sqrt{\pi} - 0 + 1.5\sqrt{\pi}$$

$$= 5.31t^2 + 2.659$$

$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

$\int_{-\infty}^{\infty} \tau e^{-\tau^2}$

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(2) The i/p s/g $x(t)$ and impulse response $h(t)$ of a continuous time system are described as

$$x(t) = e^{-3t} u(t) \qquad h(t) = u(t-1)$$

Find out the O/P.

Sol:-

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$x(\tau) = e^{-3\tau} u(\tau) \qquad u(\tau) = \begin{cases} 1 & \tau \geq 0 \\ 0 & \tau < 0 \end{cases}$$

$$h(t-\tau) = u(t-\tau-1) \qquad u(t-\tau-1) = \begin{cases} 1 & \tau \leq t-1 \\ 0 & \tau > t-1 \end{cases}$$

$$y(t) = \int_{-\infty}^{\infty} e^{-3\tau} u(\tau)\, u(t-\tau-1)\, d\tau$$

$$= \int_{0}^{t-1} e^{-3\tau} \cdot 1 \cdot 1\, d\tau \Rightarrow \int_{0}^{t-1} e^{-3\tau}\, d\tau$$

$$= \frac{-1}{3} \left[e^{-3\tau} \right]_{0}^{t-1}$$

$$\boxed{y(t) = \frac{1}{3} \left[1 - e^{-3(t-1)} \right]}$$

Q. Determine the convolution of the two continu time function given below:

$$x(t) = 3\cos 2t \quad \text{for all } t$$

$$h(t) = e^{-|t|} = \begin{cases} e^{t} & \text{for } t < 0 \\ e^{-t} & \text{for } t \geq 0 \end{cases}$$

Sol:- $$y(t) = x(t) * h(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \quad -(1)$$

$$x(\tau) = 3\cos 2\tau$$

$$h(t-\tau) = \begin{cases} e^{(t-\tau)} & \text{for } t-\tau < 0 \text{ or } \tau > t \\ e^{(\tau-t)} & \text{for } t-\tau \geq 0 \text{ or } \tau \leq t \end{cases}$$

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{\tau=-\infty}^{t} (3\cos 2\tau)\, e^{(\tau-t)}\, d\tau + \int_{\tau=t}^{\infty} (3\cos 2\tau) e^{(t-\tau)} d\tau$$

$$= 3e^{-t}\int e^{\tau} \cos 2\tau\, d\tau + 3e^{t}\int_{\tau=t}^{\infty} e^{-\tau} \cos 2\tau\, d\tau$$

$$= 3e^{-t}\left[\frac{e^{\tau}(\cos 2\tau + 2\sin 2\tau)}{5}\right]_{-\infty}^{t} + 3e^{t}\left[\frac{e^{-\tau}(-\cos 2\tau + 2\sin 2\tau)}{5}\right]_{t}^{\infty}$$

$$y(t) = \frac{6}{5}\cos 2t = 1.2\cos 2t$$

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sum of smallintervals

For discrete time o/p s/g $y[n]$

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]}$$

$$\boxed{y[n] = x[n] * h[n]}$$

Q. Obtain the convolution sum of two discrete-time s/g given below:-

$$x[n] = e^{-n^2} \qquad h[n] = 3n^2 \text{ for all } n.$$

Sol: $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$

$x[k] = e^{-k^2}$ $\qquad h[n-k] = 3[n-k]^2$

$$y[n] = \sum_{k=-\infty}^{\infty} e^{-k^2} \cdot 3[n-k]^2$$

$$= 3\sum_{k=-\infty}^{\infty} e^{-k^2}[n^2 + k^2 - 2nk]$$

$$\boxed{y[n] = 3n^2 \sum_{k=-\infty}^{\infty} e^{-k^2} + 3\sum_{k=-\infty}^{\infty} k^2 e^{-k^2} - 6nk \sum_{k=-\infty}^{\infty} k e^{-k^2}}$$

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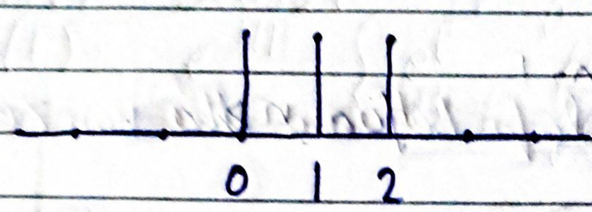
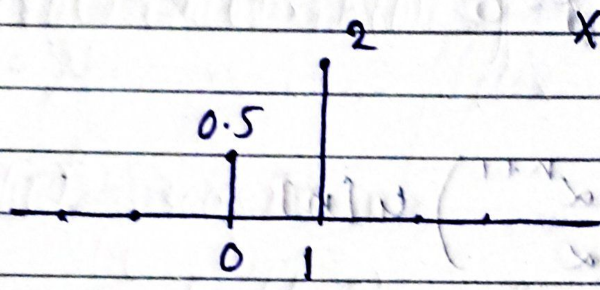
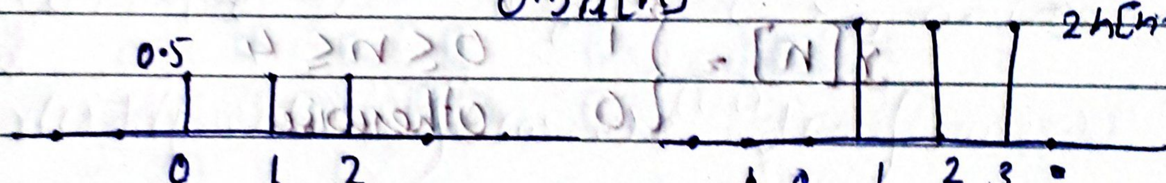
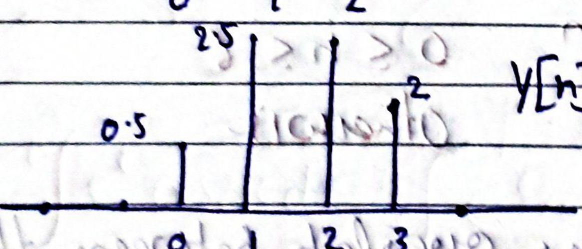
Q. $y[n] = x[0]\,h[n-0] + x[1]\,h[n-1]$
$= 0.5\,h[n] + 2h[n-1]$

Sol:-





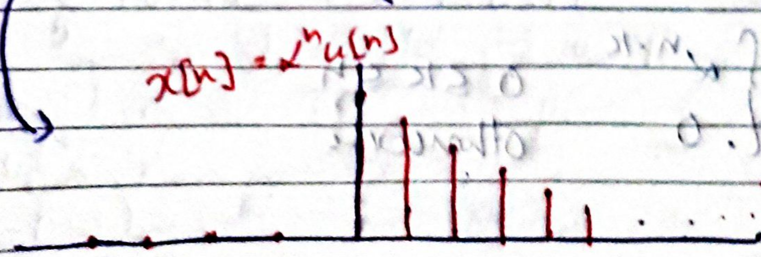


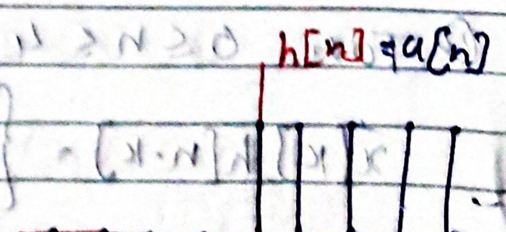


Q. $x[n] = \alpha^n u[n]$ $h[n] = u[n]$

$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$ for $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$

waveform





for $n \geq 0$

$$y[n] = \sum_{k=0}^{n} \alpha^{k}$$

$$= \frac{1 - \alpha^{n-1}}{1-\alpha} \quad \text{for } n \geq 0.$$

Thus for all n

$$y[n] = \left(\frac{1 - \alpha^{n+1}}{1-\alpha} \right) u[n]$$

Q.

$$x[n] = \begin{cases} 1 & 0 \leq n \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

$$h[n] = \begin{cases} \alpha^{n} & 0 \leq n \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

Sol ① For $n < 0$, there is no overlap between the non zero portions of $x[k]$ and $h[n-k]$

$\therefore y[n] = 0$

② For $0 \leq n \leq 4$

$$x[k]\, h[n-k] = \begin{cases} \alpha^{n-k} & 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}$$

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$$y[n] = \sum_{k=0}^{n} \alpha^{n-k}$$

By using finite sum formula, changing the variable of summation from k to $r = n-k$

$$y[n] = \sum_{r=0}^{n} \alpha^{r} = \frac{(1-\alpha^{n+1})}{1-\alpha}$$

③ For $n>4$, but $n-6 \le 0$. (i.e. $4<n\le 6$)

$$x[k]h[n-k] = \begin{cases} \alpha^{n-k} & 0 \le k \le 4 \\ 0 & \text{otherwise} \end{cases}$$

$$y[n] = \sum_{k=0}^{4} (\alpha^{n-k})$$

$$= \alpha^{n} \sum_{k=0}^{4} (\alpha^{-1})^{k} = \alpha^{n} \frac{1-(\alpha^{-1})^{5}}{1-\alpha^{-1}}$$

$$= \frac{\alpha^{n-4} - \alpha^{n+1}}{1-\alpha.}$$

④ $n>6$ but $n-6 \le 4$ $(6<n\le 10)$

$$x[k]h[n-k] = \begin{cases} \alpha^{n-k} & (n-6) \le k \le 4 \\ 0 & \text{otherwise} \end{cases}$$

$$y[n] = \sum_{k=n-6}^{4} \alpha^{n-k}.$$

Lets $r = k - n + 6$

$k = r + n - 6$

$\boxed{r=0}$

$k=4$

$r = 10-n.$

$$y[n] = \sum_{r=0}^{10-n} \alpha^{6-r} = \alpha^{6} \sum_{r=0}^{10-n} (\alpha^{-1})^{r}$$

Subject

$$= \alpha^{6} \frac{1-\alpha^{n-11}}{1-\alpha^{-1}} = \frac{\alpha^{n-4} - \alpha^{7}}{1-\alpha}$$

⑥ For $n-6 > 4$,

$n > 10$

there is no overlap between the non zero part of $x[k]$ & $h[n-k]$

$$y[n] = 0$$

Subject

Steps to find convolution Sum

1) Folding :- Fold the signal $h(k)$ about the origin i.e. at $k=0$
2) Shifting :- Shift $h(-k)$ to the right by n_0 if n_0 is +ve or shift $h(-k)$ to the left by n_0 is negative to obtain $h(n_0-k)$.
3) Multiplication :- Multiply $x(k)$ by $h(n_0-k)$ to obtain the product sequence
$$y_0(k) = x(k)\, h(n_0-k)$$
4) Summation :- Sum all the values of the product sequence $y_0(k)$ to obtain the value of the o/p at time $n=n_0$.

Circular Convolution :- (Periodic Convolution)

consider the two sequence $x_1(n)$ & $x_2(n)$ which are of finite duration. Let $X_1(k)$ & $X_2(k)$ be the N-point DFT of $x_3(k)$, ie

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2(m-n, (\text{mod } N)), \quad m = 0, 1, \ldots N-1$$

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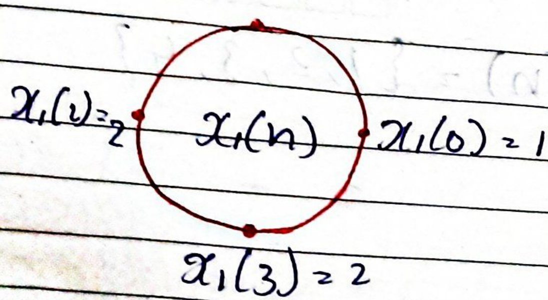
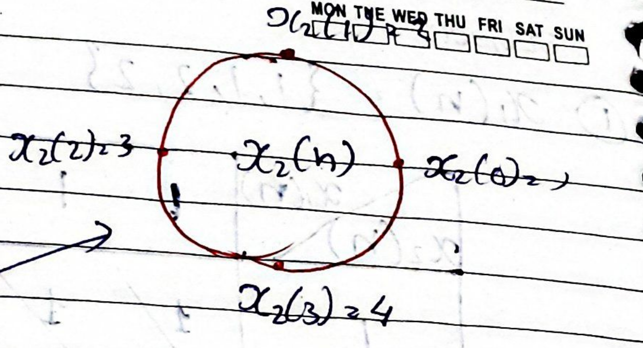
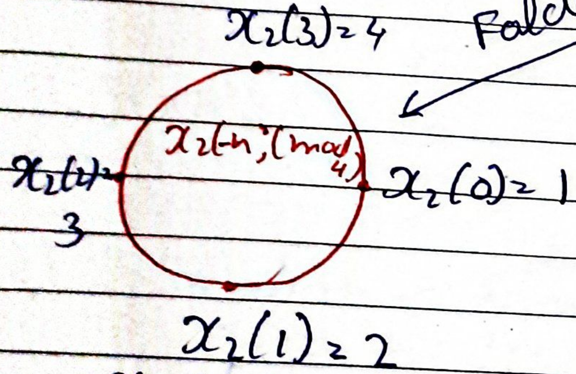




Folded seq.



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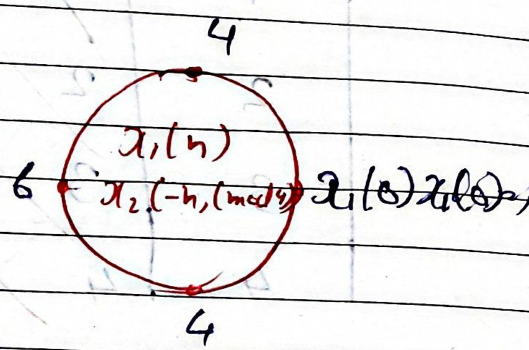
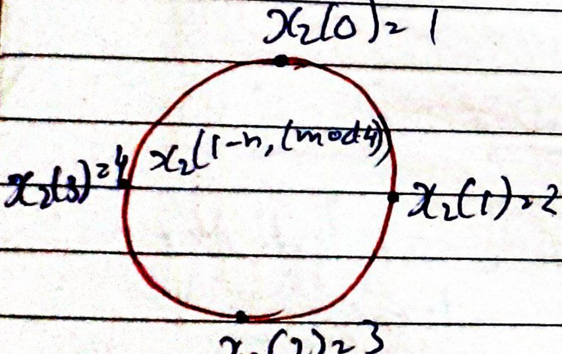
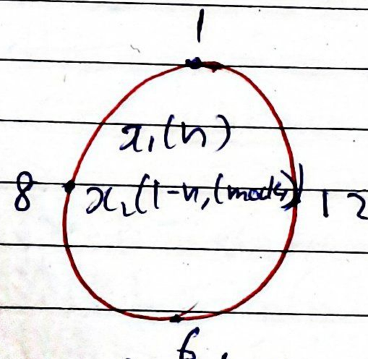


Product seq.

$=1+4+6+4=15$

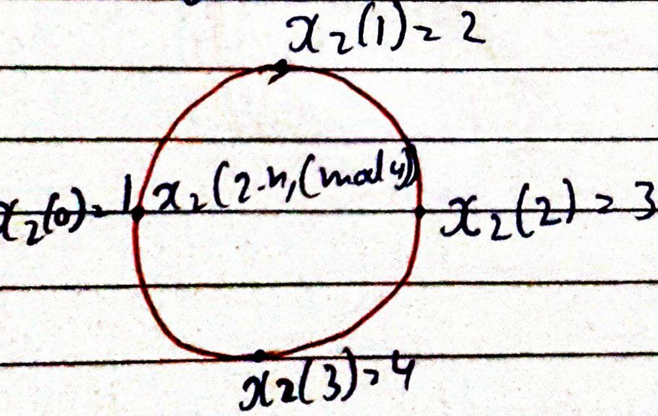


Folded seq. rot by one unit in time

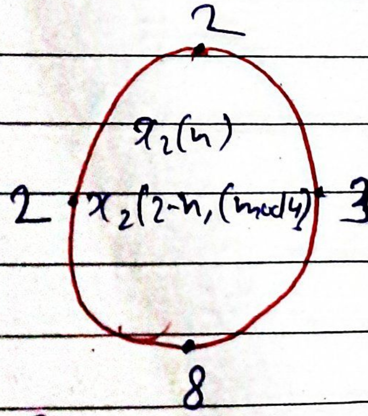


Product seq.

$=2+1+8+6=17$

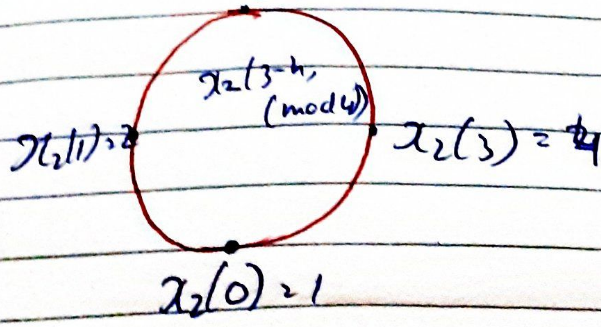


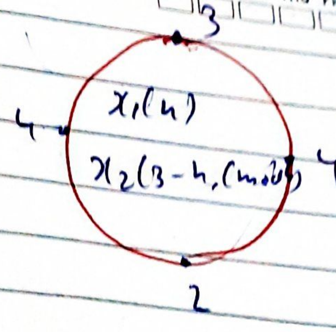
Folded seq rot by two unit in time



Product seq.

$=3+2+2+8=15$





Folded seq. rotated by 3 units

Product seq.
$= 4+3+4+2=13$

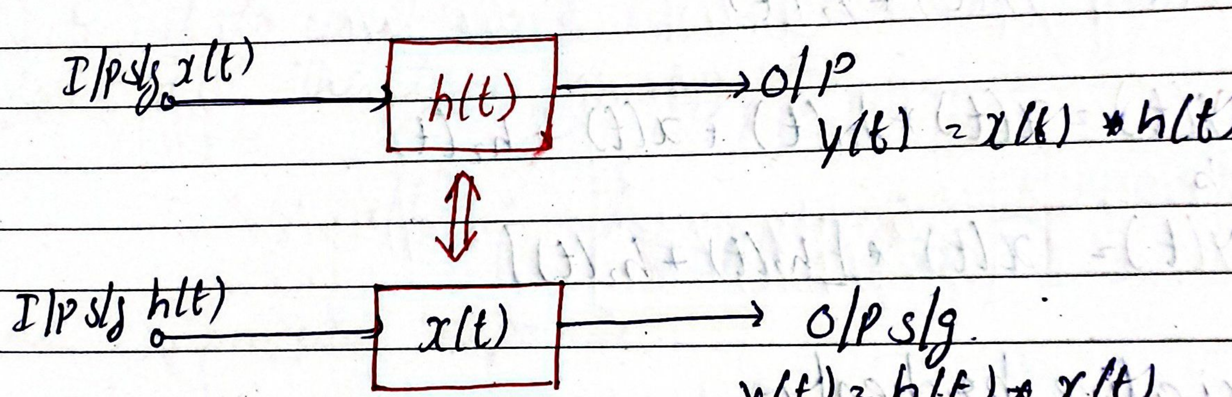
Subject

Properties of LTI system:-

① Commutative Property

$$y(t) = x(t) * h(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

or $$y(t) = h(t) * x(t) = \int_{\tau=-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$



② Distributive Property:-

For continuous time LTI system, distributive property is expressed as.

$$y(t) = x(t) * [h_1(t) + h_2(t)]$$

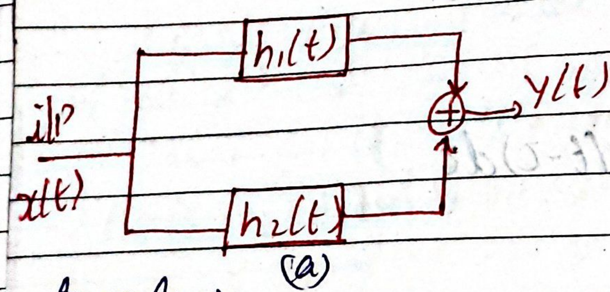
$$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$$

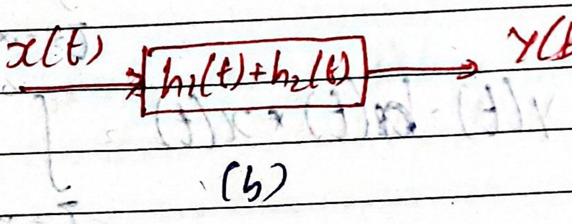
Two continuous time LTI systems, with impulse respon $h_1(t)$ & $h_2(t)$ have identical i/p & o/p are added as

Subject

$$y_1(t) = x(t) * h_1(t)$$

$$y_2(t) = x(t) * h_2(t)$$



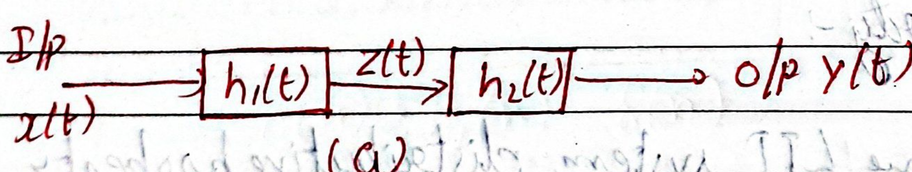


from fig a)
o/p $y(t) = y_1(t) + y_2(t)$

$$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$$

from fig b
o/p $y(t) = x(t) * [h_1(t) + h_2(t)]$

③ Associative property :-

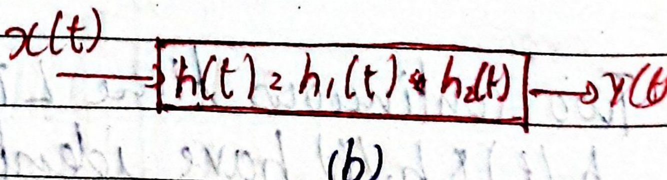


from fig a
o/p $y(t) = x(t) * [h_1(t) * h_2(t)]$

$$= [x(t) * h_1(t)] * h_2(t)$$

$$y(t) = x(t) * h_1(t) * h_2(t)$$

from fig B)



$$y(t) = z(t) * h_2(t)$$

But $z(t) = x(t) * h_1(t)$

$$y(t) = [x(t) * h_1(t)] * h_2(t)$$

$$y(t) = x(t) * h(t)$$

where $h(t) = h_1(t) * h_2(t)$

$$y(t) = x(t) * [h_1(t) * h_2(t)]$$

③ Static and Dynamic LTI system

A continuous-time LTI system is memoryless (static) if its unit-impulse respons $h(t)$ is zero for $t \neq 0$. This type of system is characterized by

$$y(t) = Kx(t) \quad \text{where } K \text{ is constant.}$$

and its impulse response.

$$h(t) = K\delta(t)$$

If $K=1$, then systems are called idenity system.

If the impulse response of a discrete-time LTI system is not identically zero for $n \neq 0$ or $t \neq 0$ then the system is called a dynamic system or system with memory.

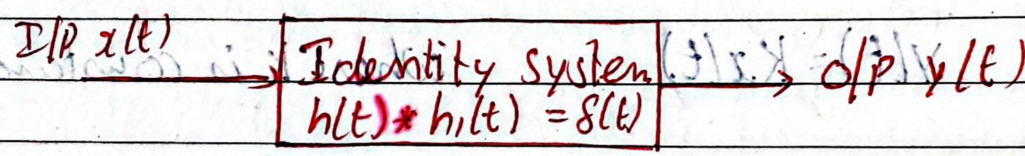
Subject

④ Invertibility of LTI system

A system is known as invertible only if an inverse system exists which, when cascaded (connected in series) with the original system, produces an o/p equal to the i/p at first system.

If an LTI system is invertible then it will have a LTI inverse system.

(a)



$$h(t) * h_1(t) = \delta(t)$$

$$h(n) * h_1(n) = \delta(n)$$

⑤ Causality for LTI system

A continuous LTI system is called causal system if its impulse response $h(t)$ is zero for $t < 0$.

$$y(t) = x(t) * h(t) = \int_{\tau=-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

$$= h(t) * x(t) = \int_{\tau=0} h(\tau)\, x(t-\tau)\, d\tau$$

A discrete time LTI system is known as causal system if its impulse response $h(n)$ is zero for $n < 0$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{n} x[k]\, h[n-k] =$$

$$= h[n] * x[n] = \sum_{k=0}^{\infty} h[k]\, x[n-k]$$

$$h(n) = u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Paley - Wiener Criterion :-

The modulus of the frequency response (or the amplitude response) is an even function of frequency where as the phase angle i.e. the phase response is an odd function of freq.

The paley wiener criterion gives the freq - domain equivalence for the causality condition of the time domain.

Necessary and sufficient condition for an amplitude response to be realizable is that

$$\int_{-\infty}^{\infty} \frac{\left|\ln |H(j\omega)|\right|}{1+\omega^2}\, d\omega < \infty$$

Subject

Or $\int_{-\infty}^{\infty} \frac{|\ln |H(f)||}{1+f^2} df < \infty$

Requirements on amplitude response $|H(j\omega)|$

① Magnitude $|H(j\omega)|$ of a realizable network can be zero at a discrete set of freq. but it cannot have zero magnitude over a finite band of freq. else Paley - Wiener criterion is invalid.

② The amp. function $|H(j\omega)|$ must not fall to zero faster than the exponential order.
Any ideal filter is always non-causal.

③ Stability for LTI system :-

A stable system is a system which produces bounded o/p for every bounded i/p. $|x(t)| < M$

$$y(t) = \int_{\tau=-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$|y(t)| = \left| \int_{\tau=-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \right| = \int_{\tau=-\infty}^{\infty} |h(\tau)|\, |x(t-\tau)|\, d\tau$$

Sub. value $|x(t-\tau)| < M$ for all values of τ & t,

$$|y(t)| \le \int_{\tau=-\infty}^{\infty} |h(\tau)| \cdot M \cdot d\tau$$

Subject

$$|y(t)| \le \int_{\tau=-\infty}^{\infty} |h(\tau)| \cdot d\tau \quad \text{for all values of } t$$

Necessary condition to stability of C.T. LTI system

$$\boxed{S = \int_{\tau=-\infty}^{\infty} |h(t)| d\tau < \infty}$$

For discrete time system

$$\boxed{S = \sum_{k=-\infty}^{\infty} |h(k)| < \infty}$$

if a system is causal, then the o/p is bounded if & only if $\sum_{k=0}^{\infty} |h(k)|$ is bounded

$$\boxed{S = \sum_{k=0}^{\infty} |h(k)| < \infty}$$

Unit step response of continuous time LTI system is found by convolution integral of $u(t)$ with unit impulse response $h(t)$ and is given by

$$g(t) = u(t) * h(t) = h(t) * u(t)$$

Numerical

Subject

1) $y[n] = x[n] + e^{\alpha} y(n-1)$

A discrete time system is characterized by the following difference eq mentioned above. check this system for BIBO stability.

Sol $y[n] = x[n] + e^{\alpha} y(n-1)$

If $x[n] = \delta[n]$, then $y[n] = h[n]$

impulse response of the system will be

$$h[n] = \delta[n] + e^{\alpha} h(n-1)$$

Now, when

$n = 0$, $h(0) = \delta(0) + e^{\alpha} \underbrace{h(-1)}_{\text{zero}} = 1$

$n = 1$, $h(1) = \delta(1) + e^{\alpha} \underbrace{h(0)}_{1} = e^{\alpha}$

$n = 2$, $h(2) = \delta(2) + e^{\alpha} \cdot h(1) = e^{2\alpha}$

Ily, $h(n) = e^{n\alpha}$

$$\boxed{\begin{array}{l}\text{Condition for BIBO stability.}\\ \displaystyle\sum_{k=0}^{\infty} |h(k)| < \infty\end{array}}$$

$= |1| + |e^{\alpha}| + |e^{2\alpha}| + \ldots + |e^{k\alpha}| + \ldots$

$$= \sum_{k=0}^{\infty} |e^{k\alpha}| = \left|\frac{1}{1-e^{\alpha}}\right|$$

$\therefore$ the given system is BIBO stable only when $e^{\alpha} < 1$ as $\alpha < 0$

Subject

Q check whether the following systems are BIBO stable or not.

① $y(n) = ax^2(n)$

Let $x(n) = \delta(n)$
$y(n) = h(n)$

$\therefore \quad h(n) = a\delta^2(n)$

when $n = 0 \quad h(0) = a\delta^2(0) = a$

$n = 1, \quad h(1) = a\delta^2(1) = 0$

$\vdots$

in general we have

$$h(n) = \begin{cases} a & \text{when } n = 0 \\ 0 & n \neq 0 \end{cases}$$

From the above eq we can say that

$$\sum_{k=0}^{\infty} |h(k)| < \infty$$

$$= |h(0)| + |h(1)| + |h(2)| + \cdots + |h(k)| + \cdots$$

$$= |a|$$

$\therefore$ Given system is BIBO stable only if $a < \infty$

Subject

(2) $y(n) = a x(n) + b$

Let $x(n) = \delta(n)$
$y(n) = h(n)$

$\therefore h(n) = a\delta(n) + b$

when
$n=0, \; h(0) = a\delta(0) + b = a+b$
$n=1, \; h(1) = a\delta(1) + b = b$

Here $h(1) = h(2) = \dots = h(k) = b$

$\therefore h(n) = \begin{cases} a+b & \text{when } n=0 \\ b & n \neq 0 \end{cases}$

$\sum_{k=0}^{\infty} |h(k)| < \infty$

$= |h(0)| + |h(1)| + |h(2)| + \dots + |h(k)| + \dots$

$= |a+b| + |b| + |b| + \dots + |b| + \dots$

If we calculate the ratio between successive terms it is 1.

$\therefore$ Hence system is BIBO unstable

Subject

Date ___/___/___

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② $y(n) = a x(n) + b$

Let $x(n) = \delta(n)$

$y(n) = h(n)$

$\therefore h(n) = a\delta(n) + b$

when

$n = 0, \; h(0) = a\delta(0) + b = a + b$

$n = 1, \; h(1) = a\delta(1) + b = b$

Here $h(1) = h(2) = \dots = h(k) = b$

$$\therefore h(n) = \begin{cases} a+b & \text{when } n = 0 \\ b & n \neq 0 \end{cases}$$

$$\sum_{k=0}^{\infty} |h(k)| < \infty$$

$= |h(0)| + |h(1)| + |h(2)| + \dots + |h(k)| + \dots$

$= |a+b| + |b| + |b| + \dots + |b| + \dots$

If we calculate the ratio between successive terms it is 1.

$\therefore$ Hence system is BIBO unstable