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The Z-Transform

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6.1 Introduction

The Z-transform is a powerful method for solving difference equations and, in general, to represent discrete systems. Although applications of Z-transforms are relatively new, the essential features of this mathematical technique date back to the early 1730s when DeMoivre introduced the concept of a generating function that is identical with that for the Z-transform. Recently, the development and extensive applications of the Z-transform are much enhanced as a result of the use of digital computers.

A. One-Sided Z-Transform

6.2 The Z-Transform and Discrete Functions

Let $f(t)$ be defined for $t \geq 0$. The Z-transform of the sequence $\{f(nT)\}$ is given by

$$Z\{f(nT)\} \doteq F(z) = \sum_{n=0}^{\infty} f(nT) z^{-n} \tag{6.2.1}$$

where T, the sampling time, is a positive number.[1]

To find the values of z for which the series converges, we use the ratio test or the root test. The ratio test states that a series of complex numbers

$$\sum_{n=0}^{\infty} a_n$$

with limit

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = A \tag{6.2.2}$$

converges absolutely if $A < 1$ and diverges if $A > 1$ the series may or may not converge.

The root test states that if

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = A \tag{6.2.3}$$

then the series converges absolutely if $A < 1$, and diverges if $A > 1$, and may converge or diverge if $A = 1$.

More generally, the series converges absolutely if

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} < 1 \tag{6.2.4}$$

where $\overline{\lim}$ denotes the **greatest** limit points of $\overline{\lim\limits_{n\to\infty}} |f(nT)|^{1/n}$, and diverges if

[1]The symbol $\doteq$ means equal by definition.

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} > 1 \tag{6.2.5}$$

If we apply the root test in (2.1) we obtain the convergence condition

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\left|f(nT)z^{-n}\right|} = \overline{\lim_{n\to\infty}} \sqrt[n]{\left|f(nT)\right|\left|z^{-1}\right|^n} < 1$$

or

$$|z| > \overline{\lim_{n\to\infty}} \sqrt[n]{\left|f(nT)\right|} = R \tag{6.2.6}$$

where R is known as the **radius of convergence** for the series. Therefore, the series will converge absolutely for all points in the z-plane that lie **outside** the circle of radius R, and is centered at the origin (with the possible exception of the point at infinity). This region is called the **region of convergence** (ROC).

Example

The radius of convergence of $f(nT) = e^{-anT}u(nT)$, a positive number, is

$$\left|z^{-1}e^{-aT}\right| < 1 \quad \text{or } |z| > e^{-aT}$$

The Z-transform of $f(nT) = e^{-anT}u(nT)$ is

$$F(z) = \sum_{n=0}^{\infty} f(nT)z^{-n} = \sum_{n=0}^{\infty}\left(e^{-aT}z^{-1}\right)^n = \frac{1}{1-e^{-aT}z^{-1}}$$

If $a = 0$

$$F(z) = \sum_{n=0}^{\infty} u(nT)z^{-n} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

Example

The function $f(nT) = a^{nT}\cos nT\omega\, u(nT)$ has the Z-transform

$$F(z) = \sum_{n=0}^{\infty} a^{nT}\frac{e^{jnT\omega}+e^{-jnT\omega}}{2}z^{-n} = \frac{1}{2}\sum_{n=0}^{\infty}\left(a^T e^{jT\omega}z^{-1}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(a^T e^{-jT\omega}z^{-1}\right)^n$$

$$= \frac{1}{2}\frac{1}{1-a^T e^{jT\omega}z^{-1}} + \frac{1}{2}\frac{1}{1-a^T e^{-jT\omega}z^{-1}} = \frac{1-a^T z^{-1}\cos T\omega}{1-2a^T z^{-1}\cos T\omega + a^{2T}z^{-2}}.$$

The ROC is given by the relations

$$\left|a^T e^{jT\omega}z^{-1}\right| < 1 \qquad \text{or} \qquad |z| > \left|a^T\right|$$

$$\left|a^T e^{-jT\omega}z^{-1}\right| < 1 \qquad \text{or} \qquad |z| > \left|a^T\right|$$

Therefore, the ROC is $|z| > \left|a^T\right|$.

6.3 Properties of the Z-Transform

Linearity

If there exists transforms of sequences $\mathcal{Z}\{c_i f_i(nT)\} = c_i F_i(z)$, c_i are complex constants, with radii of convergence $R_i > 0$ for $i = 0, 1, 2, \ldots, \ell(\ell$ finite), then

$$\mathcal{Z}\left\{\sum_{i=0}^{\ell} c_i f_i(nT)\right\} = \sum_{i=0}^{\ell} c_i F_i(z) \qquad |z| > \max R_i \tag{6.3.1}$$

Shifting Property

$$\mathcal{Z}\{f(nT - kT)\} = z^{-k}F(z), \qquad f(-nT) = 0 \qquad n = 1, 2, \ldots \tag{6.3.2}$$

$$\mathcal{Z}\{f(nT - kT)\} = z^{-k}F(z) + \sum_{n=1}^{k} f(-nT)\, z^{-(k-n)} \tag{6.3.3}$$

$$\mathcal{Z}\{f(nT + kT)\} = z^{k}F(z) - \sum_{n=0}^{k-1} f(nT)\, z^{k-n} \tag{6.3.4}$$

$$\mathcal{Z}\{f(nT + T)\} = z[F(z) - f(0)] \tag{6.3.4a}$$

Example

To find the Z-transform of $y(nT)$ we proceed as follows:

$$\frac{d^2y(t)}{dt^2} = x(t), \quad \frac{y(nT) - 2y(nT - T) + y(nT - 2T)}{T^2} = x(nT),$$

$$Y(z) - 2\left[z^{-1}Y(z) + y(-T)z^{-0}\right] + z^{-2}Y(z) + y(-T)z^{-1} + y(-2T)z^{-0} = X(z)T^2$$

or

$$Y(z) = \frac{2y(-T) - y(-T)z^{-1} - y(-2T) + X(z)T^2}{1 - 2z^{-1} + z^{-2}}$$

Time Scaling

$$\mathcal{Z}\{a^{nT} f(nT)\} = F(a^{-T}z) = \sum_{n=0}^{\infty} f(nT)\left(a^{-T}z\right)^{-n} \tag{6.3.5}$$

Example

$$\mathcal{Z}\left\{\sin\omega nTu(nT)\right\} = \frac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1} \qquad |z| > 1,$$

$$\mathcal{Z}\left\{e^{-n}\sin\omega nTu(nT)\right\} = \frac{e^{+1}z\sin\omega T}{e^{+2}z^2 - 2e^{+1}z\cos\omega T + 1} \qquad |z| > e^{-1}$$

Periodic Sequence

$$\mathcal{Z}\left\{f(nT)\right\} = \frac{z^N}{z^N - 1}\mathcal{Z}\left\{f_1(nT)\right\} = \frac{z^N}{z^N - 1}F_1(z), \quad f_1(nT) = \text{first period} \tag{6.3.6}$$

N is the number of time units in a period, $|z| > R$

where R is the radius of convergence of $F_1(z)$.

Proof

$$\begin{aligned}\mathcal{Z}\left\{f(nT)\right\} &= \mathcal{Z}\left\{f_1(nT)\right\} + \mathcal{Z}\left\{f_1(nT - NT)\right\} + \mathcal{Z}\left\{f_1(nT - 2NT)\right\} + \cdots \\ &= F_1(z) + z^{-N}F_1(z) + z^{-2N}F_1(z) + \cdots \\ &= F_1(z)\frac{1}{1 - z^{-N}} = \frac{z^N}{z^N - 1}F_1(z)\end{aligned}$$

For finite sequence of K terms

$$F(z) = F_1(z)\frac{1 - z^{-N(K+1)}}{1 - z^{-N}} \tag{6.3.6a}$$

Multiplication by *n* and *nT*

R is the radius of convergence of $F(z)$

$$\mathcal{Z}\left\{nf(nT)\right\} = -z\frac{dF(z)}{dz} \tag{6.3.7}$$

$$\mathcal{Z}\left\{nTf(nT)\right\} = -Tz\frac{dF(z)}{dz} \qquad |z| > R$$

Proof

$$\sum_{n=0}^{\infty} nT(nT)z^{-n} = Tz\sum_{n=0}^{\infty} f(nT)\left[-\frac{d}{dz}z^{-n}\right] = -Tz\frac{d}{dz}\left[\sum_{n=0}^{\infty} f(nT)z^{-n}\right]$$

$$= -Tz\frac{dF(z)}{dz}$$

Example

$$\mathcal{Z}\{u(n)\} = \frac{z}{z-1}, \quad \mathcal{Z}\{nu(n)\} = -z\frac{d}{dz}\left(\frac{z}{z-1}\right) = \frac{z}{(z-1)^2},$$

$$\mathcal{Z}\{n^2u(n)\} = -z\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right) = \frac{z(z^2-1)}{(z-1)^4}$$

Convolution

If $\mathcal{Z}\{f(nT)\} = F(z)\ |z| > R_1$ and $\mathcal{Z}\{h(nT)\} = H(z)\ |v| > R_2$, then

$$\mathcal{Z}\{f(nT) * h(nT)\} = \mathcal{Z}\left\{\sum_{m=0}^{\infty} f(mT)h(nT-mT)\right\} = F(z)H(z) \qquad |z| > \max(R_1,\ R_2) \qquad (6.3.8)$$

Proof

$$\mathcal{Z}\{f(nT) * h(nT)\} = \sum_{n=0}^{\infty}\left[\sum_{m=0}^{\infty} f(mT)h(nT-mT)\right]z^{-n}$$

$$= \sum_{m=0}^{\infty} f(mT)\sum_{n=0}^{\infty} h(nT-mT)\, z^{-n}$$

$$= \sum_{m=0}^{\infty} f(mT)\sum_{r=-m}^{\infty} h(rT)\, z^{-r}z^{-m}$$

$$= \sum_{m=0}^{\infty} f(mT)z^{-m}\sum_{r=0}^{\infty} h(rT)\, z^{-r} = F(z)H(z).$$

The value of $h(nT)$ for $n < 0$ is zero.

Additional relations of convolution are

$$\mathcal{Z}\{f(nT) * h(nT)\} = F(z)\,H(z) = \mathcal{Z}\{h(nT) * f(nT)\} = F(z)\,H(z) \qquad (6.3.8a)$$

$$\mathcal{Z}\{\{f(nT) + h(nT)\} * \{g(nT)\}\} = \mathcal{Z}\{f(nT) * g(nT)\} + Z\{h(nT) * g(nT)\}$$

$$= F(z)\ G(z) + H(z)\ G(z) \qquad (6.3.8b)$$

$$\mathcal{Z}\{\{f(nT) * h(nT)\} * g(nT)\} = \mathcal{Z}\{f(nT) * \{h(nT) * g(nT)\}\} = F(z)H(z)G(z) \quad (6.3.8c)$$

Example

The Z-transform of the output of the discrete system $y(n) = \frac{1}{2}y(n-1) + \frac{1}{2}x(n)$, when the input is the unit step function $u(n)$ given by $Y(z) = H(z)U(z)$. The Z-transform of the difference equation with a delta function input $\delta(n)$ is

$$H(z) - \frac{1}{2}z^{-1}H(z) = \frac{1}{2} \quad \text{or} \quad H(z) = \frac{1}{2}\frac{1}{1-\frac{1}{2}z^{-1}} = \frac{1}{2}\frac{z}{z-\frac{1}{2}}$$

Therefore, the output is given by

$$Y(z) = \frac{1}{2}\frac{z}{z-\frac{1}{2}}\frac{z}{z-1}$$

Example

Find the $f(n)$ if

$$F(z) = \frac{z^2}{\left(z-e^{-a}\right)\left(z-e^{-b}\right)} \qquad a,\ b \text{ are constants.}$$

From this equation we obtain

$$f_1(n) = \mathcal{Z}^{-1}\left\{\frac{z}{z-e^{-a}}\right\} = e^{-an}, \quad f_2(n) = \mathcal{Z}^{-1}\left\{\frac{z}{\left(z-e^{-b}\right)}\right\} = e^{-bn}$$

Therefore,

$$f(n) = f_1(n) * f_2(n) = \sum_{m=0}^{n} e^{-am}e^{-b(n-m)} = e^{-bn}\sum_{m=0}^{n} e^{-(a-b)m}$$

$$= e^{-bn}\frac{1-e^{-(a-b)(n+1)}}{1-e^{-(a-b)}}$$

Initial Value

$$f(0) = \lim_{z\to\infty} F(z) \quad (6.3.9)$$

The above value is obtained from the definition of the Z-transform. If $f(0) = 0$, we obtain $f(1)$ as the limit

$$\lim_{z\to\infty} zF(z) \quad (6.3.9a)$$

Final Value

$$\lim_{n\to\infty} f(n) = \lim_{z\to 1}(z-1)F(z) \qquad \text{if } f(\infty) \text{ exists} \qquad (6.3.10)$$

Proof

$$\mathcal{Z}\{f(k+1)-f(k)\} = \lim_{n\to\infty}\sum_{k=0}^{n}\left[f[(k+1)]-f(k)\right]z^{-k}$$

$$zF(z)-zf(0)-F(z) = (z-1)F(z)-zf(0) = \lim_{n\to\infty}\sum_{k=0}^{n}\left[f[(k+1)]-f(k)\right]z^{-k}$$

By taking the limit as $z \to 1$, the above equation becomes

$$\begin{aligned}\lim_{z\to 1}(z-1)F(z)-f(0) &= \lim_{n\to\infty}\sum_{k=0}^{n}\left[f[(k+1)]-f(k)\right]\\ &= \lim_{n\to\infty}\{f(1)-f(0)+f(2)-f(1)+\cdots\\ &\quad +f(n)-f(n-1)+f(n+1)-f(n)\}\\ &= \lim_{n\to\infty}\{-f(0)+f(n+1)\}\\ &= -f(0)+f(\infty)\end{aligned}$$

which is the required result.

Example

If $F(z) = 1/[(1-z^{-1})(1-e^{-1}z^{-1})]$ with $|z| > 1$ then

$$f(0) = \lim_{z\to\infty}F(z) = \frac{1}{\left(1-\frac{1}{\infty}\right)\left(1-e^{-1}\frac{1}{\infty}\right)} = 1$$

$$\lim_{n\to\infty}f(n) = \lim_{z\to 1}(z-1)\frac{1}{(1-z^{-1})(1-e^{-1}z^{-1})} = \lim_{z\to 1}\frac{z^2}{(z-e^{-1})} = \frac{1}{(1-e^{-1})}$$

Multiplication by $(nT)^k$

$$\mathcal{Z}\{n^k T^k f(nT)\} = -Tz\frac{d}{dz}\mathcal{Z}\{(nT)^{k-1}f(nT)\} \qquad k > 0 \text{ and is an integer} \qquad (6.3.11)$$

As a corollary to this theorem, we can deduce

$$\mathcal{Z}\{n^{(k)}f(n)\} = z^{-k}\frac{d^kF(z)}{d(z^{-1})^k}, \quad n^{(k)} = n(n-1)(n-2)\cdots(n-k+1) \qquad (6.3.11a)$$

The following relations are also true:

$$\mathcal{Z}\left\{(-1)^k n^{(k)} f(n-k+1)\right\} = z\frac{d^k F(z)}{dz^k} \tag{6.3.11b}$$

$$\mathcal{Z}\left\{n(n+1)(n+2)\cdots(n+k-1)f(n)\right\} = (-1)^k z^k \frac{d^k F(z)}{dz^k} \tag{6.3.11c}$$

Example

$$\mathcal{Z}\{n\} = -z\frac{d}{dz}\left(\frac{z}{z-1}\right) = \frac{z}{(z-1)^2},$$

$$\mathcal{Z}\{n^2\} = -z\frac{d}{dz}\mathcal{Z}\{n\} = -z\frac{d}{dz}\frac{z}{(z-1)^2} = \frac{z(z+1)}{(z-1)^3},$$

$$\mathcal{Z}\{n^3\} = -z\frac{d}{dz}\frac{z(z+1)}{(z-1)^3} = \frac{z(z^2+4z+1)}{(z-1)^4}$$

Initial Value of $f(nT)$

$$\mathcal{Z}\{f(nT)\} = f(0T) + f(T)z^{-1} + f(2T)z^{-2} + \cdots = F(z)$$

$$f(0T) = \lim_{z\to\infty} F(z) \qquad |z| > R \tag{6.3.12}$$

Final Value for $f(nT)$

$$\lim_{n\to\infty} f(nT) = \lim_{z\to 1}(z-1)F(z) \qquad f(\infty T) \text{ exists} \tag{6.3.13}$$

Example

For the function

$$F(z) = \frac{1}{(1-z^{-1})(1-e^{-T}z^{-1})} \qquad |z| > 1$$

we obtain

$$f(0T) = \lim_{z\to\infty} F(z) = \frac{1}{\left(1-\frac{1}{\infty}\right)\left(1-\frac{e^{-T}}{\infty}\right)} = 1$$

$$\lim_{n\to\infty} f(nT) = \lim_{z\to 1}(z-1)\frac{z}{z-1}\,\frac{z}{1-e^{-T}} = \frac{1}{1-e^{-T}}$$

Complex Conjugate Signal

$$F(z) = \sum_{n=0}^{\infty} f(nT) z^{-n} \qquad |z| > R \quad \text{or} \qquad F(z^*) = \sum_{n=0}^{\infty} f(nT)(z^*)^{-n}$$

or

$$F^*(z^*) = \sum_{n=0}^{\infty} f^*(nT) z^{-n} = \mathcal{Z}\{f^*(nT)\}$$

Hence,

$$\mathcal{Z}\{f^*(nT)\} = F^*(z^*) \qquad |z| > R \tag{6.3.14}$$

Transform of Product

If

$$\mathcal{Z}\{f(nT)\} = F(z) \qquad |z| > R_f$$

$$\mathcal{Z}\{h(nT)\} = H(z) \qquad |z| > R_h$$

then

$$\begin{aligned}\mathcal{Z}\{g(nT)\} &\doteq \mathcal{Z}\{f(nT)h(nT)\} \\ &= \sum_{n=0}^{\infty} f(nT)h(nT)z^{-n} \\ &= \frac{1}{2\pi j}\oint_C F(\tau)H\left(\frac{z}{\tau}\right)\frac{d\tau}{\tau} \qquad |z| > R_j R_h\end{aligned} \tag{6.3.15}$$

where C is a simple contour encircling counterclockwise the origin with (see Figure 6.3.1)

$$R_f < |\tau| < \frac{|z|}{R_h} \tag{6.3.15a}$$

Proof
The integration is performed in the positive sense along the circle, inside which lie all the singular points of the function $F(\tau)$ and outside which lie all the singular points of the function $H(z/\tau)$. From (6.3.15), we write

$$G(z) = \frac{1}{2\pi j}\oint_C F(\tau)\sum_{n=0}^{\infty} h(nT)\left(\frac{z}{\tau}\right)^{-n}\frac{d\tau}{\tau} \tag{6.3.16}$$

which converges uniformly for some choice of contour C and values of z. From (6.3.16), we must have

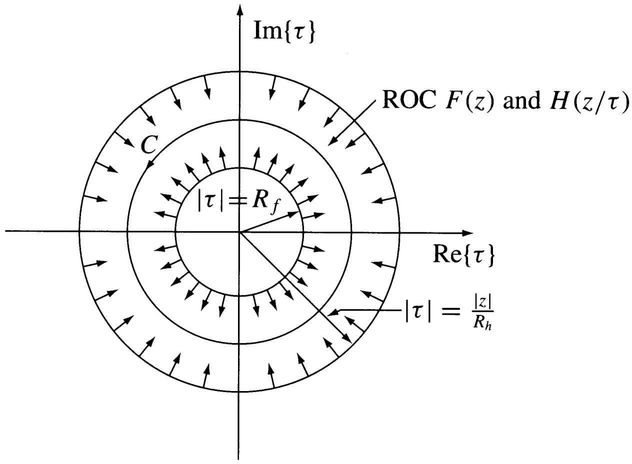

FIGURE 6.3.1

$$\left| R_h \left(\frac{z}{\tau} \right)^{-1} \right| < 1 \quad \text{or} \quad \left| \frac{z}{\tau} \right| > R_h \quad \text{or} \quad |\tau| < \frac{|z|}{R_h} \tag{6.3.17}$$

so that the sum in (6.3.16) converges. Because $|z| > R_f$ and τ takes the place of z, then (6.3.16) implies that

$$|\tau| > R_f \tag{6.3.18}$$

$$R_f < |\tau| < \frac{|z|}{R_h} \tag{6.3.19}$$

and also

$$R_f R_h < |z|.$$

Figure 6.3.1 shows the region of convergence.

The integral is solved with the aid of the residue theorem, which yields in this case

$$G(z) = \sum_{i=1}^{K} \text{res}_{\tau=\tau_i} \left\{ \frac{F(\tau) H(z/\tau)}{\tau} \right\} \tag{6.3.20}$$

where K is the number of different poles $\tau_i (i = 1, 2, \ldots, K)$ of the function $F(\tau)/\tau$. For the residue at the pole τ_i of multiplicity m of the function $F(\tau)/\tau$, we have

$$\text{res}_{\tau=\tau_i} \left\{ \frac{F(\tau) H(z/\tau)}{\tau} \right\} = \frac{1}{(m-1)!} \lim_{\tau \to \tau_i} \frac{d^{m-1}}{d\tau^{m-1}} \left[(\tau - \tau_i)^m \frac{F(\tau) H\left(\frac{z}{\tau}\right)}{\tau} \right] \tag{6.3.21}$$

Hence, for a simple pole, $m = 1$, we obtain

$$\operatorname{res}_{\tau=\tau_i}\left\{\frac{F(\tau)H(z/\tau)}{\tau}\right\} = \lim_{\tau\to\tau_i}(\tau-\tau_i)\left\{\frac{F(\tau)H\left(\frac{z}{\tau}\right)}{\tau}\right\} \tag{6.3.22}$$

Example

See Figure 6.3.2 for graphical representation of the complex integration.

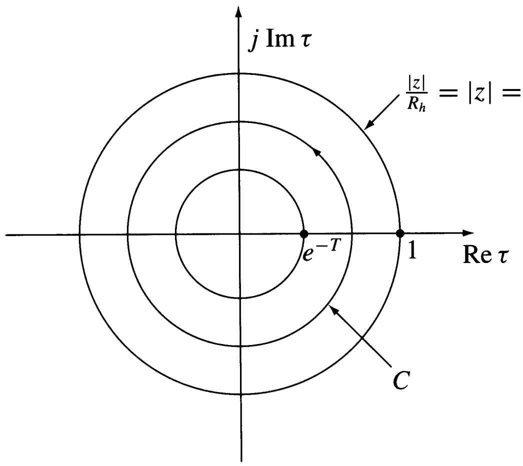

FIGURE 6.3.2

$$Z\{nT\} \doteq H(z) = \frac{z}{(z-1)^2}T \qquad |z| > 1, \qquad Z\{e^{-nT}\} \doteq F(z) = \frac{z}{z-e^{-T}} \quad |z| > e^{-T}$$

Hence,

$$Z\{nTe^{-nT}\} = \frac{1}{2\pi j}\oint_C T\frac{z}{\tau(\tau-e^{-T})\left(\frac{z}{\tau}-1\right)^2}d\tau .$$

The contour must have a radius $|\tau|$ of the value $e^{-T} < |\tau| < |z| = 1$ and we have from (6.3.22)

$$Z\{nTe^{-nT}\} = \operatorname{res}_{\tau=e^{-T}}\left\{(\tau-e^{-T})T\frac{z\tau}{(\tau-e^{-T})(z-\tau)^2}\right\} = T\frac{ze^{-T}}{(z-e^{-T})^2}$$

From (6.3.11)

$$Z\{nTe^{-nT}\} = -Tz\frac{d}{dz}\left(\frac{1}{1-e^{-T}z^{-1}}\right) = T\frac{ze^{-T}}{(z-e^{-T})^2}$$

and verifies the complex integration approach.

Parseval's Theorem

If $\mathcal{Z}\{f(nT)\} = F(z), |z| > R_f$ and $\mathcal{Z}\{h(nT)\} = H(z), |z| > R_h$ with $|z| = 1 > R_f R_h$, then

$$\sum_{n=0}^{\infty} f(nT)h(nT) = \frac{1}{2\pi j}\oint_C F(z)H(z^{-1})\frac{dz}{z} \tag{6.3.23}$$

where the contour is taken counterclockwise.

Proof

From (6.3.15) set $z = 1$ and change the dummy variable τ to z.

Example

$f(nT) = e^{-nT}u(nT)$ has the following Z-transform:

$$F(z) = \frac{1}{1-e^{-T}z^{-1}} \qquad |z| > e^{-T}$$

From (6.3.23) and with C a unit circle ($R_f = e^{-T} < 1$)

$$\sum_{n=0}^{\infty} f(nT)f(nT) = \frac{1}{2\pi j}\oint_C \frac{1}{1-e^{-T}z^{-1}}\frac{1}{1-e^{-T}z}\frac{dz}{z} = \frac{1}{2\pi j}\oint_C \frac{1}{z-e^{-T}}\frac{e^T}{e^T-z}dz$$

$$= \frac{2\pi j}{2\pi j}\sum_i \text{residues} = \frac{e^T}{e^T-e^{-T}} = \frac{1}{1-e^{-2T}}$$

Correlation

Let the Z-transform of the two consequences $\mathcal{Z}\{f(nT)\} = F(z)$ and $\mathcal{Z}\{h(nT)\} = H(z)$ exist for $|z| = 1$. Then the **cross correlation** is given by

$$g(nT) \doteq f(nT)\otimes h(nT) = \sum_{m=0}^{\infty} f(mT)h(mT-nT) = \lim_{z\to 1+}\sum_{m=0}^{\infty} f(mT)h(mT-nT)z^{-m}$$

$$= \lim_{z\to 1+} \mathcal{Z}\{f(mT)h(mT-nT)\}$$

But $\mathcal{Z}\{h(mT-nT)\} = z^{-n}H(z)$ and, therefore, (see [6.3.15])

$$\begin{aligned} g(nT) &= \lim_{z\to 1+}\frac{1}{2\pi j}\oint_C F(\tau)\left(\frac{z}{\tau}\right)^{-n} H\left(\frac{z}{\tau}\right)\frac{d\tau}{\tau} \\ &= \frac{1}{2\pi j}\oint_C F(\tau)H\left(\frac{1}{\tau}\right)\tau^{n-1}dt \qquad n \ge 1 \end{aligned} \tag{6.3.24}$$

This relation is the inverse Z-transform of $g(nT)$ and, hence,

$$\mathcal{Z}\{g(nT)\} \doteq \mathcal{Z}\{f(nT)\otimes h(nT)\} = F(z)H\left(\frac{1}{z}\right) \quad \text{for } |z| = 1 \tag{6.3.25}$$

If $f(nT) = h(nT)$ for $n \geq 0$ the **autocorrelation sequence** is

$$\begin{aligned} g(nT) &\doteq f(nT) \otimes h(nT) \\ &= \sum_{m=0}^{\infty} f(mT) f(mT - nT) \\ &= \frac{1}{2\pi j} \oint_C F(\tau) F\left(\frac{1}{\tau}\right) \tau^{n-1} d\tau \end{aligned} \tag{6.3.26}$$

and, hence,

$$G(z) = \mathcal{Z}\{g(nT)\} = \mathcal{Z}\{f(nT) \otimes h(nT)\} = F(z) F\left(\frac{1}{z}\right) \tag{6.3.27}$$

If we set $n = 0$, we obtain the Parseval's theorem in the same form it was developed above.

Example

The sequence $f(t) = e^{-nT}$, $n \geq 0$, has the Z-transform

$$\mathcal{Z}\{e^{-nT}\} = \frac{z}{z - e^{-T}} \qquad\qquad |z| > e^{-T}$$

The autocorrelation is given by (6.3.26) in the form

$$G(z) \doteq \mathcal{Z}\{f(nT) \otimes f(nT)\} = \frac{z}{z - e^{-T}} \frac{\frac{1}{z}}{\frac{1}{z} - e^{-T}} = -\frac{z}{z - e^{-T}} \frac{e^{T}}{z - e^{T}}$$

The function is regular in the region $e^{-T} < |z| < e^{T}$. Using the residue theorem from (6.3.24), we obtain

$$g(nT) = \sum_{i=1}^{K} \operatorname{res}_{\tau=\tau_i} \left\{ F(\tau) H\left(\frac{1}{\tau}\right) \right\} \tau^{n-1} \tag{6.3.28}$$

where τ_i are all poles of the integrand inside the circle $|\tau| = 1$. Similarly from (6.3.27)

$$g(nT) = \sum_{i=1}^{K} \operatorname{res}_{\tau=\tau_i} \left\{ F(\tau) F\left(\frac{1}{\tau}\right) \tau^{n-1} \right\} \tag{6.3.29}$$

where τ_i are the poles included inside the unit circle.

Example

From the previous example we obtain (only the root inside the unit circle)

$$-\frac{1}{2\pi j} \oint_C \frac{z}{z - e^{-T}} \frac{e^{T}}{z - e^{T}} z^{n-1} dz = -\operatorname{res}_{z=e^{-T}} \left\{ \frac{z e^{T}}{z - e^{T}} z^{n-1} \right\} = \frac{e^{2T}}{e^{2T} - 1} e^{-Tn}$$

which is equal to the autocorrelation of $f(nT) = e^{-nT}u(nT)$. Using the summation definitions, we obtain

$$\sum_{m=0}^{\infty} e^{-mT}u(mT)e^{-T(m-n)}u(mT-nT) = e^{Tn}\sum_{m=n}^{\infty} e^{-2mT}$$

$$= e^{Tn}\left(e^{-2nT} + e^{-2nT}e^{-2T} + e^{-2nT}e^{-4T} + \cdots\right)$$

$$= e^{-nT}\left(1 + e^{-2T} + \left(e^{-2T}\right)^2 + \cdots\right)$$

$$= e^{-nT}\frac{1}{1-e^{-2T}} = e^{-nT}\frac{e^{2T}}{e^{2T}-1}$$

Z-Transforms with Parameters

$$\mathcal{Z}\left\{\frac{\partial}{\partial a}f(nT,a)\right\} = \frac{\partial}{\partial a}F(z,a) \tag{6.3.30}$$

$$\mathcal{Z}\left\{\lim_{a\to a_0} f(nT,a)\right\} = \lim_{a\to a_0} F(z,a) \tag{6.3.31}$$

$$\mathcal{Z}\left\{\int_{a_0}^{a_1} f(nT,a)\,da\right\} = \int_{a_0}^{a_1} F(z,a)\,da \qquad \text{finite integral} \tag{6.3.32}$$

Table 1 in the Appendix contains the Z-transform properties for positive-time sequences.

6.4 Inverse Z-Transform

The **inverse Z-transform** provides the object function from its given transform. We use the symbolic solution

$$f(nT) = \mathcal{Z}^{-1}\{F(z)\} \tag{6.4.1}$$

To find the inverse transform, we may proceed as follows:

1. Use tables.
2. Decompose the expression into simpler partial forms, which are included in the tables.
3. If the transform is decomposed into a product of partial sums, the resulting object function is obtained as the convolution of the partial object function.
4. Use the inversion integral.

Power Series Method

When $F(z)$ is analytic for $|z| > R$ (and at $z = \infty$), the value of $f(nT)$ is obtained as the coefficient of z^{-n} in the power series expansion (Taylor's series of $F(z)$ as a function of z^{-1}). For example, if $F(z)$ is the ratio of two polynomials in z^{-1}, the coefficients $f(0T), \ldots, f(nT)$ are obtained as follows:

$$F(z) = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + \cdots + p_n z^{-n}}{q_0 + q_1 z^{-1} + q_2 z^{-2} + \cdots + q_n z^{-n}} = f(0T) + f(T)z^{-1} + f(2T)z^{-2} + \cdots \tag{6.4.2}$$

where

$$
\begin{aligned}
p_0 &= f(0T)q_0 \\
p_1 &= f(1T)q_0 + f(0T)q_1 \\
&\vdots \\
p_n &= f(nT)q_0 + f[(n-1)T]q_1 + f[(n-2)T]q_2 + \cdots + f(0T)q_n
\end{aligned}
\tag{6.4.3}
$$

The same can be accomplished by synthetic division.

Example

$$F(z) = \frac{1+z^{-1}}{1+2z^{-1}+3z^{-2}} = \frac{z^2+z}{z^2+2z+3} = 1 - z^{-1} - z^{-2} + 5z^{-3} + \cdots \qquad |z| > \sqrt{6}$$

From (6.4.3): $1 = f(0T)\cdot 1$ or $f(0T) = 1$, $1 = f(1T)\cdot 1 + 1\cdot 2$ or $f(1T) = -1$, $0 = f(2T)\cdot 1 + f(1T)\cdot 2 + f(0T)\cdot 3$ or $f(2T) = +2 - 3 = -1$, $0 = f(3T)\cdot 1 + f(2T)2 + f(1T)3 + f(0T)\cdot 0$ or $f(3T) = 2 + 3 = 5$, and so forth.

Partial Fraction Expansion

If $F(z)$ is a rational function of z and analytic at infinity, it can be expressed as follows:

$$F(z) = F_1(z) + F_2(z) + F_3(z) + \cdots \tag{6.4.4}$$

and therefore,

$$f(nT) = \mathcal{Z}^{-1}\{F_1(z)\} + \mathcal{Z}^{-1}\{F_2(z)\} + \mathcal{Z}^{-1}\{F_3(z)\} + \cdots \tag{6.4.5}$$

For an expansion of the form

$$F(z) = \frac{F_1(z)}{(z-p)^n} = \frac{A_1}{z-p} + \frac{A_2}{(z-p)^2} + \cdots + \frac{A_n}{(z-p)^n} \tag{6.4.6}$$

the constants A_i are given by

$$
\begin{aligned}
A_n &= (z-p)^n F(z)\Big|_{z=p} \\
A_{n-1} &= \frac{d}{dz}\left[(z-p)^n F(z)\right]\Big|_{z=p} \\
&\vdots \\
A_{n-k} &= \frac{1}{k!}\frac{d^k}{dz^k}\left[(z-p)^n F(z)\right]\Big|_{z=p} \\
&\vdots \\
A_1 &= \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[(z-p)^n F(z)\right]\Big|_{z=p}
\end{aligned}
\tag{6.4.7}
$$

Example

Let

$$F(z)=\frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}=\frac{z^2+2z+1}{z^2-\frac{3}{2}z+\frac{1}{2}}=1+\frac{7}{2}z^{-1}+\frac{23}{4}z^{-2}+\cdots \qquad |z|>1$$

Also,

$$F(z)=1+\frac{\frac{7}{2}z+\frac{1}{2}}{(z-1)\left(z-\frac{1}{2}\right)}=1+\frac{A}{z-1}+\frac{B}{z-\frac{1}{2}}$$

from which we find that

$$A=\left.\frac{(z-1)\left(\frac{7}{2}z+\frac{1}{2}\right)}{(z-1)\left(z-\frac{1}{2}\right)}\right|_{z=1}=8$$

and

$$B=\left.\frac{\left(z-\frac{1}{2}\right)\left(\frac{7}{2}z+\frac{1}{2}\right)}{(z-1)\left(z-\frac{1}{2}\right)}\right|_{z=1/2}=-\frac{9}{2}$$

Hence,

$$F(z)=1+\frac{8}{z-1}-\frac{9}{2}\frac{1}{z-\frac{1}{2}}$$

$$=1+z^{-1}\frac{8z}{z-1}-\frac{9}{2}z^{-1}\frac{z}{z-\frac{1}{2}}$$

and, therefore, its inverse transform is $f(nT)=\delta(nT)+8u(nT-T)-\frac{9}{2}\left(\frac{1}{2}\right)^{n-1}u(nT-T)$ with ROC $|z|>1$.

Example

(a) If

$$F(z)=\frac{z^2+1}{(z-1)(z-2)}=A+\frac{Bz}{z-1}+\frac{Cz}{z-2} \qquad |z|>2$$

then we obtain

$$A = \frac{0+1}{(0-1)(0-2)} = \frac{1}{2},$$

$$B = \frac{1}{z}\frac{z^2+1}{(z-2)}\bigg|_{z=1} = -2,$$

and

$$C = \frac{1}{z}\frac{z^2+1}{(z-1)}\bigg|_{z=2} = \frac{5}{2}$$

Hence,

$$F(z) = \frac{1}{2} - 2\frac{z}{z-1} + \frac{5}{2}\frac{z}{z-2}$$

and its inverse is $f(nT) = \frac{1}{2}\,\delta(nT) - 2u(nT) + \frac{5}{2}\,(2)^n\,u(nT)$.

(b) If

$$F(z) = \frac{z+1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

then we obtain

$$A = \frac{z+1}{(z-2)}\bigg|_{z=1} = -2$$

and

$$B = \frac{z+1}{(z-1)}\bigg|_{z=2} = 3$$

Hence,

$$F(z) = -2\frac{1}{(z-1)} + 3\frac{1}{(z-2)}$$

and

$$f(nT) = -2u(nT - T) + 3(2)^{n-1}u(nT - T)$$

with ROC $|z| > 2$.

Example

If $F(z) = \dfrac{z^2+1}{(z+1)(z-1)^2} = \dfrac{A}{z+1} + \dfrac{B}{z-1} + \dfrac{C}{(z-1)^2}$ with $|z| > 1$, then we find

$$A = \left.\frac{z^2+1}{(z-1)^2}\right|_{z=-1} = \frac{1}{2},$$

$$C = \left.\frac{z^2+1}{z+1}\right|_{z=1} = 1.$$

To find B we set any value of z (small for convenience) in the equality. Hence, with say $z = 2$, we obtain

$$\left.\frac{z^2+1}{(z+1)(z-1)^2}\right|_{z=2} = \frac{1}{2}\left.\frac{1}{z+1}\right|_{z=2} + B\left.\frac{1}{z-1}\right|_{z=2} + \left.\frac{1}{(z-1)^2}\right|_{z=2}$$

or $B = 1/2$. Therefore, $F(z) = \dfrac{1}{2}\dfrac{1}{z+1} + \dfrac{1}{2}\dfrac{1}{z-1} + \dfrac{1}{(z-1)^2}$ and its inverse transform is $f(nT) = \dfrac{1}{2}(-1)^{n-1}$ $u(nT-T) + \dfrac{1}{2}u(nT-T) + (nT-T)u(nT-T)$ with ROC $|z| > 1$.

Example

The function $F(z) = z^3/(z-1)^2$ with $|z| > 1$ can be expanded as follows: $F(z) = z + 2 + \dfrac{3z-2}{(z-1)^2}$ or $F(z)$ $= z + 2 + \dfrac{3z-2}{(z-1)^2} = z + 2 + \dfrac{A}{z-1} + \dfrac{B}{(z-1)^2}$. Therefore, we obtain $B = \left.\dfrac{(3z-2)(z-1)^2}{(z-1)^2}\right|_{z=1} = 1$. Set any value of z (e.g., $z = 2$) in the above equality we obtain

$$2+2+\frac{3\cdot 2-2}{(2-1)^2} = 2+2+A\frac{1}{2-1}+\frac{1}{(2-1)^2} \quad \text{or} \quad A = 3$$

Hence,

$$F(z) = z+2+\frac{3}{z-1}+\frac{1}{(z-1)^2}$$

and its inverse transform is

$$f(nT) = \delta(nT+T) + 2\delta(nT) + 3u(nT-T) + (nT-T)u(nT-T)$$

with ROC $|z| > 1$.

Tables 3 and 4 in the Appendix are useful for finding the inverse transforms.

Inverse Transform by Integration

If $F(z)$ is a regular function in the region $|z| > R$, then there exists a single sequence $\{f(nT)\}$ for which $\mathcal{Z}\{f(nT)\} = F(z)$, namely

$$f(nT) = \frac{1}{2\pi j}\oint_C F(z)z^{n-1}dz = \sum_{i=1}^{K} \operatorname{res}_{z=z_i}\left\{F(z)z^{n-1}\right\} \qquad n = 0, 1, 2, \ldots \qquad (6.4.8)$$

The contour C encloses all the singularities of $F(z)$ as shown in Figure 6.4.1 and it is taken in a counterclockwise direction.

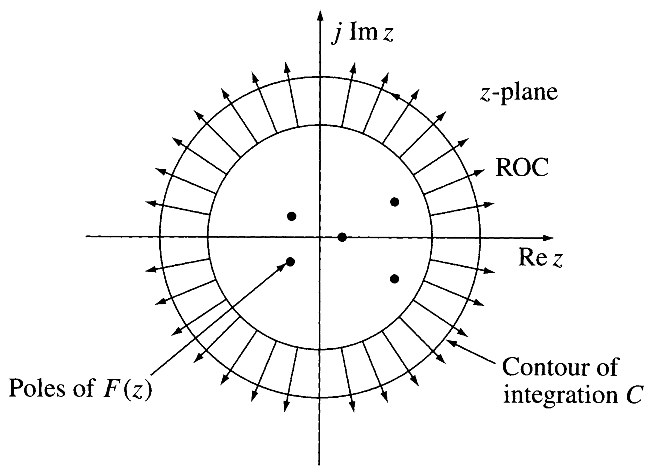

FIGURE 6.4.1

Simple Poles

If $F(z) = H(z)/G(z)$, then the residue at the singularity $z = a$ is given by

$$\lim_{z\to a}(z-a)F(z)z^{n-1} = \lim_{z\to a}\left[(z-a)\frac{H(z)}{G(z)}z^{n-1}\right] \qquad (6.4.9)$$

Multiple Poles

The residue at the pole z_i with multiplicity m of the function $F(z)z^{n-1}$ is given by

$$\operatorname{res}_{z=z_i}\left\{F(z)z^{n-1}\right\} = \frac{1}{(m-1)!}\lim_{z\to z_i}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_i)^m F(z)z^{n-1}\right] \qquad (6.4.10)$$

Simple Poles Not Factorable

The residue at the singularity a_m is

$$F(z)z^{n-1}\Big|_{z=a_m} = \left.\frac{H(z)}{\frac{dG(z)}{dz}} z^{n-1}\right|_{z=a_m} \tag{6.4.11}$$

$F(z)$ is Irrational Function of z

Let $F(z) = [(z + 1)/z]^{\alpha}$, where α is a real noninteger. By (6.4.8) we write.

$$f(nT) = \frac{1}{2\pi j}\oint_C \left(\frac{z+1}{z}\right)^{\alpha} z^{n-1} dz$$

where the closed contour C is that shown in Figure 6.4.2.

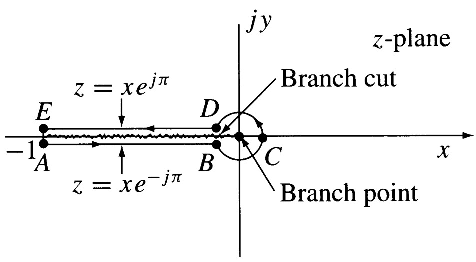

FIGURE 6.4.2

It can easily be shown that at the limit as $z \to 0$ the integral around the small circle BCD is zero (set $z = re^{j\theta}$ and take the limit $r \to 0$). Also, the integral along EA is also zero. Because along AB $z = xe^{-j\pi}$ and along DE $z = xe^{j\pi}$, which implies that x is positive, we obtain

$$\begin{aligned} f(nT) &= \frac{1}{2\pi j}\left[\int_1^0 \left(\frac{xe^{-j\pi}+1}{xe^{-j\pi}}\right)^{\alpha} x^{n-1}e^{-j\pi n}dx + \int_0^1 \left(\frac{xe^{j\pi}+1}{xe^{j\pi}}\right)^{\alpha} x^{n-1}e^{j\pi n}dx\right] \\ &= \frac{1}{2\pi j}\left[-\int_0^1 (1-x)^{\alpha} x^{n-1-\alpha} e^{-j\pi(n-\alpha)}dx + \int_0^1 (1-x)^{\alpha} x^{n-1-\alpha} e^{j\pi(n-\alpha)}dx\right] \\ &= \frac{\sin[(n-\alpha)\pi]}{\pi}\int_0^1 x^{n-1-\alpha}(1-x)^{\alpha}dx \end{aligned} \tag{6.4.12}$$

But the beta function is given by

$$B(m, k) = \frac{\Gamma(m)\Gamma(k)}{\Gamma(m+k)} = \int_0^1 x^{m-1}(1-x)^{k-1}dx \tag{6.4.13}$$

and, hence,

$$f(nT) = \frac{\sin[(n-\alpha)\pi]}{\pi} \frac{\Gamma(n-\alpha)\Gamma(\alpha+1)}{\Gamma(n+1)} \tag{6.4.14}$$

But,

$$\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin \pi m} \tag{6.4.15}$$

and, therefore,

$$f(nT) = \frac{\Gamma(n-\alpha)\Gamma(\alpha+1)}{\Gamma(n+1)} \frac{1}{\Gamma(n-\alpha)\Gamma(\alpha-n+1)} = \frac{\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha-n+1)} \tag{6.4.16}$$

The Taylor's expansion of $F(z)$ is given as follows:

$$F(z) = \left(\frac{z+1}{z}\right)^{\alpha} = (1+z^{-1})^{\alpha} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n (1+z^{-1})^{\alpha}}{(dz^{-1})^n}\bigg|_{z^{-1}=0} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1) z^{-n} \tag{6.4.17}$$

But,

$$\Gamma(\alpha+1) = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)\Gamma(\alpha-n+1), \qquad \Gamma(n+1) = n! \tag{6.4.18}$$

and, therefore, (6.4.17) becomes

$$F(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha-n+1)} z^{-n} \tag{6.4.19}$$

The above equation is a Z-transform expansion and, hence, the function $F(nT)$ is that given in (6.4.16).

Example

To find the inverse of the transform

$$F(z) = \frac{(z-1)}{(z+2)\left(z-\frac{1}{2}\right)} \qquad |z| > 2$$

we proceed with the following approaches:

1. By fraction expansion

$$\frac{(z-1)}{(z+2)\left(z-\frac{1}{2}\right)} = \frac{A}{z+2} + \frac{B}{z-\frac{1}{2}},$$

$$A=\left.\frac{(z-1)}{\left(z-\frac{1}{2}\right)}\right|_{z=-2}=\frac{6}{5},\quad B=\left.\frac{(z-1)}{(z+2)}\right|_{z=\frac{1}{2}}=-\frac{1}{5}$$

$$f(nT)=\mathcal{Z}^{-1}\left\{\frac{6}{5}\frac{1}{z+2}-\frac{1}{5}\frac{1}{z-\frac{1}{2}}\right\}=\frac{6}{5}(-2)^{n-1}-\frac{1}{5}\left(\frac{1}{2}\right)^{n-1}\qquad n\geq 1$$

2. By integration

$$f(nT)=\operatorname{res}_{z=-2}\left\{(z+2)\frac{z-1}{(z+2)\left(z-\frac{1}{2}\right)}z^{n-1}\right\}+\operatorname{res}_{z=\frac{1}{2}}\left\{\left(z-\frac{1}{2}\right)\frac{z-1}{(z+2)\left(z-\frac{1}{2}\right)}z^{n-1}\right\}$$

$$=\frac{6}{5}(-2)^{n-1}-\frac{1}{5}\left(\frac{1}{2}\right)^{n-1}\qquad n\geq 1$$

3. By power expansion

$$\frac{z-1}{z^2+\frac{3}{2}z-1}=z^{-1}-\frac{5}{2}z^{-2}+\frac{19}{4}z^{-3}+\cdots=z^{-1}\left(1-\frac{5}{2}z^{-1}+\frac{19}{4}z^{-2}+\cdots\right)$$

The multiplier z^{-1} indicates one time-unit shift and, hence, $\{f(nT)\}=\left\{1,-\frac{5}{2},\frac{19}{4},\ldots\right\}$ $n=1,2,\ldots$.

Example

1. By expansion
   By $F(z)$ has the region of convergence $|z|>5$, then

$$F(z)=\frac{5z}{(z-5)^2}=\frac{5z}{z^2-10z+25}=5z^{-1}+50z^{-2}+375z^{-3}+\cdots$$

$$=0\cdot 5^0z^{-0}+1\cdot 5z^{-1}+2\cdot 5^2z^{-2}+3\cdot 5^3z^{-3}+\cdots$$

   Hence, $f(nT)=n5^n$ $n=0,1,2,\ldots$, which sometimes is difficult to recognize using the expansion method.
2. By fraction expansion

$$F(z)=\frac{5z}{(z-5)^2}=\frac{Az}{z-5}+\frac{Bz^2}{(z-5)^2},$$

$$B = \left.\frac{5}{z}\right|_{z=5} = 1,$$

$$\frac{5\times 6}{(6-1)^2} = \frac{A\times 6}{6-5} + \frac{6^2}{(6-5)^2} \quad \text{or} \quad A = -1.$$

Hence,

$$F(z) = -\frac{z}{z-5} + \frac{z^2}{(z-5)^2}$$

and $f(nT) = -(5)^n + (n+1)5^n = n5^n$, $n \ge 0$.

3. By integration

$$\frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\left[(z-5)^2 \frac{5z}{(z-5)^2} z^{n-1}\right]\Bigg|_{z=5} = 5nz^{n-1}\Big|_{z=5} = n5^n, \quad n \ge 0.$$

Figure 6.4.3 shows the relation between pole location and type of poles and the behavior of causal signals; m stands for pole multiplicity. Table 5 (Appendix) gives the Z-transform of a number of sequences.

B. Two-Sided Z-Transform

6.5 The Z-Transform

If a function $f(z)$ is defined by $-\infty < t < \infty$, then the Z-transform of its discrete representation $f(nT)$ is given by

$$\mathcal{Z}_{II}\{f(nT)\} \doteq F(z) = \sum_{n=-\infty}^{\infty} f(nT)z^{-n} \qquad R_+ > |z| < R_- \tag{6.5.1}$$

where R_+ is the radius of convergence for the positive time of the sequence, and R_- is the radius of convergence for the negative time of the sequence.

Example

$$F(z) = \mathcal{Z}_{II}\left\{e^{-|nT|}\right\} = \sum_{n=-\infty}^{-1} e^{nT}z^{-n} + \sum_{n=0}^{\infty} e^{-nT}z^{-n} = \sum_{n=-\infty}^{0} e^{nT}z^{-n} - 1 + \sum_{n=0}^{\infty} e^{-nT}z^{-n}$$

$$= \sum_{n=0}^{\infty} e^{-nT}z^{n} - 1 + \sum_{n=0}^{\infty} e^{-nT}z^{-n} = \frac{1}{1-e^{-T}z} - 1 + \frac{1}{1-e^{-T}z^{-1}}$$

The first sum (negative time) converges if $|e^{-T}z| < 1$ or $|z| < e^{T}$. The second sum (positive time) converges if $|e^{-T}z^{-1}| < 1$ or $e^{-T} < |z|$. Hence, the region of convergence is $R_+ = e^{-T} < |z| < R_- = e^{T}$. The two poles of $F(z)$ are $z = e^{T}$ and $z = e^{-T}$.

Single Real Poles—Causal Signals

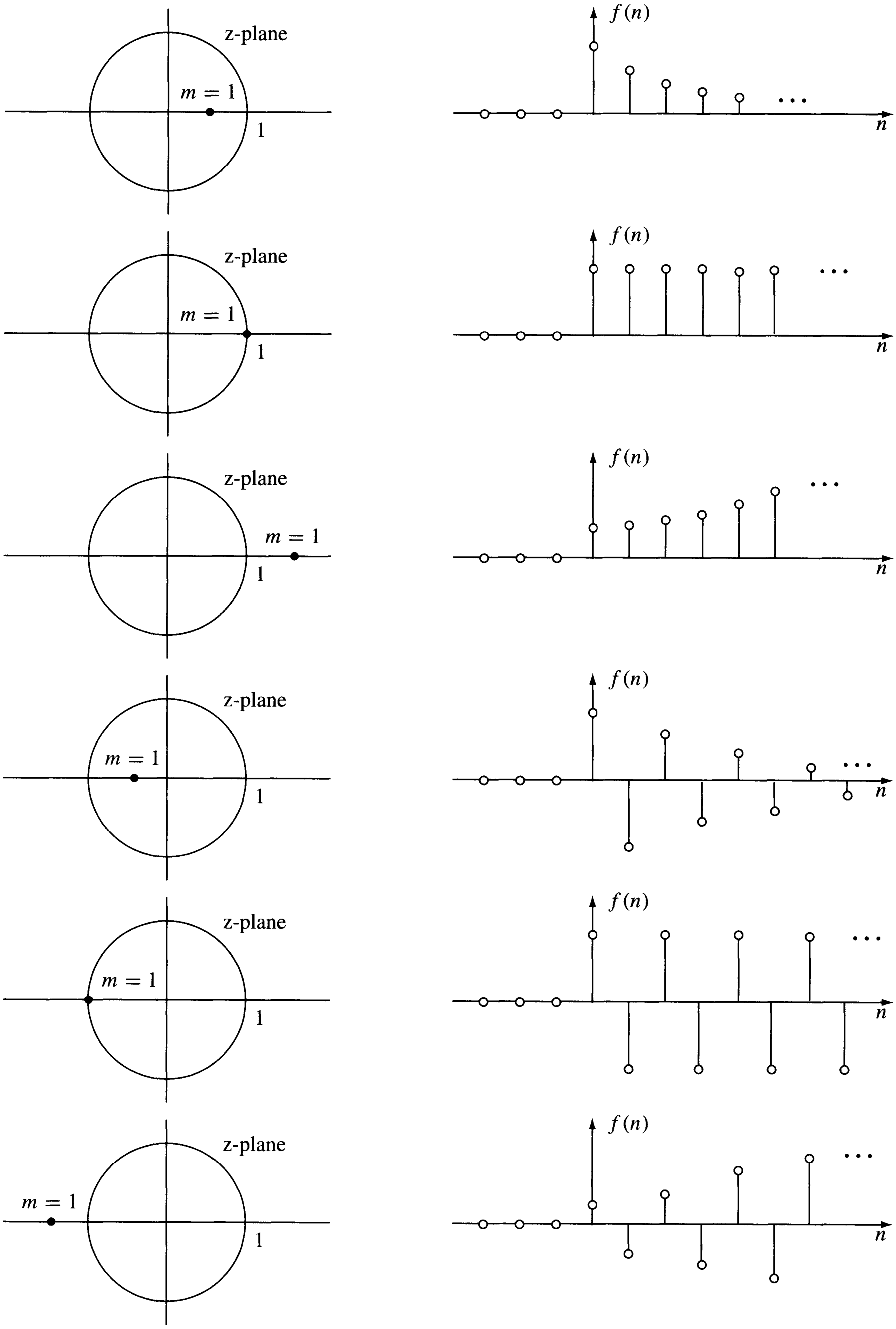

FIGURE 6.4.3

Double Real Poles—Causal Signals

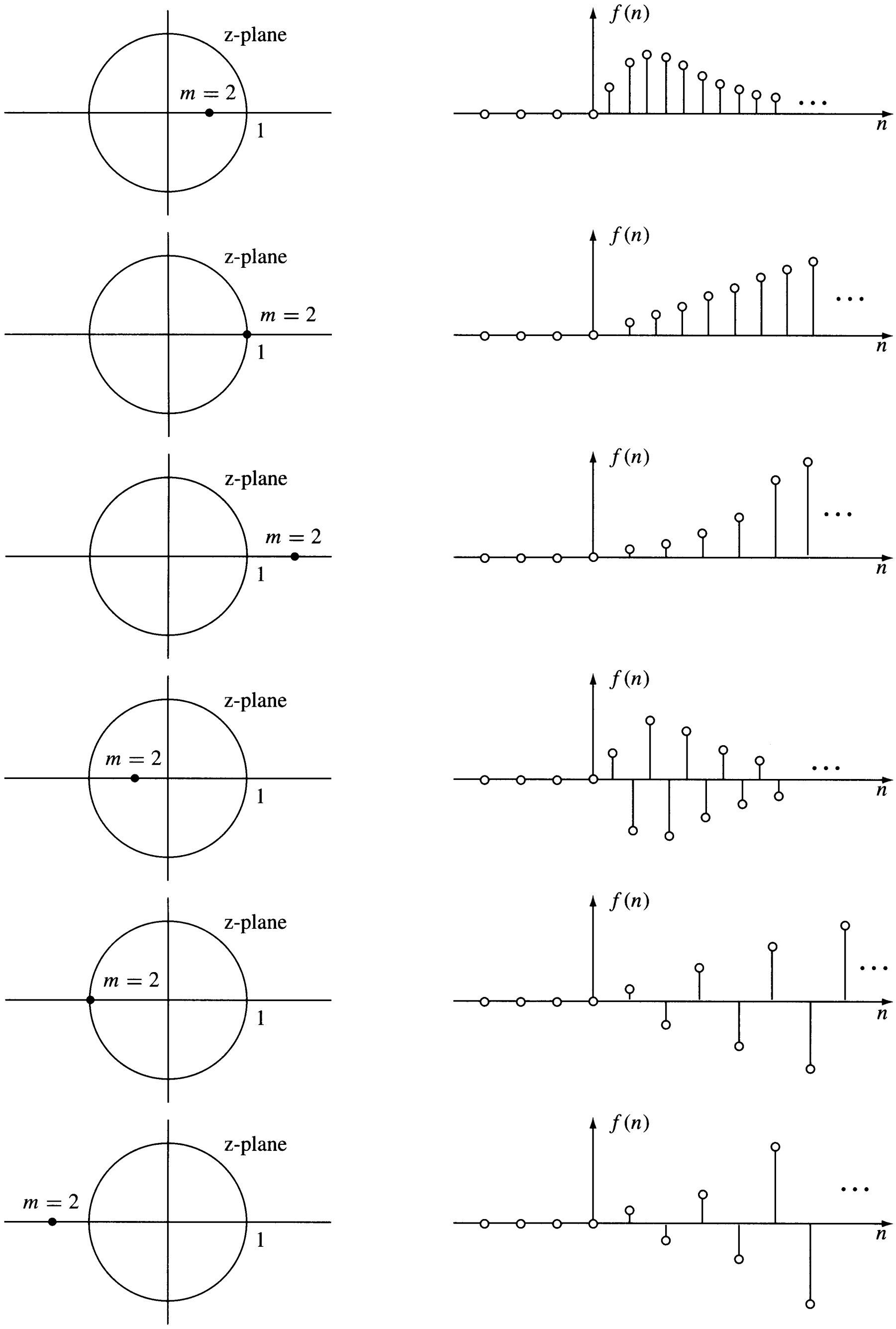

FIGURE 6.4.3 (continued)

Example

The Z-transform of the functions of $u(nT)$ and $-u(-nT-T)$ are

Complex-Conjugate Poles—Causal Signals

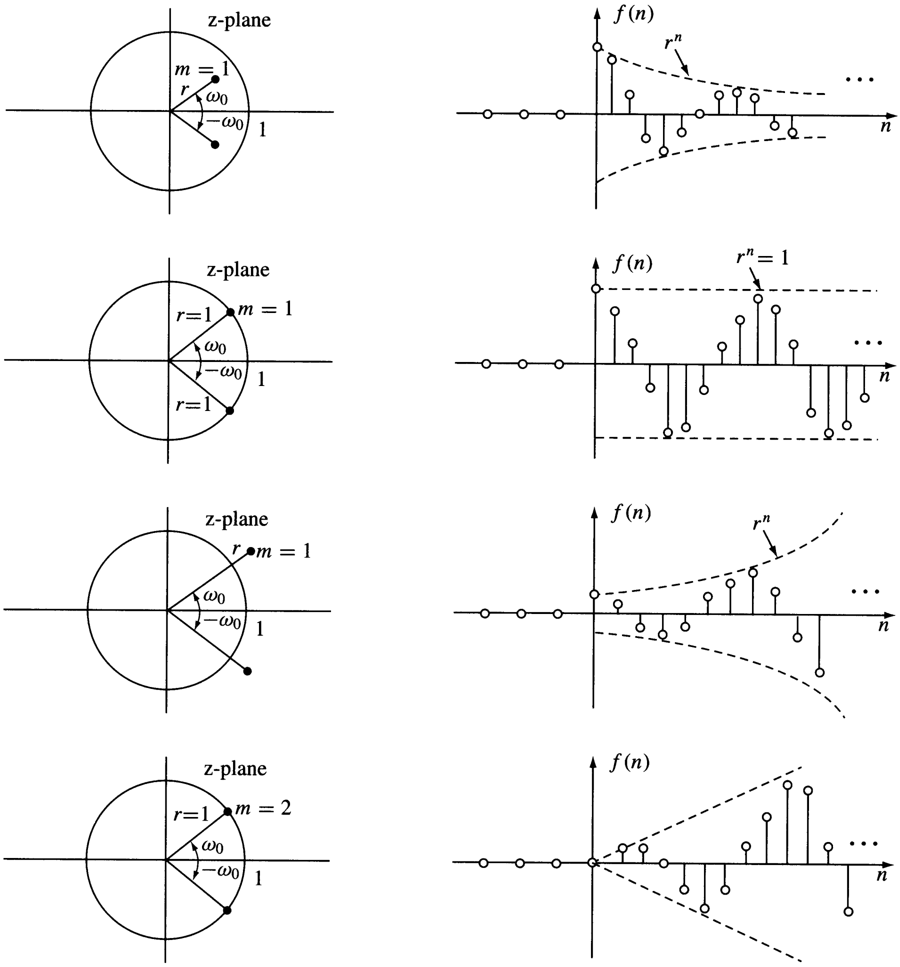

FIGURE 6.4.3 (continued)

$$\mathcal{Z}_{II}\{u(nT)\} = \sum_{n=0}^{\infty} u(nT)z^{-n} = \frac{1}{1-z^{-1}} = \frac{z}{z-1} \qquad |z|>1$$

$$\mathcal{Z}_{II}\{-u(-nT-T)\} = -\sum_{n=-\infty}^{-1} u(-nT-T)z^{-n}$$

$$= -\left[\sum_{n=-\infty}^{0} z^{-n} - 1\right]$$

$$= 1 - \sum_{n=0}^{\infty} z^{n} = 1 - \frac{1}{1-z} = \frac{z}{z-1} \qquad |z|<1$$

Although their Z-transform is identical their ROC is different. Therefore, to find the inverse Z-transform the region of convergence must also be given.

Figure 6.5.1 shows signal characteristics and their corresponding region of convergence.

Assuming that the algebraic expression for the Z-transform $F(z)$ is a rational function and that $f(nT)$ has finite amplitude, except possibly at infinities, the properties of the region of convergence are

1. The ROC is a ring or disc in the z-plane and centered at the origin, and $0 \le R_+ < |z| < R_- \le \infty$.
2. The Fourier transform converges also absolutely if and only if the ROC of the Z-transform of $f(nT)$ includes the unit circle.
3. No poles exist in the ROC.
4. The ROC of a finite sequence $\{f(nT)\}$ is the entire z-plane except possibly for $z = 0$ or $z = \infty$.
5. If $f(nT)$ is right handed, $0 \le n < \infty$, the ROC extends outward from the outermost pole of $F(z)$ to infinity.
6. If $f(nT)$ is left handed, $-\infty < n < 0$, the ROC extends inward from the innermost pole of $F(z)$ to zero.
7. An infinite-duration two-sided sequence $\{f(nT)\}$ has a ring as its ROC, bounded on the interior and exterior by a pole., The ring contains no poles.
8. The ROC must be a connected region.

6.6 Properties

Linearity

The proof is similar to the one-sided Z-transform.

Shifting

$$Z_{II}\{f(nT \pm kT)\} = z^{\pm k}F(z) \tag{6.6.1}$$

Proof

$$Z_{II}\{f(nT-kT)\} = \sum_{n=-\infty}^{\infty} f(nT-kT)z^{-n} = z^{-k}\sum_{m=-\infty}^{\infty} f(mT)z^{-m}$$

The last step results from setting $m = n - k$. Proceed similarly for the positive sign. The ROC of the shifted functions is the same as that of the unfinished function except at $z = 0$ for $k > 0$ and $z = \infty$ for $k < 0$.

Example

To find the transfer function of the system $y(nT) - y(nT - T) + 2y(nT - 2T) = x(nT) + 4x(nT - T)$, we take the Z-transform of both sides of the equation. Hence, we find

$$Y(z) - z^{-1}Y(z) + 2z^{-2}Y(z) = X(z) + 4z^{-1}X(z)$$

or

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 4z^{-1}}{1 - z^{-1} + 2z^{-2}}$$

Example

Consider the Z-transform

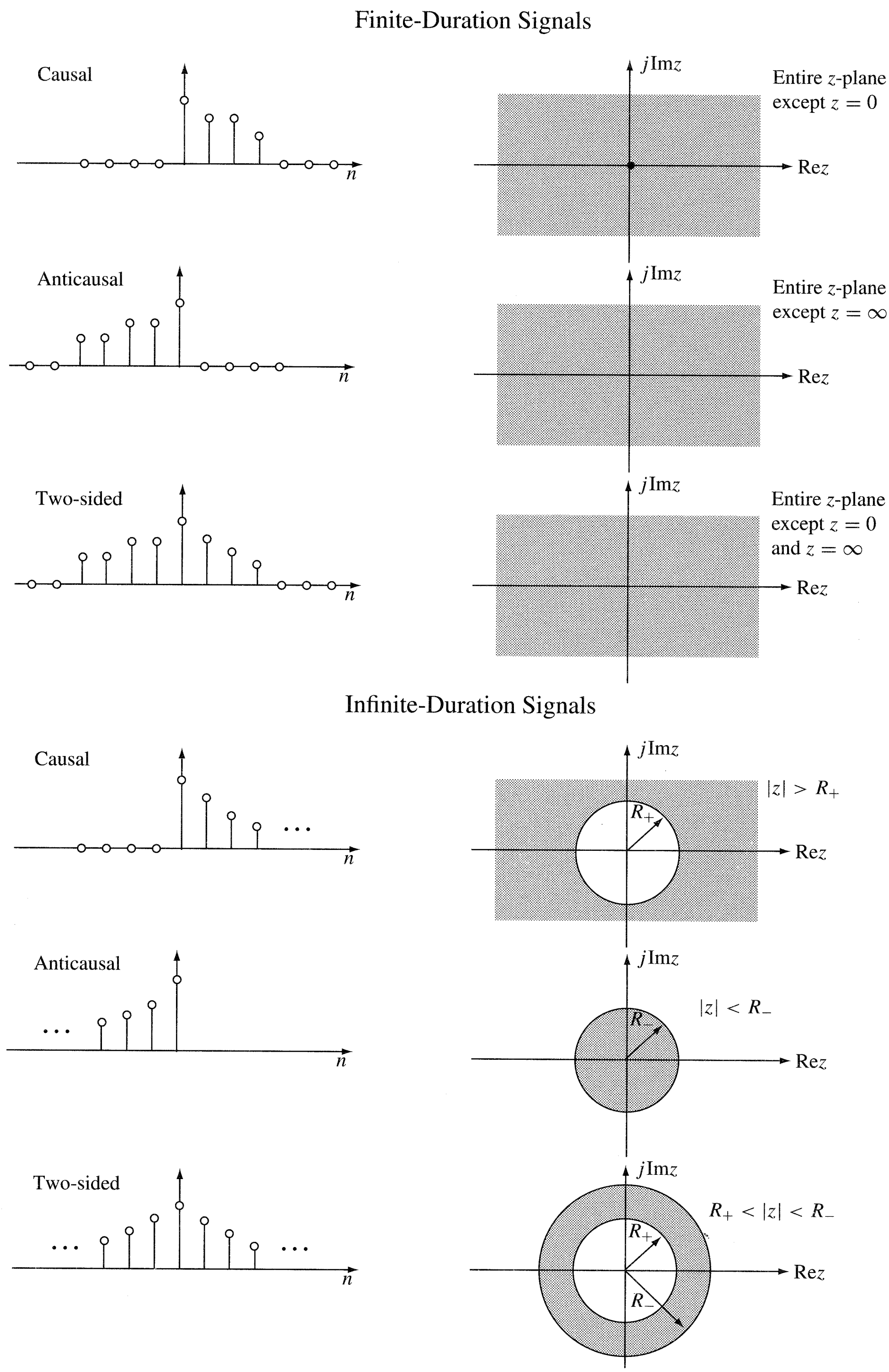
Finite-Duration Signals
Causal
n
jImz
Entire z-plane except z = 0
Rez
Anticausal
n
jImz
Entire z-plane except z = ∞
Rez
Two-sided
n
jImz
Entire z-plane except z = 0 and z = ∞
Rez
Infinite-Duration Signals
Causal
n
jImz
|z| > R+
R+
Rez
Anticausal
n
jImz
|z| < R−
R−
Rez
Two-sided
n
jImz
R+ < |z| < R−
R+
R−
Rez

FIGURE 6.5.1

$$F(z) = \frac{1}{z - \frac{1}{2}} \qquad |z| > \frac{1}{2}$$

Because the pole is inside the ROC, it implies that the function is causal. We next write the function in the form

$$F(z) = z^{-1}\frac{z}{z - \frac{1}{2}} = z^{-1}\frac{1}{1 - \frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$$

which indicates that it is a shifted function (because of the multiplier z^{-1}). Hence, the inverse transform is $f(n) = \left(\frac{1}{2}\right)^{n-1} u(n-1)$ because the inverse transform of $1\Big/\left(1 - \frac{1}{2}z^{-1}\right)$ is equal to $\left(\frac{1}{2}\right)^{n}$.

Scaling

If

$$\mathcal{Z}_{II}\{f(nT)\} = F(z) \qquad R_+ < |z| < R_-$$

then

$$\mathcal{Z}_{II}\{a^{nT} f(nT)\} = F(a^{-T} z) \qquad |a^T| R_+ < |z| < |a^T| R_- \tag{6.6.2}$$

Proof

$$\mathcal{Z}_{II}\left\{a^{nT} f(nT)\right\} = \sum_{n=-\infty}^{\infty} a^{nT} f(nT) z^{-n} = \sum_{n=-\infty}^{\infty} f(nT)\left(a^{-T} z\right)^{-n} = F\left(a^{-T} z\right)$$

Because the ROC of $F(z)$ is $R_+ < |z| < R_-$, the ROC of $F(a^{-T} z)$ is

$$R_+ < |a^{-T} z| < R_- \qquad \text{or} \qquad R_+|a^T| < |z| < |a^T| R_-$$

Example

If the Z-transform of $f(nT) = \exp(-|nT|)$ is

$$F(z) = \frac{1}{1 - e^{-nT} z} + \frac{1}{1 - e^{-nT} z^{-1}} - 1 \qquad e^{-T} < |z| < e^{T}$$

then the Z-transform of $g(nT) = a^{nT} f(nT)$ is

$$G(z) = \frac{1}{1 - e^{-nT} a^{-T} z} + \frac{1}{1 - e^{-nT} a^{T} z^{-1}} - 1 \qquad a^T e^{-T} < |z| < e^T a^T$$

Time Reversal

If

$$\mathcal{Z}_{II}\{f(nT)\} = F(z) \qquad R_+ < |z| < R_-$$

then

$$\mathcal{Z}_{II}\left\{f(-nT)\right\} = F\left(z^{-1}\right) \qquad \frac{1}{R_-} < |z| < \frac{1}{R_+} \tag{6.6.3}$$

Proof

$$\mathcal{Z}_{II}\left\{f(-nT)\right\} = \sum_{n=-\infty}^{\infty} f(nT)\,z^{-n} = \sum_{n=-\infty}^{\infty} f(nT)\left(z^{-1}\right)^{-n} = F\left(z^{-1}\right)$$

and

$$R_+ < \left|z^{-1}\right| < R_- \quad \text{or} \quad |z| > \frac{1}{R_-} \quad \text{and} \quad |z| < \frac{1}{R_+}$$

The above means that if z_0 belongs to the ROC of $F(z)$ then $1/z_0$ is in the ROC of $F(z^{-1})$. The reflection in the time domain corresponds to inversion in the z-domain.

Example

The Z-transform of $f(n) = u(n)$ is $z/(z-1)$ for $|z| > 1$. Therefore, the Z-transform of $f(-n) = u(-n)$ is

$$\frac{\frac{1}{z}}{\frac{1}{z} - 1} = \frac{1}{1-z}$$

Also, from the definition of the Z-transform, we write

$$\mathcal{Z}\left\{u(-n)\right\} = \sum_{n=-\infty}^{0} z^{-n} = \sum_{n=0}^{\infty} z^{n} = \frac{1}{1-z}$$

Multiplication by nT

If

$$\mathcal{Z}_{II}\{f(nT)\} = F(z) \qquad R_+ < |z| < R_-$$

then

$$\mathcal{Z}_{II}\left\{nTf(nT)\right\} = -zT\frac{dF(z)}{dz} \qquad R_+ < |z| < R_- \tag{6.6.4}$$

Proof

A Laurent series can be differentiated term-by-term in its ROC and the resulting series has the same ROC. Therefore, we have

$$\frac{dF(z)}{dz} = \frac{d}{dz}\sum_{n=-\infty}^{\infty} f(nT)z^{-n} = \sum_{n=-\infty}^{\infty} -nf(nT)z^{-n-1} \qquad \text{for} \quad R_+ < |z| < R_-$$

Multiply both sides by $-zT$

$$-zT\frac{dF(z)}{dz} = \sum_{n=-\infty}^{\infty} nTf(nT)z^{-n} = \mathcal{Z}\{nTf(nT)\} \qquad \text{for} \quad R_+ < |z| < R_-$$

Example

If $F(z) = \log(1 + az^{-1})$ $|z| > |a|$, then

$$\frac{dF(z)}{dz} = \frac{-az^{-2}}{1+az^{-1}} \quad \text{or} \quad -z\frac{dF(z)}{dz} = az^{-1}\frac{1}{1-(-a)z^{-1}} \qquad |z| > |a|$$

The z^{-1} implies a time shift, and the inverse transform of the fraction is $(-a)^n$. Hence, the inverse transform is $a(-a)^{n-1}\, u(n-1)$. From the differentiation property (with $T = 1$), we obtain

$$nf(n) = a(-a)^{n-1}u(n-1) \quad \text{or} \quad f(n) = (-1)^{n-1}\frac{a^n}{n}u(n-1)$$

Example

If $f(nT) = au(nT)$ then its Z-transform is $F(z) = a/(1 - z^{-1})$ for $|z| > 1$. Therefore,

$$\mathcal{Z}\{nTau(nT)\} = -zTa\frac{dF(z)}{dz} = aT\frac{z}{(z-1)^2} \qquad |z| > 1$$

Convolution

If

$$\mathcal{Z}_{II}\{f_1(nT)\} = F_1(z) \quad \text{and} \quad \mathcal{Z}_{II}\{f_2(nT)\} = F_2(z)$$

then

$$F(z) = \mathcal{Z}_{II}\,\{f_1(nT) * f_2(nT)\} = F_1(z)\,F_2(z) \tag{6.6.5}$$

The ROC of $F(z)$ is, at least, the intersection of that for $F_1(z)$ and $F_2(z)$.

Proof

$$F(z)=\sum_{n=-\infty}^{\infty} f(nT)z^{-n}=\sum_{n=-\infty}^{\infty}\left[\sum_{m=-\infty}^{\infty} f_1(mT)f_2(nT-mT)\right]z^{-n}$$

$$=\sum_{m=-\infty}^{\infty} f_1(mT)\left[\sum_{n=-\infty}^{\infty} f_2(nT-mT)z^{-n}\right]$$

$$=\sum_{m=-\infty}^{\infty} f_1(mT)z^{-m}F_2(z)=F_1(z)F_2(z)$$

where the shifting property was invoked.

Example

The Z-transform of the convolution of $e^{-n}u(n)$ and $u(n)$ is

$$Z_{II}\left\{\left(e^{-n}u(n)\right)*u(n)\right\}=Z\left\{\sum_{m=0}^{n} e^{-m}u(n-m)\right\}=Z\left\{e^{-n}\right\}Z\left\{u(n)\right\}=\frac{z}{z-e^{-1}}\frac{z}{z-1}$$

Also, from the convolution, definition we find

$$Z\left\{\sum_{m=0}^{n} e^{-m}u(n-m)\right\}=Z\left\{\frac{1-e^{-n-1}}{1-e^{-1}}\right\}$$

$$=Z\left\{\frac{1}{1-e^{-1}}-\frac{e^{-1}}{1-e^{-1}}e^{-n}\right\}$$

$$=\frac{1}{1-e^{-1}}\left(\frac{z}{z-1}-e^{-1}\frac{z}{z-e^{-1}}\right)$$

$$=\frac{z^2}{(z-1)(z-e^{-1})}$$

which verifies the convolution property. The ROC for $e^{-n}u(n)$ is $|z| > e^{-1}$ and the ROC of $u(n)$ is $|z| > 1$. The ROC of $e^{-n}u(n) * u(n)$ is the intersection of these two ROCs and, hence, the ROC is $|z| > 1$.

Example

The convolution of $f_1(n) = \{2, 1, -3\}$ for $n = 0$, 1, and 2, and $f_2(n) = \{1, 1, 1, 1\}$ for $n = 0$, 1, 2, and 3 is

$$G(z) = F_1(z)F_2(z) = (2 + z^{-1} - 3z^{-2})(1 + z^{-1} + z^{-2} + z^{-3}) = 2 + 3z^{-1} - 2z^{-4} - 3z^{-5}$$

which indicates that the output is $g(n) = \{2, 3, 0, 0, -2, -3\}$ which can easily be found by simply convoluting $f_1(n)$ and $f_2(n)$.

Correlation

If

$$\mathcal{Z}_{II}\{f_1(nT)\} = F_1(z) \quad \text{and} \quad \mathcal{Z}_{II}\{f_2(nT)\} = F_2(z)$$

then

$$\mathcal{Z}_{II}\left\{r_{f_1 f_2}(\ell T)\right\} \doteq \mathcal{Z}_{II}\left\{f_1(nT)\otimes f_2(nT)\right\} = \mathcal{Z}_{II}\left\{\sum_{n=-\infty}^{\infty} f_1(nT) f_2(nT-\ell T)\right\}$$
$$= R_{f_1 f_2}(z) = F_1(z)F_2(z^{-1}) \tag{6.6.6}$$

The ROC of $R_{f_1 f_2}(z)$ is at least the intersection of that for $F_1(z)$ and $F_1(z^{-1})$.

Proof

But $r_{f_1 f_2}(\ell T) = f_1(\ell T) * f_2(-T\ell)$ and, hence, from the convolution property and the time-reversal property $R_{f_1 f_2}(z) = F_1(z)F_2(z^{-1})$.

Example

The transform of the autocorrelation sequencing $f(nT) = a^{nT}\, u(n)$, $-1 < a < 1$ is

$$R_{ff}(z) \doteq \mathcal{Z}_{II}\left\{r_{ff}(\ell T)\right\} = F(z)F(z^{-1})$$

But,

$$F(z) = \frac{1}{1-a^T z^{-1}} \qquad |z| > |a|^T \quad \text{causal signal}$$

and

$$F(z^{-1}) = \frac{1}{1-a^T z} \qquad |z| < \frac{1}{|a|^T} \quad \text{anticausal signal}$$

Hence,

$$R_{ff}(z) = \frac{1}{1-a^T(z+z^{-1})+a^{2T}} \qquad \text{ROC } |a|^T < |z| < \frac{1}{|a|^T}$$

Because the ROC of $R_{ff}(z)$ is a ring, it implies that $r_{ff}(\ell T)$ is a two-sided signal.

We proceed to find the autocorrelation first

$$r_{ff}(nT) = \sum_{m=n}^{\infty} a^{mT} a^{(m-n)T} = a^{-nT}\sum_{m=0}^{\infty} a^{2Tm} - a^{-nT}\sum_{m=0}^{n-1} a^{2Tm}$$

$$= a^{-nT}\frac{1}{1-a^{2T}} - a^{-nT}\frac{1-a^{2Tn}}{1-a^{2T}} = \frac{a^{nT}}{1-a^{2T}} \qquad n \ge 0$$

$$r_{ff}(nT) = \sum_{m=0}^{\infty} a^{mT} a^{(m-n)T} = a^{-nT}\frac{1}{1-a^{2T}} \qquad n \le 0$$

and then compare by inverting the function $F(z)F(z^{-1})$.

Multiplication by e^{-anT}

If

$$\mathcal{Z}_{II}\{f(nT)\} = F(z) \qquad R_+ < |z| < R_-$$

then

$$\mathcal{Z}_{II}\{e^{-anT}f(nT)\} = F(e^{-aT}z) \quad |e^{-aT}|R_+ < |z| < |e^{-aT}|R_- \tag{6.6.7}$$

Proof

$$\mathcal{Z}_{II}\left\{e^{-anT}f(nT)\right\} = \sum_{n=-\infty}^{\infty} f(nT)\left(e^{aT}z\right)^{-n} = F\left(e^{aT}z\right) \qquad R_+ < \left|e^{aT}z\right| < R_-$$

Frequency Translation

If the region of convergence of $F(z)$ includes the unit circle and $g(nT) = e^{j\omega_0 nT} f(nT)$, then

$$G(\omega) = F(\omega - \omega_0) \tag{6.6.8}$$

Proof

From (6.6.7) $G(z) = F(e^{-j\omega_0 T}z)$ and has the same region of convergence as $F(z)$ because $|\exp(j\omega_0 T)| = 1$. Therefore,

$$G(\omega) = G(z)\big|_{z=e^{j\omega T}} = F(e^{j(\omega-\omega_0)T}) = F(\omega - \omega_0)$$

Product

If

$$\mathcal{Z}_{II}\{f(nT)\} = F(z) \qquad R_{+f} < |z| < R_{-f} \tag{6.6.9}$$

$$\mathcal{Z}_{II}\{h(nT)\} = H(z) \qquad R_{+h} < |z| < R_{-h} \tag{6.6.10}$$

$$g(nT) = f(nT)\,h(nT)$$

then

$$\begin{aligned}\mathcal{Z}_{II}\left\{f(nT)h(nT)\right\} &\doteq G(z) = \sum_{n=-\infty}^{\infty} f(nT)h(nT)z^{-n} \\ &= \frac{1}{2\pi j}\oint_C F(\tau)H\left(\frac{z}{\tau}\right)\frac{d\tau}{\tau} \qquad R_{+f}R_{+h} < |z| < R_{-f}R_{-h}\end{aligned} \tag{6.6.11}$$

where C is any simple closed curve encircling the origin counterclockwise with

$$\max\left(R_{+f}, \frac{|z|}{R_{-h}}\right) < |\tau| < \min\left(R_{-f}, \frac{|z|}{R_{+h}}\right) \tag{6.6.12}$$

Proof

The series in (6.6.11) will converge to an analytic function $G(z)$ for $R_{+g} < |z| < R_{-g}$. Using the root test (see Section 6.2), we obtain

$$
\begin{aligned}
R_{+g} &= \overline{\lim_{n\to\infty}}\left(\left|f(nT)\,h(nT)\right|\right)^{1/n} \\
&\le \overline{\lim_{n\to\infty}}\left(\left|f(nT)\right|\right)^{1/n}\overline{\lim_{n\to\infty}}\left(\left|h(nT)\right|\right)^{1/n} = R_{+f}R_{+h}
\end{aligned}
\tag{6.6.13}
$$

for positive n. However,

$$
F(z) = \sum_{n=-\infty}^{0} f(nT)\,z^{-n} = \sum_{n=0}^{\infty} f(-nT)\,z^{n} \tag{6.6.14}
$$

and this series converges if

$$
|z| < \frac{1}{\overline{\lim_{n\to\infty}}\left(\left|f(-nT)\right|\right)^{1/n}} = R_{-f} \tag{6.6.15}
$$

Hence,

$$
\begin{aligned}
R_{-g} &= \frac{1}{\overline{\lim_{n\to\infty}}\left(\left|f(-nT)\,h(-nT)\right|\right)^{1/n}} \\
&\ge \frac{1}{\overline{\lim_{n\to\infty}}\left(\left|f(-nT)\right|\right)^{1/n}\overline{\lim_{n\to\infty}}\left(\left|h(-nT)\right|\right)^{1/n}} \\
&\ge R_{-f}R_{-h}
\end{aligned}
\tag{6.6.16}
$$

Replacing $f(nT)$ in the summation of (6.6.11) by its inversion formula (6.4.8), we find

$$
G(z) = \sum_{n=-\infty}^{\infty}\frac{1}{2\pi j}\oint_C F(\tau)\,\tau^{n}\frac{d\tau}{\tau}\,h(nT)\,z^{-n} = \frac{1}{2\pi j}\oint_C F(\tau)\sum_{n=-\infty}^{\infty} h(nT)\left(\frac{z}{\tau}\right)^{-n}\frac{d\tau}{\tau} \tag{6.6.17}
$$

The interchange of the sum and integral is justified if the integrand converges uniformly for some choice of C and z. The contour must be chosen so that

$$
R_{+f} < |\tau| < R_{-f} \tag{6.6.18}
$$

If

$$
R_{+h} < \left|\frac{z}{\tau}\right| < R_{-h} \quad \text{or} \quad \frac{|z|}{R_{-h}} < |\tau| < \frac{|z|}{R_{+h}} \tag{6.6.19}
$$

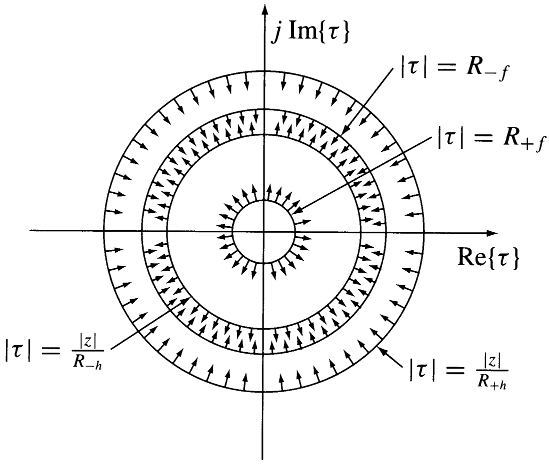

FIGURE 6.6.1

the series in the integrand of (6.17) will converge uniformly to $H(z/\tau)$, and otherwise will diverge. Figure 6.6.1 shows the region of convergence for $F(\tau)$ and $H(z/\tau)$. From (6.6.18) and (6.6.19) we obtain

$$\frac{|z|}{R_{-h}} < R_{-f} \quad \text{or} \quad |z| < R_{-f}R_{-h}$$

$$\frac{|z|}{R_{+h}} > R_{+f} \quad \text{or} \quad |z| > R_{+f}R_{+h}$$

or equivalently

$$R_{+f}R_{+h} < |z| < R_{-f}\,R_{-h} \tag{6.6.20}$$

When z satisfies the above equation, the intersection of the domain identified by (6.6.18) and (6.6.19) is

$$\left(R_{+f} < |\tau| < R_{-f}\right) \cap \left(\frac{|z|}{R_{-h}} < |\tau| < \frac{|z|}{R_{+h}}\right) = \max\left(R_{+f}, \frac{|z|}{R_{-h}}\right) < |\tau| < \min\left(R_{-f}, \frac{|z|}{R_{+h}}\right) \tag{6.6.21}$$

The contour must be located inside the intersection.

When signals are causal, $R_{-f} = R_{-h} = \infty$ and the conditions (6.6.20) and (6.6.21) reduce to

$$R_{+f}R_{+h} < |z| \tag{6.6.22}$$

$$R_{+f} < |\tau| < \frac{|z|}{R_{+h}} \tag{6.6.23}$$

Hence, all of the poles of $F(\tau)$ lie inside the contour and all the poles of $H(z/\tau)$ lie outside the contour.

Example

The Z-transform of $u(nT)$ is

$$F(z) = \frac{1}{1-z^{-1}} \qquad |z| > 1 = R_{+f},\ R_{-f} = \infty$$

and the Z-transform of $h(nT) = \exp(-|nT|)$ is

$$H(z) = \frac{1-e^{-2T}}{\left(1-e^{-T}z^{-1}\right)\left(1-e^{-T}z\right)} \qquad R_{+h} = e^{-T} < |z| < e^{T} = R_{-h}$$

But $R_{-f} = \infty$ and, hence, from (6.11) $1 \cdot \exp(-T) < |z| < \infty$. The contour must lie in the region max $(1, |z|e^{-T}) < |\tau| < \min(-\infty, |z|e^{-T})$ as given by (6.6.21). The pole-zero configuration and the contour are shown in Figure 6.6.2.

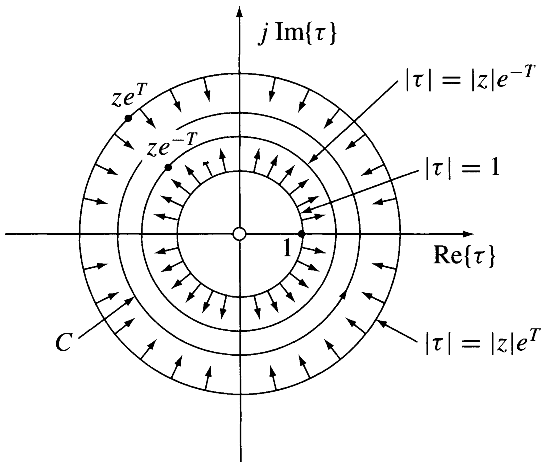

FIGURE 6.6.2

If we choose $|z| > e^{T}$, then the contour is that shown in the figure. Therefore, (6.6.11) becomes

$$Z_{II}\left\{u(nT)\,h(nT)\right\} \doteq G(z) = \frac{1}{2\pi j}\oint_C \frac{1}{1-\tau^{-1}} \frac{1-e^{-2T}}{\left(1-e^{-T}\dfrac{\tau}{z}\right)\left(1-e^{-T}\dfrac{z}{\tau}\right)} \frac{d\tau}{\tau}$$

The poles of $H(z/\tau)$ are at $\tau = z\exp(-T)$ and $\tau = z\exp(T)$. Hence, the contour encloses the poles $\tau = 1$ and $\tau = z\exp(-T)$. Applying the residue theorem next we obtain

$$G(z) = \frac{1}{1-e^{-T}z^{-1}} \qquad |z| > e^{-T}$$

which has the inverse function $g(nT) = e^{-nT}u(nT)$, as expected.

Parseval's Theorem

If

$$\mathcal{Z}_{II}\{f(nT)\} = F(z) \qquad R_{+f} < |z| < R_{-f}$$

$$\mathcal{Z}_{II}\{h(nT)\} = H(z) \qquad R_{+h} < |z| < R_{-h} \tag{6.6.24}$$

with

$$R_{+f}\, R_{+h} < |z| = 1 < R_{-f}\, R_{-h} \tag{6.6.25}$$

then we have

$$\sum_{n=-\infty}^{\infty} f(nT)\, h(nT) = \frac{1}{2\pi j}\oint_C F(z)\, H(z^{-1})\frac{dz}{z} \tag{6.6.26}$$

where the contour encircles the origin with

$$\max\left(R_{+f}, \frac{1}{R_{-h}}\right) < |z| < \min\left(R_{-f}, \frac{1}{R_{+h}}\right) \tag{6.6.27}$$

Proof

In (6.6.11) and (6.6.12) set $z = 1$ and replace the dummy variable τ and z to obtain (6.6.26) and (6.6.27).

For complex signals Parseval's relation (6.6.26) is modified as follows:

$$\sum_{n=-\infty}^{\infty} f(nT)\, h^*(nT) = \frac{1}{2\pi j}\oint_C F(z)\, H^*\left(\frac{1}{z^*}\right)\frac{dz}{z} \tag{6.6.28}$$

If $f(nT)$ and $h(nT)$ converge on the unit circle, we can use the unit circle as the contour. We then obtain

$$\sum_{n=-\infty}^{\infty} f(nT)\, h^*(nT) = \frac{1}{\omega_s}\int_{-\omega_s/2}^{\omega_s/2} F(e^{j\omega T})\, H^*(e^{j\omega T})\, d\omega \qquad \omega_s = \frac{2\pi}{T} \tag{6.6.29}$$

where we set $z = e^{j\omega T}$. If $f(nT) = h(nT)$ then

$$\sum_{n=-\infty}^{\infty} \left|f(nT)\right|^2 = \frac{1}{\omega_s}\int_{-\omega_s/2}^{\omega_s/2} \left|F(e^{j\omega T})\right|^2 d\omega \tag{6.6.30}$$

Example

The Z-transform of $f(nT) = \exp(-nT)u(nT)$ is $F(z) = 1/(1 - e^{-T}z^{-1})$ for $|z| > e^{-T}$. From (6.6.26) we obtain

$$\sum_{n=-\infty}^{\infty} f^2(nT) = \sum_{n=0}^{\infty} f^2(nT) = \frac{1}{2\pi j}\oint_C \frac{1}{1-e^{-T}z^{-1}}\,\frac{1}{1-e^{-T}z}\,\frac{dz}{z}$$

From (6.6.27) we see that $\max(e^{-T}, 0) < |z| < \min(\infty, e^{T})$. The contour encircles the pole at $z = e^{-T}$ so that

$$\sum_{n=0}^{\infty} f^2(nT) = \text{res}\left\{\left[\frac{z - e^{-T}}{(z - e^{-T})(1 - e^{-T}z)}\right]\right\}_{z=e^{-T}} = \frac{1}{1 - e^{-2T}}$$

Also we find directly

$$\sum_{n=0}^{\infty} e^{-nT} e^{-nT} = \sum_{n=0}^{\infty} e^{-2nT} = \left(1 + e^{-2T} + \left(e^{-2T}\right)^2 + \cdots\right) = \frac{1}{1 - e^{-2T}}$$

Complex Conjugate Signal

If

$$\mathcal{Z}_{II}\{f(nT)\} = F(z) \qquad R_{+f} < |z| < R_{-f}$$

then

$$\mathcal{Z}_{II}\{f^*(nT)\} = F^*(z^*) \qquad R_{+f} < |z| < R_{-f} \tag{6.6.31}$$

Proof
By definition we have

$$F(z) = \sum_{n=-\infty}^{\infty} f(nT) z^{-n}$$

Replacing z with z^* and taking the conjugate of both sides of the above equation, we obtain (6.6.31).

6.7 Inverse Z-Transform

Power Series Expansion

The inverse Z-transform in operational form is given by

$$f(nT) = \mathcal{Z}_{II}^{-1}\{F(z)\}$$

If $F(z)$ corresponds to a causal signal, then the signal can be found by dividing the denominator into the numerator to generate a power series in z^{-1} and recognizing that $f(nT)$ is the coefficient of z^{-n}. Similarly, if it is known that $f(nT)$ is zero for positive time (n positive), the value of $f(nT)$ can be found by dividing the denominator into the numerator to generate a power series in z.

Example

If $F(z) = [z(z + 1)]/(z^2 - 2z + 1) = (1 + z^{-1})/(1 - 2z^{-1} + z^{-2})$ and the ROC is $|z| > 1$, then

$$
\begin{array}{r|l}
 & 1+3z^{-1}+5z^{-2}+7z^{-3}+\cdots \\
\hline
1-2z^{-1}+z^{-2} & 1+z^{-1} \\
 & \underline{1-2z^{-1}+z^{-2}} \\
 & 3z^{-1}-z^{-2} \\
 & \underline{3z^{-1}-6z^{-2}+3z^{-3}} \\
 & 5z^{-2}-3z^{-3} \\
 & \cdots
\end{array}
$$

and by continuing the division we recognize that

$$
f(nT)=\begin{cases} 0 & n<0 \\ (2n+1) & n\geq 0 \end{cases}
$$

If $f(nT)$ is known to be zero for positive n, that the ROC is $|z|<1$, then

$$
\begin{array}{r|l}
 & z+3z^{2}+5z^{3}+\cdots \\
\hline
z^{-2}-2z^{-1}+1 & z^{-1}+1 \\
 & \underline{z^{-1}-2+z} \\
 & 3-z \\
 & \underline{3-6z+3z^{2}} \\
 & 5z-3z^{2} \\
 & \cdots
\end{array}
$$

This series is recognized as

$$
f(nT)=\begin{cases} -(2n+1) & n<0 \\ 0 & n\geq 0 \end{cases}
$$

Example

If $F(z)=\log(1+2z^{-1})$, $|z|>2$, then using power series expansion for $\log(1+x)$, with $|x|<1$, we obtain

$$
F(z)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^{n}z^{-n}}{n}
$$

which indicates that

$$
f(nT)=\begin{cases} (-1)^{n+1}\dfrac{2^{n}}{n} & n\geq 0 \\ 0 & n\leq 0 \end{cases}
$$

In general, any **improper** rational function ($M\geq N$) can be expressed as

$$F(z)=\frac{N(z)}{D(z)}=\frac{b_0+b_1z^{-1}+\cdots+b_Mz^{-M}}{1+a_1z^{-1}+\cdots+a_Nz^{-N}}$$

$$=c_0+c_1z^{-1}+\cdots+c_{M-N}z^{-(M-N)}+\frac{N_1(z)}{D(z)} \tag{6.7.1}$$

where the inverse Z-transform of the polynomial can easily be found by inspection.

A **proper** function ($M < N$) is of the form

$$F(z)=\frac{N(z)}{D(z)}=\frac{b_0+b_1z^{-1}+\cdots+b_Mz^{-M}}{1+a_1z^{-1}+\cdots+a_Nz^{-N}} \qquad a_N\neq 0,\ M<N$$

or

$$F(z)=\frac{N(z)}{D(z)}=\frac{b_0z^N+b_1z^{N-1}+\cdots+b_Mz^{N-M}}{z^N+a_1z^{N-1}+\cdots+a_N} \tag{6.7.2}$$

Because $N > M$, the function

$$\frac{F(z)}{z}=\frac{b_0z^{N-1}+b_1z^{N-2}+\cdots+b_Mz^{N-M-1}}{z^N+a_1z^{N-1}+\cdots+a_N} \tag{6.7.3}$$

is always a proper function.

Partial Fraction Expansion

Distinct Poles

If the poles $p_1, p_2, \ldots, p_N$ of a proper function $F(z)$ are all different, then we expand it in the form

$$\frac{F(z)}{z}=\frac{A_1}{z-p_1}+\frac{A_2}{z-p_2}+\cdots+\frac{A_N}{z-p_N} \tag{6.7.4}$$

where all A_i are unknown constants to be determined.

The inverse Z-transform of the kth term of (6.7.4) is given by

$$Z^{-1}\left\{\frac{1}{1-p_kz^{-1}}\right\}=\begin{cases}(p_k)^n u(nT) & \text{if ROC}: |z| > |p_k| \ (\text{causal signal})\\ -(p_k)^n u(-nT-T) & \text{if ROC}: |z| < |p_k| \ (\text{anticausal signal})\end{cases} \tag{6.7.5}$$

If the signal is causal, the ROC is $|z| > p_{max}$, where $p_{max} = \max\{|p_1|, |p_2|, \ldots, |p_N|\}$. In this case, all terms in (6.7.4) result in causal signal components.

Example

(a) If $F(z) = z(z + 3)/(z^2 - 3z + 2)$ with $|z| > 2$ then

$$\frac{F(z)}{z} = \frac{z+3}{(z-2)(z-1)} = \frac{A_1}{z-2} + \frac{A_2}{z-1}$$

$$A_1 = \left.\frac{(z+3)(z-2)}{(z-2)(z-1)}\right|_{z=2} = 5, \quad A_2 = \left.\frac{(z+3)(z-1)}{(z-2)(z-1)}\right|_{z=1} = -4$$

Therefore,

$$F(z) = 5\frac{z}{z-2} - 4\frac{z}{z-1} \quad \text{or} \quad f(nT) = 5(2)^n - 4(1)^n \quad n \geq 0$$

(b) If $F(z) = z(z+3)/(z^2 - 3z + 2)$ with $1 < |z| > 2$, then following exactly the same procedure

$$F(z) = 5\frac{z}{z-2} - 4\frac{z}{z-1}$$

However, the pole at $z = 2$ belongs to the negative-time sequence and the pole at $z = 1$ belongs to the positive-time sequence. Hence,

$$f(nT) = \begin{cases} -4(1)^n & n \geq 0 \\ -5(2)^n & n \leq -1 \end{cases}$$

Example

To detrmine the inverse Z-transform of $F(z) = 1/(1 - 1.5z^{-1} + 0.5z^{-2})$ if (a) ROC: $|z| > 1$, (b) ROC: $|z| < 0.5$, and (c) ROC: $0.5 < |z| < 1$, we proceed as follows:

$$F(z) = \frac{z^2}{z^2 - 1.5z + 0.5} = \frac{z^2}{(z-1)\left(z - \frac{1}{2}\right)} = A + \frac{Bz}{z-1} + \frac{Cz}{z - \frac{1}{2}}$$

or

$$F(z) = 2\frac{z}{z-1} - \frac{z}{z - \frac{1}{2}}$$

(a) $f(nT) = 2(1)^n - (1/2)^n$, $n \geq 0$ because both poles are outside the region of convergence $|z| > 1$ (inside the unit circle).

(b) $f(nT) = -2(1)^n\, u(-nT - T) + (1/2)^n\, u(-nT - T)$, $n \leq -1$ because both poles are outside the region of convergence (outside the circle $|z| = 0.5$).

(c) Pole at 1/2 provides the causal part and the pole at 1 provides the anticausal. Hence,

$$f(nT) = -2(1)^n u(-nT - T) - \left(\frac{1}{2}\right)^n u(nT) \qquad -\infty < n < \infty$$

Multiple Poles

If $F(z)$ has repeated poles, we must modify the form of the expansion. Suppose $F(z)$ has a pole of multiplicity m at $z = p_i$. Then one form of expansion is of the form

$$A_1 \frac{z}{z - p_i} + A_2 \frac{z^2}{(z - p_i)^2} + \cdots + A_m \frac{z^m}{(z - p_i)^m} \tag{6.7.6}$$

The following example shows how to find A_i's.

Example

Let the transfer function of each of two cascade systems be $1/(1 - (1/2)z^{-1})$. If the input to this system is the unit step function $1/(1 - z^{-1})$, then its output is

$$F(z) = \frac{1}{1 - z^{-1}} \frac{1}{\left(1 - \frac{1}{2} z^{-1}\right)^2} = \frac{z^3}{(z-1)\left(z - \frac{1}{2}\right)^2}$$

$$= A_0 + \frac{A_1 z}{z - 1} + \frac{A_2 z}{z - \frac{1}{2}} + \frac{A_3 z^2}{\left(z - \frac{1}{2}\right)^2} \qquad |z| > 1$$

If we set $z = 0$ in both sides, we find that $A_0 = 0$. Next we find A_3 by multiplying both sides by $(z - 1/2)^2$ and setting $z = 1/2$. Hence,

$$A_3 = \left. \frac{z^3\left(z - \frac{1}{2}\right)^2}{z^2(z-1)\left(z - \frac{1}{2}\right)^2} \right|_{z=\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2} - 1} = -1$$

and then we write

$$\frac{z^3}{(z-1)\left(z - \frac{1}{2}\right)^2} = \frac{A_1 z}{z - 1} + \frac{A_2 z}{z - \frac{1}{2}} - \frac{z^2}{\left(z - \frac{1}{2}\right)^2}$$

$$= \frac{A_1 z\left(z - \frac{1}{2}\right)^2 + A_2 z(z-1)\left(z - \frac{1}{2}\right) - z^2(z-1)}{(z-1)\left(z - \frac{1}{2}\right)^2}$$

$$= \frac{(A_1 + A_2 - 1)z^3 + \left(1 - \frac{3}{2} A_2 - A_1\right) z^2 + \left(\frac{1}{4} A_1 + \frac{1}{2} A_2\right) z}{(z-1)\left(z - \frac{1}{2}\right)^2}$$

Equating coefficients of equal powers, we obtain the system

$$A_1 + A_2 - 1 = 1, \quad 1 - A_1 - \frac{3}{2}A_2 = 0, \quad A_1 = 4, \quad \text{and} \quad A_2 = -2$$

Hence,

$$\frac{z^3}{(z-1)\left(z-\frac{1}{2}\right)^2} = 4\frac{z}{z-1} - 2\frac{z}{z-\frac{1}{2}} - \frac{z^2}{\left(z-\frac{1}{2}\right)^2}$$

and the output is

$$f(nT) = 4(1)^n - 2\left(\frac{1}{2}\right)^n - (n+1)\left(\frac{1}{2}\right)^n \qquad n \ge 0$$

Another form of expansion of a proper function (the degree of the denominator is one less than the numerator) is of the form

$$\frac{A_1}{z-p_i} + \frac{A_2 z}{(z-p_i)^2} + \frac{A_3 z(z+p_i)}{(z-p_i)^3} \tag{6.7.7}$$

and the following example explains its use (see Table 4 in the Appendix).

Example

Using the previous example for $F(z)$ with $|z| > 1$, we obtain

$$F(z) = \frac{z^3}{(z-1)\left(z-\frac{1}{2}\right)^2} = 1 + \frac{2z^2 - \frac{5}{4}z + \frac{1}{4}}{(z-1)\left(z-\frac{1}{2}\right)^2}$$

$$= 1 + \frac{A_1}{z-1} + \frac{A_2}{\left(z-\frac{1}{2}\right)} + \frac{A_3 z}{\left(z-\frac{1}{2}\right)^2}$$

Hence,

$$A_1 = \left.\frac{\left(2z^2 - \frac{5}{4}z + \frac{1}{4}\right)(z-1)}{(z-1)\left(z-\frac{1}{2}\right)^2}\right|_{z=1} = 4,$$

$$A_3 = \frac{1}{z}\left.\frac{\left(2z^2 - \frac{5}{4}z + \frac{1}{4}\right)\left(z-\frac{1}{2}\right)^2}{(z-1)\left(z-\frac{1}{2}\right)^2}\right|_{z=\frac{1}{2}} = -\frac{1}{2},$$

$$A_2 = -\frac{3}{2}$$

where A_2 was found by setting an arbitrary value of z, that is, $z = -1$, in both sides of the equation. Therefore, the inverse Z-transform is given by

$$f(nT) = \begin{cases} \delta(n) & n = 0 \\ 4(1)^{n-1} - \frac{3}{2}\left(\frac{1}{2}\right)^{n-1} - \frac{1}{2}n\left(\frac{1}{2}\right)^{n-1} & n \geq 1 \end{cases}$$

Example

Now let us assume the same example but with $|z| < 1/2$. This indicates that the output signal is anticausal. Hence, from

$$F(z) = 4\frac{z}{z-1} - 2\frac{z}{z-\frac{1}{2}} - \frac{z^2}{\left(z-\frac{1}{2}\right)^2}$$

and Table 3 (Appendix), we obtain

$$f(nT) = -4(1)^n + 2\left(\frac{1}{2}\right)^n + (n+1)\left(\frac{1}{2}\right)^n \qquad n \leq -1$$

Similarly from

$$F(z) = 1 + 4\frac{1}{z-1} - \frac{3}{2}\frac{1}{z-\frac{1}{2}} - \frac{1}{2}\frac{z}{\left(z-\frac{1}{2}\right)^2}$$

and Table 4 (Appendix), we obtain

$$f(nT) = \begin{cases} \delta(n) & n = 0 \\ -4(1)^{n-1} + \frac{3}{2}\left(\frac{1}{2}\right)^{n-1} + \frac{1}{2}n\left(\frac{1}{2}\right)^{n-1} & n \leq -1 \end{cases}$$

Integral inversion formula

Theorem 7.1

If

$$F(z)=\sum_{m=-\infty}^{\infty} f(mT)z^{-m} \tag{6.7.8}$$

converges to an analytic function in the annular domain $R_+ < |z| < R_-$, then

$$f(nT)=\frac{1}{2\pi j}\oint_C F(z)z^n \frac{dz}{z} \tag{6.7.9}$$

where C is any simple closed curve separating $|z| = R_+$ from $|z| = R_-$ and it is traced in the counterclockwise direction.

Proof

Multiply (6.7.8) by z^{n-1} and integrate around C. Then

$$\frac{1}{2\pi j}\oint_C F(z)z^n \frac{dz}{z} = \sum_{m=-\infty}^{\infty} f(mT)\frac{1}{2\pi j}\oint_C z^{n-m}\frac{dz}{z} \tag{6.7.10}$$

Set $z = \mathrm{R}e^{j\theta}$ with $R_+ < R < R_-$ to obtain

$$\begin{aligned}\frac{1}{2\pi j}\oint_C z^{n-m}\frac{dz}{z} &= \frac{1}{2\pi j}\int_0^{2\pi} R^{n-m-1}e^{j\theta(n-m-1)}Rje^{j\theta}d\theta \\ &= \frac{1}{2\pi}R^k\int_0^{2\pi} e^{j\theta k}d\theta \\ &= \begin{cases}1 & k=0\\ 0 & \text{elsewhere}\end{cases}\end{aligned} \tag{6.7.11}$$

Hence, the summation on the right-hand side of (6.7.10) reduces to $f(nT)$.

Let $\{a_k\}$ be the set of poles of $F(z)z^{n-1}$ inside the contour C and $\{b_k\}$ be the set of poles of $F(z)z^{n-1}$ outside C in a finite region of the z-plane. By Cauchy's residue theorem

$$f(nT)=\sum_k \mathrm{Res}\left\{F(z)z^{n-1}, a_k\right\} \quad n\ge 0 \tag{6.7.12}$$

$$f(nT)=-\sum_k \mathrm{Res}\left\{F(z)z^{n-1}, b_k\right\} \quad n< 0 \tag{6.7.13}$$

Example

Let

$$F(z)=\frac{1}{(1-z^{-1})(1-a^T z^{-1})} \qquad a<1,\ |z|>1$$

The function $F(z)z^{n-1} = z^{n+1}/(z-1)(z-a^T)$ has two poles enclosed by C for $n \ge 0$. Hence,

$$f(nT) = \text{Res}\{F(z)z^{n-1}, 1\} + \text{Res}\{F(z)z^{n-1}, a\}$$
$$= \frac{1}{1-a^T} + \frac{a^{(n+1)T}}{a^T-1} \qquad n \ge 0$$

Example

Let

$$F(z) = \frac{1-0.8^2}{(1-0.8z)(1-0.8z^{-1})} \qquad 0.8 < |z| < 0.8^{-1}$$

For $n \ge 0$ the contour C encloses only the pole $z = 0.8$ of the function $F(z)z^{n-1}$. Therefore,

$$f(nT) = \text{Res}\{F(z)z^{n-1}\}\Big|_{z=0.8} = \frac{(1-0.8^2)z^n(z-0.8)}{(1-0.8z)(z-0.8)}\Bigg|_{z=0.8} = 0.8^n \qquad n \ge 0$$

For $n < 0$ only the pole $z = 1/0.8$ is outside C. Hence,

$$f(nT) = -\text{Res}\{F(z)z^{n-1}\}\Big|_{z=1/0.8}$$
$$= -\frac{(1-0.8^2)0.8^{-1}z^n(z-0.8^{-1})}{-(1-0.8^{-1})(z-0.8)}\Bigg|_{z=0.8^{-1}} = 0.8^{-n} \qquad n \le -1$$

The residue for a multiple pole of order k at z_0 is given by

$$\text{Res}\{F(z)z^{n-1}\}\Big|_{z=z_0} = \lim_{z\to z_0} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}}\left[(z-z_0)^k F(z)z^{n-1}\right] \qquad (6.7.14)$$

C. Applications

6.8 Solutions of Difference Equations with Constant Coefficients

Based on the relation

$$\mathcal{Z}\{f(n-m)\} = \sum_{\ell=-m}^{-1} f(\ell)z^{-(\ell+m)} = z^{-m}F(z) \qquad (6.8.1)$$

where $\mathcal{Z}\{f(n)\} = F(z)$, we can solve a difference equation of the form

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{L} b_k f(n-k) \tag{6.8.2}$$

using the Z-transform approach.

Example

To find the solution to $y(n) = y(n-1) + 2y(n-2)$ with initial conditions $y(0) = 1$ and $y(1) = 2$, we proceed as follows:

From the difference equation

$$y(0) = y(-1) + 2y(-2) = 1$$

$$y(1) = y(0) + 2y(-1) = 2$$

Hence, $y(-1) = \frac{1}{2}$ and $y(-2) = \frac{1}{4}$. The Z-transform of the difference equation is given by

$$\begin{aligned} Y(z) &= \sum_{\ell=-1}^{-1} y(\ell) z^{-(\ell+1)} + z^{-1}Y(z) + 2\left(\sum_{\ell=-2}^{-1} y(\ell) z^{-(\ell+2)} + z^{-2}Y(z)\right) \\ &= y(-1) + z^{-1}Y(z) + 2\left(y(-2) + y(-1)z^{-1} + z^{-2}Y(z)\right) \\ &= \frac{1}{2} + z^{-1}Y(z) + \frac{1}{2} + z^{-1} + 2z^{-2}Y(z) = 1 + z^{-1} + z^{-1}Y(z) + 2z^{-2}Y(z) \end{aligned}$$

Hence,

$$Y(z) = \frac{1}{1 - z^{-1} - 2z^{-2}} + \frac{z^{-1}}{1 - z^{-1} - 2z^{-2}} = \frac{z^2}{z^2 - z - 2} + \frac{z}{z^2 - z - 2}$$

and

$$\mathcal{Z}^{-1}\{Y(z)\} \doteq y(n) = \mathcal{Z}^{-1}\left\{\frac{z^2}{z^2 - z - 2}\right\} + \mathcal{Z}^{-1}\left\{\frac{z}{z^2 - z - 2}\right\}$$

Example

The solution of the difference equation $y(n) - ay(n-1) = u(n)$ with initial condition $y(-1) = 2$ and $|a| < 1$ proceeds as follows:

$$Y(z) - ay(-1) - az^{-1}Y(z) = \frac{z}{z-1}$$

$$\begin{aligned} Y(z) &= \frac{2a}{1 - az^{-1}} + \frac{z}{z-1}\frac{1}{1 - az^{-1}} = \frac{2a}{1 - az^{-1}} + \frac{z^2}{(z-1)(z-a)} \\ &= \frac{2a}{1 - az^{-1}} + \frac{1}{1-a}\frac{1}{1 - z^{-1}} + \frac{a}{a-1}\frac{1}{1 - az^{-1}} \end{aligned}$$

Hence, the inverse Z-transform gives

$$y(n) = \underbrace{2a \cdot a^n}_{\text{zero input}} + \underbrace{\frac{1}{1-a} u(n) + \frac{a}{a-1} a^n}_{\text{zero state}} = \underbrace{\frac{1}{1-a} u(n)}_{\text{steady state}} + \underbrace{\frac{2a-1}{a-1} a^{n+1}}_{\text{transient}} \qquad n \geq 0$$

6.9 Analysis of Linear Discrete Systems

Transfer Function

From (6.8.2) we obtain the **transfer function** by ignoring initial conditions. The result is

$$H(z) = \frac{Y(z)}{F(z)} = \frac{\sum_{k=0}^{L} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \text{transfer function} \tag{6.9.1}$$

where $H(z)$ is the **transform of the impulse response of a discrete system**.

Stability

Using the convolution relation between input and output of a discrete systems, we obtain

$$|y(n)| = \left| \sum_{k=0}^{n} h(k) f(n-k) \right| \leq M \sum_{k=0}^{\infty} |h(k)| < \infty \tag{6.9.2}$$

where M is the maximum value of $f(n)$. The above inequality specifies that a discrete system is stable if to a finite input the absolute sum of its impulse response is finite. From the properties of the Z-transform, the ROC of the impulse response satisfying (6.9.2) is $|z| > 1$. Hence, all the poles of $H(z)$ of a stable system lie inside the unit circle.

The modified Schur-Cohn criterion establishes if the zeros of the denominator of the rational transfer function $H(z) = N(z)/D(z)$ are inside or outside the unit circle.

The first step is to form the polynomial

$$D_{rp}(z) = z^N D(z^{-1}) = d_0 z^N + \cdots + d_{N-1} z + d_N$$

where $D(z^{-1}) = d_0 + \cdots + d_{N-1} z^{N-1} + d_N z^N$. This $D_{rp}(z)$ is called the **reciprocal polynomial** associated with $D(z)$. The roots of $D_{rp}(z)$ are the reciprocals of the roots of $D(z)$ and $|D_{rp}(z)| = |D(z)|$ on the unit circle. Next, we must divide $D_{rp}(z)$ by $D(z)$ starting at the high power and obtain the quotient $\alpha_0 = d_0/d_N$ and the remainder $D_{1rp}(z)$ of degree $N-1$ or less, so that

$$\frac{D_{rp}(z)}{D(z)} = \alpha_0 + \frac{D_{1rp}(z)}{D(z)}$$

The division is repeated with $D_{1rp}(z)$ and its reciprocal polynomial $D_1(z)$ and the sequence $\alpha_0, \alpha_1, \ldots, \alpha_{N-2}$ is generated according to the rule

$$\frac{D_{krp}(z)}{D_k(z)} = \alpha_k + \frac{D_{(k+1)rp}(z)}{D_k(z)} \qquad \text{for } k = 0, 1, 2, \ldots, N-2$$

The zeros of $D(z)$ are all inside the unit circle (stable system) if and only if the following three conditions are satisfied:

1. $D(1) > 0$
2. $D(-1) \begin{cases} < 0 & N \text{ odd} \\ > 0 & N \text{ even} \end{cases}$
3. $|\alpha_k| < 1 \quad$ for $k = 0, 1, \ldots, N-2$

Check conditions (1) and (2) before proceeding to (3). If they are not satisfied, the system is unstable.

Example

$$D(z) = z^3 - 0.2z^2 + z - 0.2, \quad D_{rp}(z) = -0.2z^3 + z^2 - 0.2z + 1$$

$$\alpha_0 = \frac{-0.2z^3 + z^2 - 0.2z + 1}{z^3 - 0.2z^2 + z - 0.2} = -0.2 + \frac{0.8z^2 + 0.96}{D(z)}, \quad \alpha_1 = \frac{0.96z^2 + 0.96}{0.96z^2 + 0.96} = 1$$

Because $|\alpha_1| = 1$, condition (3) is not satisfied and the system is unstable.

The transfer function of a feedback system with forward (open-loop) gain $D(z)G(z)$ and unit feedback gain is given by

$$H(z) = \frac{D(z)G(z)}{1 + D(z)G(z)}$$

Assuming that all the individual systems are causal and have rational transfer function, the open-loop gain $D(z)G(z)$ can be written as

$$D(z)G(z) = \frac{A(z)}{B(z)}$$

where

$$A(z) = a_L z^L + \cdots + a_0, \quad B(z) = z^M + b_{M-1}z^{M-1} + \cdots + b_0, \quad L \le M$$

Hence, the total transfer function becomes

$$H(z) = \frac{A(z)}{B(z) + A(z)}$$

which indicates that the system will be stable if $B(z) + A(z)$ or $1 + D(z)G(z)$ has zeros inside the unit circle.

Causality

A system is causal if $h(n) = 0$ for $n < 0$. From the properties of the Z-transform, $H(z)$ is regular in the ROC and at the infinity point. For rational functions the numerator polynomial has to be at most of the same degree as the polynomial in the denominator.

The Paley-Wiener theorem provides the necessary and sufficient conditions that a frequency response characteristic $H(\omega)$ must satisfy in order for the resulting filter to be causal.

Paley-Wiener Theorem

If $h(n)$ has finite energy and $h(n) = 0$ for $n < 0$, then

$$\int_{-\pi}^{\pi} \left|\ell n \left|H(\omega)\right|\right| d\omega < \infty$$

Conversely, if $|H(\omega)|$ is square integrable and if the above integral is finite, then we can associate with $|H(\omega)|$ a phase response with $\varphi(\omega)$ so that the resulting filter with frequency response

$$H(\omega) = |H(\omega)| e^{j\varphi(\omega)}$$

is causal.

The relationship between the real and imaginary parts of an absolutely summable, causal, and real sequence is given by the relation

$$H_i(\omega) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} H_r(\lambda) \cot \frac{\omega - \lambda}{2} d\lambda$$

which is known as the **discrete Hilbert transform**.

Summary of Causality

1. $H(\omega)$ cannot be zero except at a finite set of points.
2. $|H(\omega)|$ cannot be constant in any finite range of frequencies.
3. The transition from pass band to stop band cannot be infinitely sharp.
4. The real and imaginary parts of $H(\omega)$ are independent and are related by the discrete Hilbert transform.
5. $|H(\omega)|$ and $\varphi(\omega)$ cannot be chosen arbitrarily.

Frequency Characteristics

With input $f(n) = e^{j\omega n}$, the output is

$$y(n) = \sum_{k=0}^{\infty} h(k) e^{j\omega(n-k)} = e^{j\omega n} \sum_{k=0}^{\infty} h(k) e^{-j\omega k} = e^{j\omega n} H(e^{j\omega}) \qquad (6.9.3)$$

where

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = H_r(e^{j\omega}) + jH_i(e^{j\omega}) = A(\omega)\, e^{j\varphi(\omega)} \qquad (6.9.4)$$

$$A(\omega) = [H_r^2(e^{j\omega}) + H_i^2(e^{j\omega})]^{1/2} = \text{amplitude response} \qquad (6.9.5)$$

$$\varphi(\omega) = \tan^{-1}[H_i(e^{j\omega})/H_r(e^{j\omega})] = \text{phase response} \qquad (6.9.6)$$

$$\tau(\omega) = -\frac{d\varphi(\omega)}{d\omega} = -\mathrm{Re}\left\{ z\frac{d}{dz}\ell nH(z) \right\}\Bigg|_{z=e^{j\omega}} = \begin{array}{l}\text{group delay}\\ \text{characteristic}\end{array} \tag{6.9.7}$$

Because $H(e^{j\omega}) = H(e^{j(\omega+2\pi k)})$ it implies that the **frequency characteristics of discrete systems are periodic with period** 2π.

Z-Transform and Discrete Fourier Transform (DFT)

If $x(n)$ has a finite duration of length N or less, the sequence can be recovered from its N-point DFT. Hence, its Z-transform is uniquely determined by its N-point DFT. Hence, we find

$$\begin{aligned} X(z) &= \sum_{n=0}^{N-1} x(n) z^{-n} = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \right] z^{-n} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} \left(e^{j2\pi k/N} z^{-1} \right)^n = \frac{1-z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1-e^{j2\pi k/N} z^{-1}} \end{aligned} \tag{6.9.8}$$

Set $z = e^{j\omega}$ (evaluated on the unit circle) to find

$$X(e^{j\omega}) \doteq X(\omega) = \frac{1-e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1-e^{-j(\omega-2\pi k/N)}} \tag{6.9.9}$$

$X(\omega)$ is the Fourier transform of the finite-duration sequence in terms of its DFT.

6.10 Digital Filters

Infinite Impulse Response (IIR) Filters

A discrete, linear, and time invariant system can be described by a higher-order difference equation of the form

$$y(n) - \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{N} b_k x(n-k) \tag{6.10.1}$$

Taking the Z-transform of the above equation and solving for the ratio $Y(z)/X(z)$, we obtain

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} \tag{6.10.2}$$

The block diagram representation of (6.10.1), in the form of the following pair of equations:

$$\upsilon(n) = \sum_{k=0}^{M} b_k x(n-k) \tag{6.10.3}$$

$$y(n)=\sum_{k=1}^{N}a_k y(n-k)+v(n) \tag{6.10.4}$$

is shown in Figure 6.10.1. Each appropriate rearrangement of the block diagram represents a **different** computational algorithm for implementing the **same** system.

Figure 6.10.1 can be viewed as an implementation of $H(z)$ through the decomposition

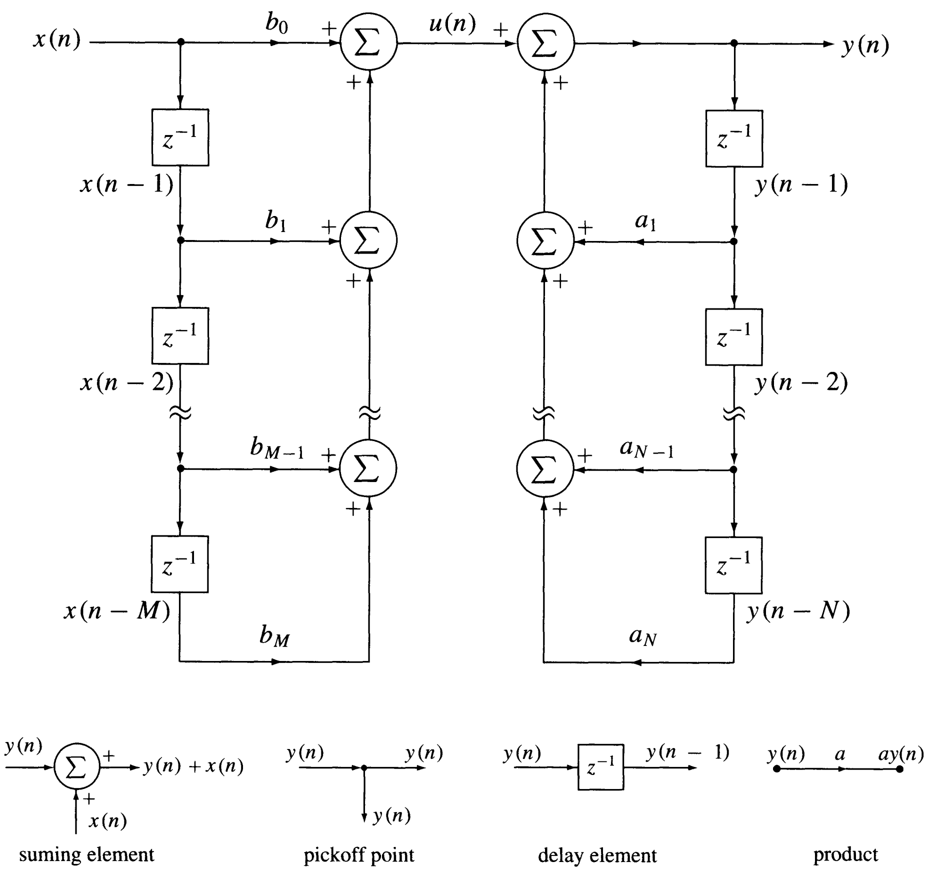

FIGURE 6.10.1

$$H(z)=H_2(z)H_1(z)=\left(\frac{1}{1-\sum_{k=1}^{N}a_k z^{-k}}\right)\left(\sum_{k=0}^{M}b_k z^{-k}\right) \tag{6.10.5}$$

or through the pair of equations

$$V(z)=H_1(z)X(z)=\left(\sum_{k=0}^{M}b_k z^{-k}\right)X(z) \tag{6.10.6}$$

$$Y(z) = H_2(z)V(z) = \left(\frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}} \right) V(z) \tag{6.10.7}$$

If we arrange (6.10.5), we can create the following two equations:

$$W(z) = H_2(z)X(z) = \left(\frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}} \right) X(z) \tag{6.10.8}$$

$$Y(z) = H_1(z)W(z) = \left(\sum_{k=1}^{M} b_k z^{-k} \right) W(z) \tag{6.10.9}$$

The last two equations are presented graphically in Figure 6.10.2 ($M = N$).

The time domain of Figure 6.10.2 is the pair of equations

$$w(n) = \sum_{k=1}^{N} a_k w(n-k) + x(n) \tag{6.10.10}$$

$$y(n) = \sum_{k=0}^{M} b_k w(n-k) \tag{6.10.11}$$

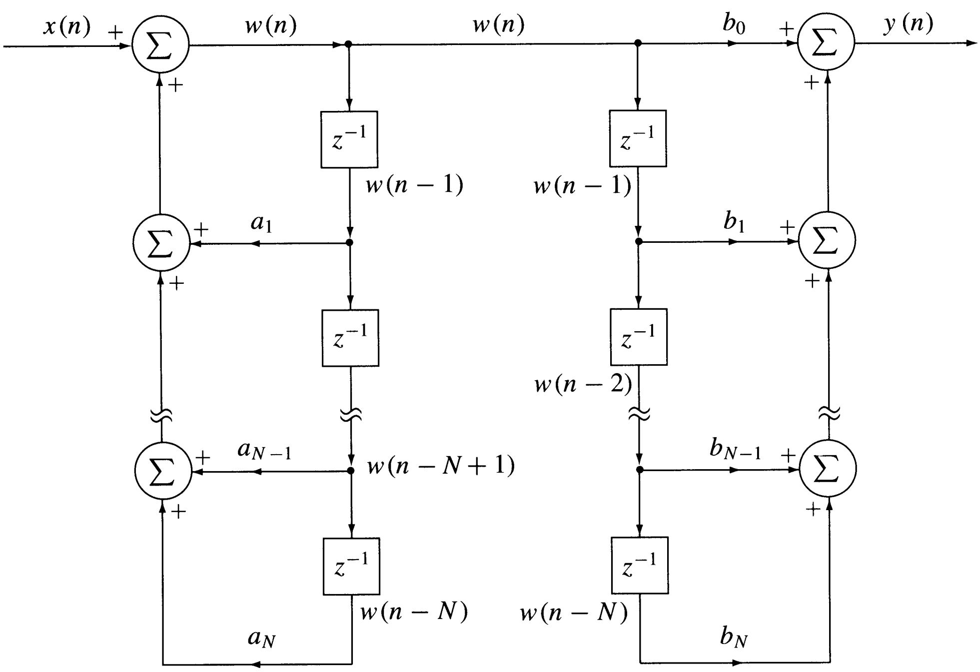

FIGURE 6.10.2

Because the two internal branches of Figure 6.10.2 are identical, they can be combined in one branch so that Figure 6.10.3. Figure 6.10.1 represents the **direct form I** of the general Nth-order system and Figure 6.10.3 is often referred to as the **direct form II** or **canonical direct form** implementation.

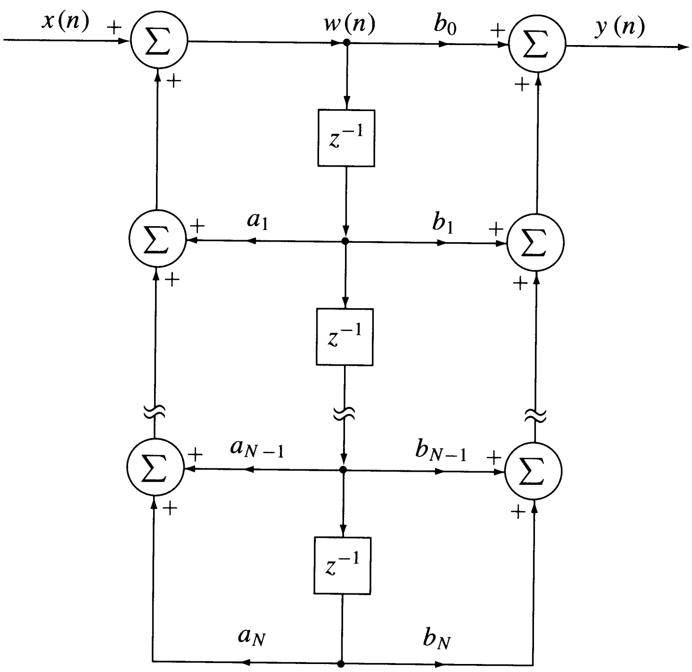

FIGURE 6.10.3

Finite Impulse Responses (FIR) Filters

For causal FIR systems, the difference equation describing such a system is given by

$$y(n)=\sum_{k=0}^{M} b_k x(n-k) \tag{6.10.12}$$

which is recognized as the discrete convolution of $x(n)$ with the impulse response

$$h(n)=\begin{cases} b_n & n=0,1,\ldots,M \\ 0 & \text{otherwise} \end{cases} \tag{6.10.13}$$

The direct form I and direct form II structures are shown in Figures 6.10.4 and 6.10.5. Because of the chain of delay elements across the top of the diagram, this structure is also referred to as a **tapped delay line** structure or a **transversal filter** structure.

6.11 Linear, Time-Invariant, Discrete-Time, Dynamical Systems

The mathematical models describing dynamical systems are almost always of finite-order difference equations. If we know the initial conditions at $t = t_0$, their behavior can be uniquely determined for $t \geq t_0$. To see how to develop a dynamic, let us consider the example below.

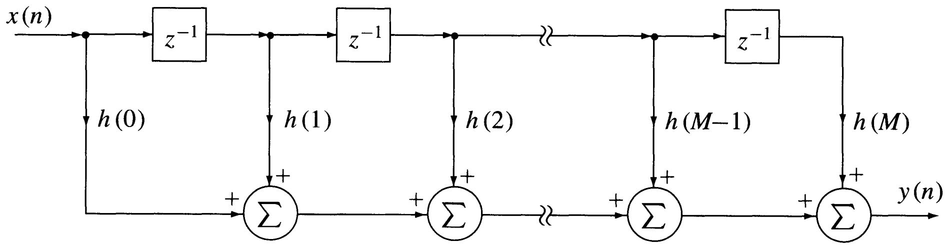

FIGURE 6.10.4

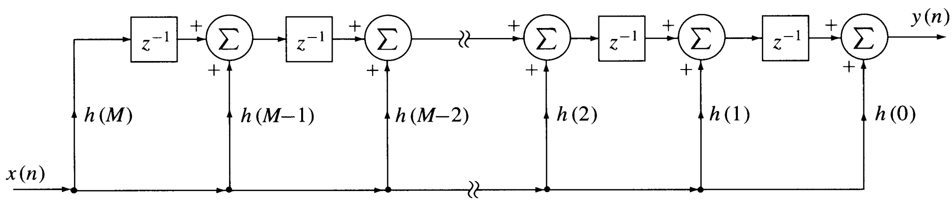

FIGURE 6.10.5

Example

Let a discrete system with input $v(n)$ and output $y(n)$ be described by the difference equation

$$y(n) + 2y(n-1) + y(n-2) = \upsilon(n) \tag{6.11.1}$$

If $y(n_0 - 1)$ and $y(n_0 - 2)$ are the initial conditions for $n > n_0$, then $y(n)$ can be found recursively from (6.11.1). Let us take the pair $y(n-1)$ and $y(n-2)$ as the state of the system at time n. Let us call the vector

$$\underline{x}(n) = \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} = \begin{bmatrix} y(n-2) \\ y(n-1) \end{bmatrix} \tag{6.11.2}$$

the **state vector** for the system. From the definition above, we obtain

$$x_1(n+1) = y(n+1-2) = y(n-1) \tag{6.11.3}$$

and

$$x_2(n+1) = y(n) = \upsilon(n) - y(n-2) - 2y(n-1) \tag{6.11.4}$$

or

$$x_2(n+1) = \upsilon(n) - x_1(n) - 2x_2(n) \tag{6.11.5}$$

Equations (6.11.3) and (6.11.5) can be written in the form

$$\begin{bmatrix} x_1(n+1) \\ x_2(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \upsilon(n) \tag{6.11.6}$$

or

$$\underline{x}(n+1) = \underline{A}\ \underline{x}(n) + \underline{B}\,\upsilon(n) \tag{6.11.7}$$

But (11.4) can be written in the form

$$y(n) = \upsilon(n) - x_1(n) - 2x_2(n) = \begin{bmatrix}-1 & -2\end{bmatrix}\begin{bmatrix}x_1(n)\\ x_2(n)\end{bmatrix} + \upsilon(n)$$

or

$$y(n) = \underline{C}\ \underline{x} + \upsilon(n) \tag{6.11.8}$$

Hence, the system can be described by vector-matrix difference equation (6.11.7) and an output equation (6.11.8) rather than by the second-order difference equation (6.11.1).

A time-invariant, linear, and discrete dynamic system is described by the state equation

$$\underline{x}(nT+T) = \underline{A}\ \underline{x}(nT) + \underline{B}\,\upsilon(nT) \tag{6.11.9}$$

and the output equation is of the form

$$\underline{y}(nT) = \underline{C}\ \underline{x}(nT) + \underline{D}\,\upsilon(nT) \tag{6.11.10}$$

where

$$\underline{x}(nT) = N\text{-dimensional column vector}$$

$$\underline{\upsilon}(nT) = M\text{-dimensional column vector}$$

$$\underline{y}(nT) = R\text{-dimensional column vector}$$

$$\underline{A} = N \times N \text{ nonsingular matrix}$$

$$\underline{B} = N \times M \text{ matrix}$$

$$\underline{C} = R \times N \text{ matrix}$$

$$\underline{D} = R \times M \text{ matrix}$$

When the input is identically zero, (6.11.9) reduces to

$$\underline{x}(nT+T) = \underline{A}\ \underline{x}(nT) \tag{6.11.11}$$

so that

$$\underline{x}(nT+2T) = \underline{A}\ \underline{x}(nT+T) = \underline{A}\ \underline{A}\ \underline{x}(nT) = \underline{A}^2\underline{x}(nT)$$

and so on. In general we have

$$\underline{x}(nT+kT) = \underline{A}^k\,\underline{x}(nT) \tag{6.11.12}$$

The **state transition matrix** from n_1T to n_2T ($n_2 > n_1$) is given by

$$\underline{\varphi}\,(n_2T,\ n_1T) = \underline{A}^{n_2-n_1} \tag{6.11.13}$$

This is a function only of the time difference $n_2T - n_1T$. Therefore, it is customary to name the matrix

$$\underline{\varphi}\,(nT) = \underline{A}^{n} \tag{6.11.14}$$

the state transition matrix with the understanding that $n = n_2 - n_1$. It follows that the system states at two times, n_2T and n_1T, are related by the relation

$$\underline{x}\,(n_2T) = \underline{\varphi}\,(n_2T,\ n_1T)\ \underline{x}\,(n_1T) \tag{6.11.15}$$

when the input is zero. From (6.11.13) we obtain the following relationships:

(a) $$\underline{\varphi}\,(nT,\ nT) = \underline{I} = \text{identity matrix} \tag{6.11.16}$$

(b) $$\underline{\varphi}\,(n_2T,\ n_1T) = \underline{\varphi}^{-1}\,(n_1T,\ n_2T) \tag{6.11.17}$$

(c) $$\underline{\varphi}\,(n_3T,\ n_2T)\ \underline{\varphi}\,(n_2T,\ n_1T) = \underline{\varphi}\,(n_3T,\ n_1T) \tag{6.11.18}$$

If the input is not identically zero and $\underline{x}(nT)$ is known, then the progress (later states) of the system can be found recursively from (6.11.9). Proceeding with the recursion, we obtain

$$\begin{aligned}
\underline{x}(nT+2T) &= \underline{A}\,\underline{x}(nT+T) + \underline{B}\,\underline{\upsilon}(nT+T) \\
&= \underline{A}\,\underline{A}\,\underline{x}(nT) + \underline{A}\,\underline{B}\,\underline{\upsilon}(nT) + \underline{B}\,\underline{\upsilon}(nT+T) \\
&= \underline{\varphi}(nT+2T, nT)\underline{x}(nT) + \underline{\varphi}(nT+2T, nT+T)\underline{B}\,\underline{\upsilon}(nT) + \underline{B}\,\underline{\upsilon}(nT+T)
\end{aligned}$$

In general, for $k > 0$ we have the solution

$$\underline{x}(nT+kT) = \underline{\varphi}(nT+kT, nT)\underline{x}(nT) + \sum_{i=n}^{n+k-1} \underline{\varphi}(nT+kT, iT+T)\underline{B}\,\underline{\upsilon}(iT) \tag{6.11.19}$$

From (6.11.15), when the input is zero, we obtain the relation

$$\underline{x}(n_2T) = \underline{\varphi}(n_2T - n_1T)\underline{x}(n_1T) = \underline{A}^{n_2-n_1}\,\underline{x}(n_1T) \tag{6.11.20}$$

According to (6.11.19), the solution to the dynamic system when the input is not zero is given by

$$\underline{x}(nT+kT) = \underline{\varphi}(nT+kT-nT)\underline{x}(nT) + \sum_{i=n}^{n+k-1} \underline{\varphi}\big[(n+k-i-1)T\big]\underline{B}\,\underline{\upsilon}(iT) \tag{6.11.21}$$

or

$$\underline{x}(nT+kT) = \underline{\varphi}(kT)\underline{x}(nT) + \sum_{i=n}^{n+k-1} \underline{\varphi}\big[(n+k-i-1)T\big]\underline{B}\,\underline{\upsilon}(iT) \qquad k>0 \tag{6.11.22}$$

To find the solution using the Z-transform method, we define the one-sided Z-transform of an $R \times S$ matrix function $\underline{f}(nT)$ as the $R \times S$ matrix

$$\underline{F}(z) = \sum_{n=0}^{\infty} \underline{f}(nT) z^{-n} \tag{6.11.23}$$

The elements of $\underline{F}(z)$ are the transforms of the corresponding elements of $\underline{f}(nT)$. Taking the Z-transform of both sides of the state equation (6.11.9), we find

$$z\,\underline{X}(z) - z\,\underline{x}(0) = \underline{A}\;\underline{X}(z) + \underline{B}\;\underline{V}(z)$$

or

$$\underline{X}(z) = (z\,\underline{I} - \underline{A})^{-1}\,z\,\underline{x}(0) + (z\,\underline{I} - \underline{A})^{-1}\,\underline{B}\;\underline{V}(z) \tag{6.11.24}$$

From the output equation (6.11.10), we see that

$$\underline{Y}(z) = \underline{C}\;\underline{X}(z) + \underline{D}\;\underline{V}(z) \tag{6.11.25}$$

The state of the system $\underline{x}(nT)$ and its output $\underline{y}(nT)$ can be found for $n \geq 0$ by taking the inverse Z-transform of (6.11.24) and (6.11.25).

For a zero input, (6.11.24) becomes

$$\underline{X}(z) = (z\,\underline{I} - \underline{A})^{-1}\,z\,\underline{x}(0) \tag{6.11.26}$$

so that

$$\underline{x}(nT) = \mathcal{Z}^{-1}\{(z\,\underline{I} - \underline{A})^{-1}z\}\;\underline{x}(0) \tag{6.11.27}$$

If we let $n_1 = 0$ and $n_2 = n$, then (6.11.20) becomes

$$\underline{x}(nT) = \underline{\varphi}(nT)\;\underline{x}(0) = \underline{A}^n\,\underline{x}(0) \tag{6.11.28}$$

Comparing (6.11.27) and (6.11.28) we observe that

$$\underline{\varphi}(nT) = \underline{A}^n = \mathcal{Z}^{-1}\{(z\,\underline{I} - \underline{A})^{-1}z\} \qquad n \geq 0 \tag{6.11.29}$$

or equivalently,

$$\underline{\Phi}(z) = \mathcal{Z}\{\,\underline{A}^n\,\} = (z\,\underline{I} - \underline{A})^{-1}z \tag{6.11.30}$$

The Z-transform provides a straightforward method for calculating the state transition matrix.

Next combine (6.11.30) and (6.11.24) to find

$$\underline{X}(z) = \underline{\Phi}(z)\;\underline{x}(0) + \underline{\Phi}(z)z^{-1}\;\underline{B}\;\underline{V}(z) \tag{6.11.31}$$

By applying the convolution theorem and the fact that

$$\mathcal{Z}^{-1}\{\,\underline{\Phi}(z)z^{-1}\} = \underline{\varphi}(nT - T)u(nT - T) \tag{6.11.32}$$

the inverse Z-transform of (6.11.31) is given by

$$\underline{x}(kT)=\underline{\varphi}(kT)\underline{x}(0)+\sum_{i=0}^{k-1}\underline{\varphi}\left[\left(k-i-1\right)T\right]\underline{B}\underline{\upsilon}(iT) \tag{6.11.33}$$

The above equation is identical to (6.11.22) with $n = 0$.

The behavior of the system with zero input depends on the location of the poles of

$$\underline{\Phi}\,(z) = (z\,\underline{I} - \underline{A}\,)^{-1}z \tag{6.11.34}$$

Because

$$\left(z\underline{I}-\underline{A}\right)^{-1}=\frac{\mathrm{adj}\left(z\underline{I}-\underline{A}\right)}{\det\left(z\underline{I}-\underline{A}\right)} \tag{6.11.35}$$

where adj(·) denotes the regular adjoint in matrix theory, these poles can only occur at the roots of the polynomial

$$D(z) = \det(z\,\underline{I} - \underline{A}\,) \tag{6.11.36}$$

$D(z)$ is known as the **characteristic polynomial** for $\underline{A}$ (for the system) and its roots are known as the **characteristic values** or **eigenvalues** of $\underline{A}$. If all roots are inside the unit circle, the system is stable. If even one root is outside the unit circle, the system is unstable.

Example

Consider the system

$$\begin{bmatrix} x_1(nT+T) \\ x_2(nT+T) \end{bmatrix}=\begin{bmatrix} 0 & 2 \\ 0.22 & 2 \end{bmatrix}\begin{bmatrix} x_1(nT) \\ x_2(nT) \end{bmatrix}+\begin{bmatrix} 0 \\ 1 \end{bmatrix}\upsilon(nT)$$

$$y(nT)=\begin{bmatrix} 0.22 & 2 \end{bmatrix}\begin{bmatrix} x_1(nT) \\ x_2(nT) \end{bmatrix}+\upsilon(nT)$$

For this system we have

$$A=\begin{bmatrix} 0 & 2 \\ 0.22 & 2 \end{bmatrix},\quad \underline{B}=\begin{bmatrix} 0 \\ 1 \end{bmatrix},\quad \underline{C}=\begin{bmatrix} 0.22 & 2 \end{bmatrix},\quad \underline{D}=\begin{bmatrix} 1 \end{bmatrix}$$

The characteristic polynomial is

$$D(z)=\det\left(z\underline{I}-\underline{A}\right)=\det\left[\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}-\begin{bmatrix} 0 & 2 \\ 0.22 & 2 \end{bmatrix}\right]=\det\begin{bmatrix} z & -2 \\ -0.22 & z-2 \end{bmatrix}$$
$$=z(z-2)-0.44=z^2-2z-0.44=(z-2.2)(z+0.2)$$

Hence, we obtain (see [6.11.34])

$$\underline{\Phi}(z)=\frac{z}{(z-2.2)(z+0.2)}\begin{bmatrix} z-2 & 2 \\ 0.22 & z \end{bmatrix}=\begin{bmatrix} \dfrac{z(z-2)}{(z-2.2)(z+0.2)} & \dfrac{2z}{(z-2.2)(z+0.2)} \\ \dfrac{0.22z}{(z-2.2)(z+0.2)} & \dfrac{z^2}{(z-2.2)(z+0.2)} \end{bmatrix}$$

Because $D(z)$ has a root outside the unit circle at 2.2, the system is unstable. Taking the inverse transform we find that

$$\underline{\varphi}(nT)=\begin{bmatrix} \frac{1}{12}(2.2)^n+\frac{11}{12}(-0.2)^n & \frac{5}{6}(2.2)^n-\frac{5}{6}(-0.2)^n \\ \frac{11}{120}(2.2)^n-\frac{11}{120}(-0.2)^n & \frac{11}{12}(2.2)^n+\frac{1}{12}(-0.2)^n \end{bmatrix} \qquad n\ge 0$$

To check, set $n = 0$ to find $\underline{\varphi}(0) = \underline{I}$ and $\underline{\varphi}(T) = \underline{A}$.

Let $\underline{x}(0) = \underline{0}$ and the input be, the unit impulse $\upsilon(nT) = \delta(nT)$ so that $V(z) = 1$. Hence, according to (6.11.31)

$$\underline{X}(z)=\underline{\Phi}(z)z^{-1}\underline{B}\,\underline{V}(z)=\frac{1}{(z-2.2)(z+0.2)}\begin{bmatrix} z-2 & 2 \\ 0.22 & z \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$=\frac{1}{(z-2.2)(z+0.2)}\begin{bmatrix} 2 \\ z \end{bmatrix}$$

The inverse Z-transform gives

$$\underline{x}(nT)=\frac{5}{6}\begin{bmatrix} (2.2)^{n-1}-(-0.2)^{n-1} \\ \frac{1}{2}(2.2)^n-\frac{1}{2}(-0.2)^n \end{bmatrix} \qquad n>0$$

and the output is given by

$$y(nT)=\underline{C}\,\underline{x}(nT)+\underline{D}\upsilon(nT)$$
$$=\begin{cases} 1 & n=0 \\ \frac{5}{12}(2.2)^{n+1}-\frac{5}{12}(-0.2)^{n+1} & n>0 \end{cases}$$

6.12 Z-Transform and Random Processes

Power Spectral Densities

The Z-transform of the autocorrelation function $R_{xx}(\tau) = E\{x(t+\tau)\,x(t)\}$ sampled uniformly at nT times is given by

$$S_{xx}(z)=\sum_{n=-\infty}^{\infty}R_{xx}(nT)z^{-n} \tag{6.12.1}$$

where the Fourier transform of $R_{xx}(\tau)$ is designated by $S_{xx}(\omega)$. The **sampled power spectral density** for $x(nT)$ is defined to be

$$S_{xx}\left(e^{j\omega T}\right)=S_{xx}\left(z\right)\Big|_{z=e^{j\omega T}}=\sum_{n=-\infty}^{\infty}R_{xx}\left(nT\right)e^{-j\omega nT} \tag{6.12.2}$$

However, from the sampling theorem we have

$$S_{xx}\left(e^{j\omega T}\right)=\frac{1}{T}\sum_{n=-\infty}^{\infty}S_{xx}\left(\omega-n\omega_s\right),\quad \omega_s=2\pi/T \tag{6.12.3}$$

Because $S_{xx}(\omega)$ is real, nonnegative, and even, it follows from (6.12.3) that $S_{xx}(e^{j\omega T})$ is also real, nonnegative, and even. If the envelope of $R_{xx}(\tau)$ decays exponentially for $|\tau| > 0$, then the region of convergence for $S_{xx}(z)$ includes the unit circle. If $R_{xx}(\tau)$ has undamped periodic components the series in (6.12.2) converges in the distribution sense that contains impulse function.

The average power in $x(nT)$ is

$$E\left\{x^2\left(nT\right)\right\}=R_{xx}\left(0\right)=\frac{1}{2\pi j}\oint_C S_{xx}\left(z\right)\frac{dz}{z} \tag{6.12.4}$$

where C is a simple, closed contour lying in the region of convergence and the integration is taken in counterclockwise sense. If C is the unit circle, then

$$R_{xx}\left(0\right)=\frac{1}{\omega_s}\int_{-\omega_s/2}^{\omega_s/2}S_{xx}\left(e^{j\omega T}\right)d\omega \qquad \omega_s=\frac{2\pi}{T} \tag{6.12.5}$$

$$S_{xx}\left(e^{j\omega T}\right)\frac{d\omega}{\omega_s}=\text{average power in } d\omega \tag{6.12.6}$$

$S_{xy}(z)$ is called the **cross power spectral density** for two jointly wide-sense stationary processes $x(t)$ and $y(t)$. It is defined by the relation

$$S_{xy}\left(z\right)=\sum_{n=-\infty}^{\infty}R_{xy}\left(nT\right)z^{-n} \tag{6.12.7}$$

Because $R_{xy}(nT) = R_{yx}(-nT)$ it follows that

$$S_{xy}(z) = S_{yx}(z^{-1}), \qquad S_{xx}(z) = S_{xx}(z^{-1}) \tag{6.12.8}$$

Equivalently, we have

$$S_{xx}(e^{j\omega T}) = S_{xx}(e^{-j\omega T}) \tag{6.12.9}$$

If $S_{xx}(z)$ is a rational polynomial, it can be factored in the form

$$S_{xx}(z) = \frac{N(z)}{D(z)} = \gamma^2 G(z) G(z^{-1}) \tag{6.12.10}$$

where

$$G(z) = \frac{\prod_{k=1}^{L} (1 - \alpha_k z^{-1})}{\prod_{k=1}^{M} (1 - \beta_k z^{-1})} = \frac{\sum_{k=0}^{L} a_k z^{-k}}{\sum_{k=0}^{M} b_k z^{-k}}$$

$$\gamma^2 > 0, \quad |\alpha_k| < 1, \quad |b_k| < 1, \quad a_k \text{ and } b_k \text{ are real}$$

Linear Discrete-Time Filters

Let $R_{xx}(nT)$, $R_{yy}(nT)$, and $R_{xy}(nT)$ be known. Let two systems have transfer functions $H_1(z)$ and $H_2(z)$, respectively. The output of these filters, when the inputs are $x(nT)$ and $y(nT)$ (see Figure 6.12.1), are

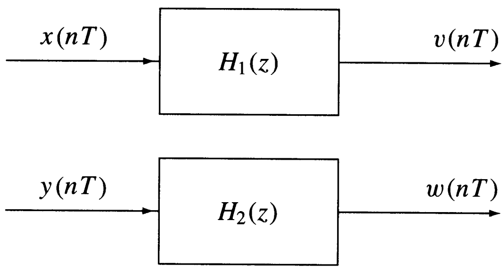

FIGURE 6.12.1

$$v(nT) = \sum_{k=-\infty}^{\infty} h_1(kT) x(nT - kT) \tag{6.12.11}$$

$$w(nT) = \sum_{k=-\infty}^{\infty} h_2(kT) y(nT - kT) \tag{6.12.12}$$

Let $n = n + m$ in (6.12.11), multiply by $y(nT)$, and take the ensemble average to find

$$\begin{aligned} R_{vy}(mT) &= \sum_{k=-\infty}^{\infty} h_1(kT) E\{x(mT + nT - kT) y(nT)\} \\ &= \sum_{k=-\infty}^{\infty} h_1(kT) R_{xy}(mT - kT) \end{aligned} \tag{6.12.13}$$

Hence, by taking the Z-transform we obtain

$$S_{vy}(z) = H_1(z) S_{xy}(z) \tag{6.12.14}$$

Similarly from (6.12.12) we obtain

$$R_{\upsilon w}(mT) = \sum_{k=-\infty}^{\infty} h_2(kT) R_{\upsilon y}(mT + kT) \tag{6.12.15}$$

and

$$S_{\upsilon w}(z) = H_2(z^{-1}) S_{\upsilon y}(z) \tag{6.12.16}$$

From (6.12.14) and (6.12.16), we obtain

$$S_{\upsilon w}(z) = H_1(z) H_2(z^{-1}) S_{xy}(z) \tag{6.12.17}$$

Also, for $x(nT) = y(nT)$ and $h_1(nT) = h_2(nT) = h(nT)$, (6.12.17) becomes

$$S_{\upsilon\upsilon}(z) = H(z)\, H(z^{-1}) S_{xx}(z) \tag{6.12.18}$$

and

$$\begin{aligned} S_{\upsilon\upsilon}(e^{j\omega T}) &= H(e^{j\omega T}) H(e^{-j\omega T}) S_{xx}(e^{j\omega T}) \\ &= \left| H(e^{j\omega T}) \right|^2 S_{xx}(e^{j\omega T}) \end{aligned} \tag{6.12.19}$$

Optimum Linear Filtering

Let $y(nT)$ be an observed wide-sense stationary process and $x(nT)$ be a desired wide-sense stationary process. The process $y(nT)$ could be the result of the desired signal $x(nT)$ and a noise signal $\upsilon(nT)$. It is desired to find a system with transfer function $H(z)$ such that the error $e(nT) = x(nT) - \hat{x}(nT) = x(nT) - \mathcal{Z}^{-1}\{Y(z) H(z)\}$ is minimized. Referring to Figure 6.12.2 and to (6.12.18), we can write

$$S_{aa}(z) = \frac{1}{H_1(z) H_1(z^{-1})} S_{yy}(z) = \gamma^2 \tag{6.12.20}$$

where $a(nT)$ is taken as white noise (uncorrelated process). We, therefore, can write

$$R_{aa}(mT) = \gamma^2 \delta(mT) \tag{6.12.21}$$

The signal $a(nT)$ is known as the **innovation** process associated with $y(nT)$. From Figure 6.12.2, we obtain

$$\hat{x}(nT) = \sum_{k=-\infty}^{\infty} g(kT) a(nT - kT) \tag{6.12.22}$$

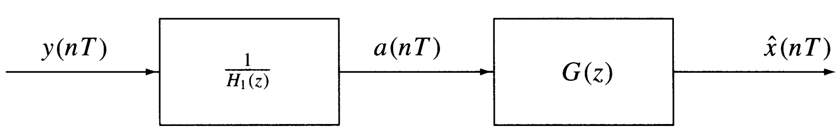

FIGURE 6.12.2

The mean square error is given by

$$E\{e^2(nT)\} = E\left\{\left[x(nT) - \sum_{k=-\infty}^{\infty} g(kT)a(nT-kT)\right]^2\right\}$$

$$= E\{x^2(nT)\} - 2E\left\{\sum_{k=-\infty}^{\infty} g(kT)x(nT)a(nT-kT)\right\}$$

$$+ E\left\{\left[\sum_{k=-\infty}^{\infty} g(kT)a(nT-kT)\right]^2\right\}$$

$$= R_{xx}(0) - 2\sum_{k=-\infty}^{\infty} g(kT)R_{xa}(kT) + \gamma^2 \sum_{k=-\infty}^{\infty} g^2(kT)$$

$$= R_{xx}(0) + \sum_{k=-\infty}^{\infty}\left[\gamma\, g(kT) - \frac{R_{xa}(kT)}{\gamma}\right]^2 - \frac{1}{\gamma^2}\sum_{k=-\infty}^{\infty} R_{xa}^2(kT)$$

To minimize the error we must set the quantity in the brackets equal to zero. Hence,

$$g(nT) = \frac{1}{\gamma^2} R_{xa}(nT) \qquad -\infty < n < \infty$$

and its Z-transform is

$$G(z) = \frac{1}{\gamma^2} S_{xa}(z)$$

but from (6.12.17) (because $\upsilon(nT) = x(nT)$ implies that $H_1(z) = 1$) we have

$$S_{xy}(z) = H_1(z^{-1})S_{xa}(z) \quad \text{or} \quad S_{xa}(z) = \frac{S_{xy}(z)}{H_1(z^{-1})} \tag{6.12.23}$$

$$G(z) = \frac{1}{\gamma^2}\frac{S_{xy}(z)}{H_1(z^{-1})} \tag{6.12.24}$$

From Figure 6.12.2, the optimum filter is given by (see also [6.12.20])

$$H(z) = \frac{1}{H_1(z)}G(z) = \frac{S_{xy}(z)}{\gamma^2 H_1(z)H_1(z^{-1})} = \frac{S_{xy}(z)}{S_{yy}(z)} \tag{6.12.25}$$

The mean square error for an optimum filter is

$$E\{e^2(nT)\} = R_{xx}(0) - \frac{1}{\gamma^2}\sum_{k=-\infty}^{\infty} R_{xa}^2(kT) \tag{6.12.26}$$

Applying Parseval's theorem in the above equation, we obtain

$$\begin{aligned} E\{e^2(nT)\} &= \frac{1}{2\pi j}\oint_C \left[S_{xx}(z) - \frac{1}{\gamma^2} S_{xa}(z) S_{xa}(z^{-1}) \right] \frac{dz}{z} \\ &= \frac{1}{2\pi j}\oint_C \left[S_{xx}(z) - \frac{S_{xy}(z) S_{xy}(z^{-1})}{S_{yy}(z)} \right] \frac{dz}{z} \\ &= \frac{1}{2\pi j}\oint_C \left[S_{xx}(z) - H(z) S_{xy}(z^{-1}) \right] \frac{dz}{z} \end{aligned} \tag{6.12.27}$$

where C can be the unit circle.

6.13 Relationship Between the Laplace and Z-Transform

The one-sided Laplace transform and its inverse are given by the following two equations:

$$F(s) \doteq \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t) e^{-st} dt \qquad \operatorname{Re}\{s\} > \sigma_c \tag{6.13.1}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s) e^{st} ds \qquad c > \sigma_c \tag{6.13.2}$$

where σ_c is the abscissa of convergence.

The Laplace transform of a sampled function

$$f_s(t) = f(t)\sum_{k=-\infty}^{\infty} \delta(t-nT) \doteq f(t)\,\mathrm{comb}_T(t) = \sum_{k=-\infty}^{\infty} f(nT)\delta(t-nT) \tag{6.13.3}$$

is given by

$$F_s(s) \doteq \mathcal{L}\{f_s(t)\} = \sum_{k=-\infty}^{\infty} f(nT) e^{-nTs} \tag{6.13.4}$$

because

$$\mathcal{L}\{\delta(t-nT)\} = \int_{-\infty}^{\infty} \delta(t-nT) e^{-st} dt = e^{-snT} \tag{6.13.5}$$

From (6.13.4) we obtain

$$F(z) = F_s(s)\big|_{s=T^{-1}\ell n\, z} \tag{6.13.6}$$

and, hence,

$$F(z)\big|_{z=e^{Ts}} = F_s(s) \doteq \mathscr{L}\{f_s(t)\} = \mathscr{L}\{f(t)\,\text{comb}_T(t)\} \tag{6.13.7}$$

If the region of convergence for $F(z)$ includes the unit circle, $|z| = 1$, then

$$F_s(\omega) = F(z)\big|_{z=e^{j\omega T}} = \sum_{n=-\infty}^{\infty} f(nT)e^{-j\omega nT} \tag{6.13.8}$$

$$F_s(s+j\omega_s) = F_s(s) = \text{periodic} \qquad \omega_s = \frac{2\pi}{T} \tag{6.13.9}$$

The knowledge of $F_s(s)$ in the strip $-\omega_s/2 < \omega \le \omega_s/2$ determines $F_s(s)$ for all s. The transformation $z = e^{sT}$ maps this strip uniquely onto the complex z-plane. Therefore, $F(z)$ contains all the information in $F_s(s)$ without redundancy. Letting $\sigma = s + j\omega$, then

$$z = e^{\sigma T}e^{j\omega T} \tag{6.13.10}$$

Because $|z| = e^{\sigma T}$, we obtain

$$|z| = \begin{cases} <1 & \sigma < 0 \\ =1 & \sigma = 0 \\ >1 & \sigma > 0 \end{cases} \tag{6.13.11}$$

Therefore, we have the following correspondence between the s- and z-planes:

1. Points in the left half of the s-plane are mapped inside the unit circle in the z-plane.
2. Points on the $j\omega$-axis are mapped onto the unit circle.
3. Points in the right half of the s-plane are mapped outside the unit circle.
4. Lines parallel to the $j\omega$-axis are mapped into circles with radius $|z| = e^{\sigma T}$.
5. Lines parallel to the σ-axis are mapped into rays of the form $\arg z = \omega T$ radians from $z = 0$.
6. The origin of the s-plane corresponds to $z = 1$.
7. The σ-axis corresponds to the positive $u = \text{Re}\, z$-axis.
8. As ω varies between $-\omega_s/2$ and $\omega_s/2$, $\arg z = \omega T$ varies between $-\pi$ and π radians.

Let $f(t)$ and $g(t)$ be causal functions with Laplace transforms $F(s)$ and $G(s)$ that converge absolutely for $\text{Re}\, s > \sigma_f$ and $\text{Re}\, s > \sigma_g$, respectively; then

$$\mathscr{L}\{f(t)g(t)\} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(p)G(s-p)\,dp \tag{6.13.12}$$

The contour is parallel to the imaginary axis in the complex p-plane with

$$\sigma = \text{Re}\, s > \sigma_f + \sigma_g \quad \text{and} \quad \sigma_f < c < \sigma - \sigma_g \tag{6.13.13}$$

With this choice the poles $G(s-p)$ lie to the right of the integration path.
For causal $f(t)$, its sampling form is given by

$$f_s(t)=f(t)\sum_{n=0}^{\infty}\delta(t-nT)\doteq f(t)\,\mathrm{comb}_T(t)=\sum_{n=0}^{\infty}f(nT)\delta(t-nT) \tag{6.13.14}$$

If

$$g(t)=\mathrm{comb}_T(t)\doteq\sum_{n=0}^{\infty}\delta(t-nT) \tag{6.13.15}$$

then its Laplace transform is

$$G(s)=\mathscr{L}\{g(t)\}=\sum_{n=0}^{\infty}e^{-nTs}=\frac{1}{1-e^{-Ts}} \qquad \mathrm{Re}\,s>0 \tag{6.13.16}$$

Because $\sigma_g = 0$, then (6.13.12) becomes

$$F_s(s)=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}\frac{F(p)}{1-e^{-(s-p)T}}\,dp \qquad \sigma>\sigma_f,\ \sigma_f<c<\sigma \tag{6.13.17}$$

The distance p in Figure 6.13.1 is given by

$$p = c + Re^{j\theta} \qquad \pi/2 \le \theta \le 3\pi/2 \tag{6.13.18}$$

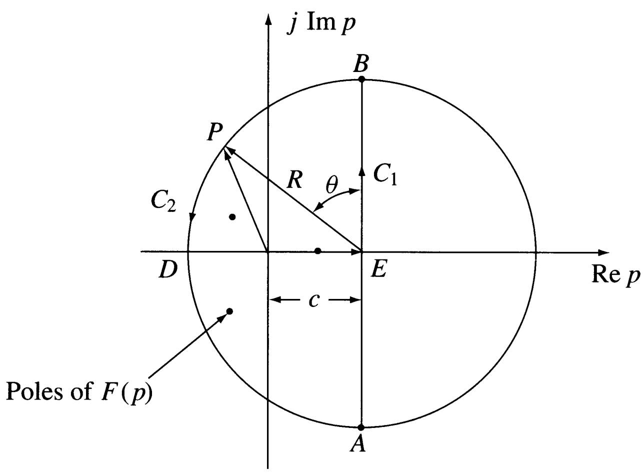

FIGURE 6.13.1

If the function $F(p)$ is analytic for some $|p|$ greater than a finite number R_0 and has a zero at infinity, then in the limit as $R\to\infty$ the integral along the path BDA is identically zero and the integral along the path AEB averages to $F_s(s)$. The contour $C_1 + C_2$ encloses all the poles of $F(p)$. Because of these assumptions, $F(p)$ must have a Laurent series expansion of the form

$$F(p)=\frac{a_{-1}}{p}+\frac{a_{-2}}{p2}+\cdots=\frac{a_{-1}}{p}+\frac{Q(p)}{p2} \qquad |p|>R_0 \qquad (6.13.19)$$

$Q(p)$ is analytic in this domain and

$$|Q(p)| < M < \infty \qquad |p| > R_0 \qquad (6.13.20)$$

Therefore, from (6.13.19)

$$a_{-1}=\lim_{p\to\infty} pF(p) \qquad (6.13.21)$$

From the initial value theorem

$$a_{-1} = f(0+) \qquad (6.13.22)$$

Applying Cauchy's residue theorem to (6.13.17), we obtain

$$F_s(s)=\sum_k \mathrm{Res}\left\{\frac{F(p)}{1-e^{pT}e^{-sT}}\right\}\Bigg|_{p=p_k} - \lim_{R\to\infty}\frac{1}{2\pi j}\int_{C_2}\frac{F(p)}{1-e^{pT}e^{-sT}}dp \qquad (6.13.23)$$

where $\{p_k\}$ are the poles of $F(p)$ and $\sigma = \mathrm{Re}\{s\} > \sigma_f$.

Introducing (6.13.22) and (6.13.19) into the above equation, it can be shown (see Jury, 1973)

$$F_s(s)=\sum_k \mathrm{Res}\left\{\frac{F(p)}{1-e^{pT}e^{-sT}}\right\}\Bigg|_{p=p_k} - \frac{f(0+)}{2} \qquad (6.13.24)$$

By letting $z = e^{sT}$, the above equation becomes

$$F(z)=F_s(s)\Big|_{s=\frac{1}{T}\ell n z}=\sum_k \mathrm{Res}\left\{\frac{F(p)}{1-e^{pT}z^{-1}}\right\}\Bigg|_{p=p_k} - \frac{f(0+)}{2}, \qquad |z|>e^{\sigma_f T} \qquad (6.13.25)$$

Example

The Laplace transform of $f(t) = tu(t)$ is $1/s^2$. The integrand $|te^{-\sigma t}e^{-j\omega t}| < \infty$ for $\sigma > 0$ implies that the region of convergence is $\mathrm{Re}\{s\} > 0$. Because $f(t)$ has a double pole at $s = 0$, (6.13.25) becomes

$$F(z)=\mathrm{Res}\left\{\frac{1}{p^2\left(1-e^{pT}z^{-1}\right)}\right\}\Bigg|_{p=0} - \frac{0}{2}$$

$$=\frac{d}{dp}\frac{p^2}{p^2\left(1-e^{pT}z^{-1}\right)}\Bigg|_{p=0}=\frac{Tz^{-1}}{\left(1-z^{-1}\right)^2}$$

Example

The Laplace transform of $f(t) = e^{-a\,t}\ u(t)$ is $1/(s + a)$. The ROC is Res > – a and from (6.13.25) we obtain

$$F(z) = \mathrm{Res}\left\{\frac{1}{(p+a)\left(1-e^{pT}z^{-1}\right)}\right\}\Bigg|_{p=-a} - \frac{1}{2} = \frac{1}{1-e^{-aT}z^{-1}} - \frac{1}{2}$$

The inverse transform is

$$f(nT) = -\frac{1}{2}\delta(n) + e^{-anT}u(nT)$$

If we had proceeded to find the Z-transform from $f(nT) = \exp(-anT)u(nT)$, we would have found $F(z) = 1/(1 - e^{-a\,T} - z^{-1})$. Hence, to make a causal signal $f(t)$ consistent with $F(s)$ and the inversion formula, $f(0)$ should be assigned the value $f(0+)/2$.

It is conventional in calculating with the Z-transform of causal signals to assign the value of $f(0+)$ to $f(0)$. With this convention the formula for calculating $F(z)$ from $F(s)$ reduces to

$$F(z) = \sum_{\mathrm{k}} \mathrm{Res}\left\{\frac{F(p)}{1-e^{pT}z^{-1}}\right\}\Bigg|_{p=p_k}, \quad |z| > e^{\sigma_f T} \tag{6.13.26}$$

6.14 Relationship to the Fourier Transform

The sampled signal can be represented by

$$f_s(t) = \sum_{n=-\infty}^{\infty} f(nT)\delta(t-nT) \tag{6.14.1}$$

with corresponding Laplace and Fourier transforms

$$F_s(s) = \sum_{n=-\infty}^{\infty} f(nT)e^{-snT} \tag{6.14.2}$$

$$F_s(\omega) = \sum_{n=-\infty}^{\infty} f(nT)e^{-j\omega nT} \tag{6.14.3}$$

If we set $z = e^{sT}$ in the definition of the Z-transform, we see that

$$F_s(s) = F(z)\big|_{z=e^{sT}} \tag{6.14.4}$$

If the region of convergence for $F(z)$ includes the unit circle, $|z| = 1$, then

$$F_s(\omega) = F(z)\big|_{z=e^{j\omega T}} \tag{6.14.5}$$

Because $F_s(s)$ is periodic with period $\omega_s = 2\pi / T$, we need only consider the strip $-\omega_s/2 < \omega \le \omega_s/2$, which uniquely determines $F_s(s)$ for all s. The transformation $z = \exp(sT)$ maps this strip uniquely onto the complex z-plane so that $F(z)$ contains all the information in $F_s(s)$ without the redundancy.

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Appendix: Tables

TABLE 1 Z-Transform Properties for Positive-Time Sequences

1. Linearity

$$\mathcal{Z}\{c_i f_i(nT)\} = c_i F_i(z) \qquad |z| > R_i, \; c_i \text{ are constants}$$

$$\mathcal{Z}\left\{\sum_{i=0}^{\ell} c_i f_i(nT)\right\} = \sum_{i=0}^{\ell} c_i F_i(z) \qquad |z| > \max R_i$$

2. Shifting Property

$$\mathcal{Z}\{f(nT - kT)\} = z^{-k}F(z), \qquad f(-nT) = 0 \qquad \text{for } n = 1, 2, \ldots$$

$$\mathcal{Z}\{f(nT - kT)\} = z^{-k}F(z) + \sum_{n=1}^{k} f(-nT) z^{-(k-n)}$$

$$\mathcal{Z}\{f(nT + kT)\} = z^{k}F(z) - \sum_{n=0}^{k-1} f(nT) z^{k-n}$$

$$\mathcal{Z}\{f(nT + T)\} = z[F(z) - f(0)]$$

3. Time Scaling

$$\mathcal{Z}\{a^{nT} f(nT)\} = F(a^{-T}z) = \sum_{n=0}^{\infty} f(nT)(a^{-T}z)^{-n} \qquad |z| > a^T$$

4. Periodic Sequence

$$\mathcal{Z}\{f(nT)\} = \frac{z^N}{z^N - 1} F_{(1)}(z) \qquad |z| > R$$

N = number of time units in a period

R = radius of convergence of $F_{(1)}(z)$

$F_{(1)}(z)$ = Z-transform of the first period

TABLE 1 Z-Transform Properties for Positive-Time Sequences (continued)

5. Multiplication by n and nT

$$\mathcal{Z}\{nf(nT)\} = -z\frac{dF(z)}{dz} \qquad |z| > R$$

$$\mathcal{Z}\{nTf(nT)\} = -zT\frac{dF(z)}{dz} \qquad |z| > R$$

R = radius of convergence of $F(z)$

6. Convolution

$$\mathcal{Z}\{f(nT)\} = F(z) \qquad |z| > R_1$$

$$\mathcal{Z}\{h(nT)\} = H(z) \qquad |z| > R_2$$

$$\mathcal{Z}\{f(nT) * h(nT)\} = F(z)H(z) \qquad |z| > \max(R_1, R_2)$$

7. Initial Value

$$f(0T) = \lim_{z\to\infty} F(z) \qquad |z| > R \qquad \text{if } F(\infty) \text{ exists}$$

8. Final Value

$$\lim_{n\to\infty} f(nT) = \lim_{z\to 1}(z-1)F(z) \qquad \text{if } f(\infty T) \text{ exists}$$

9. Multiplication by $(nT)^k$

$$\mathcal{Z}\{n^kT^kf(nT)\} = -Tz\frac{d}{dz}\mathcal{Z}\{(nT)^{k-1}f(nT)\} \qquad k > 0 \text{ and is an integer}$$

10. Complex Conjugate Signals

$$\mathcal{Z}\{f(nT)\} = F(z) \qquad |z| > R$$

$$\mathcal{Z}\{f^*(nT)\} = F^*(z^*) \qquad |z| > R$$

11. Transform of Product

$$\mathcal{Z}\{f(nT)\} = F(z) \qquad |z| > R_f$$

$$\mathcal{Z}\{h(nT)\} = H(z) \qquad |z| > R_h$$

$$\mathcal{Z}\{f(nT)h(nT)\} = \frac{1}{2\pi j}\oint_C F(\tau)H\left(\frac{z}{\tau}\right)\frac{d\tau}{\tau}, \qquad |z| > R_fR_h,\; R_f < |\tau| < \frac{|z|}{R_h}$$

counterclockwise integration

12. Parseval's Theorem

$$\mathcal{Z}\{f(nT)\} = F(z) \qquad |z| > R_f$$

$$\mathcal{Z}\{h(nT)\} = H(z) \qquad |z| > R_h$$

$$\sum_{n=0}^{\infty} f(nT)h(nT) = \frac{1}{2\pi j}\oint_C F(z)H(z^{-1})\frac{dz}{z} \qquad |z| = 1 > R_fR_h$$

counterclockwise integration

13. Correlation

$$f(nT) \otimes h(nT) = \sum_{m=0}^{\infty} f(mT)h(mT - nT) = \frac{1}{2\pi j}\oint_C F(\tau)H\left(\frac{1}{\tau}\right)\tau^{n-1}\,d\tau \qquad n \geq 1$$

Both $f(nT)$ and $h(nT)$ must exist for $|z| > 1$. The integration is taken in counterclockwise direction.

TABLE 1 Z-Transform Properties for Positive-Time Sequences (continued)

14. Transforms with Parameters

$$\mathcal{Z}\left\{\frac{\partial}{\partial a} f(nT, a)\right\} = \frac{\partial}{\partial a} F(z, a)$$

$$\mathcal{Z}\left\{\lim_{a \to a_0} f(nT, a)\right\} = \lim_{a \to a_0} F(z, a)$$

$$\mathcal{Z}\left\{\int_{a_0}^{a_1} f(nT, a)\, da\right\} = \int_{a_0}^{a_1} F(z, a)\, da \quad \text{finite interval}$$

TABLE 2 Z-Transform Properties for Positive- and Negative-Time Sequences

1. Linearity

$$\mathcal{Z}_{II}\left\{\sum_{i=0}^{\ell} c_i f_i(nT)\right\} = \sum_{i=0}^{\ell} c_i F_i(z) \qquad \max R_{i+} < |z| < \min R_{i-}$$

2. Shifting Property

$$\mathcal{Z}_{II}\{f(nT \pm kT)\} = z^{\pm k} F(z) \qquad R_+ < |z| < R_-$$

3. Scaling

$$\mathcal{Z}_{II}\{f(nT)\} = F(z) \qquad R_+ < |z| < R_-$$

$$\mathcal{Z}_{II}\{a^{nT} f(nT)\} = F(a^{-T} z) \qquad |a^T| R_+ < |z| < |a^T| R_-$$

4. Time Reversal

$$\mathcal{Z}_{II}\{f(nT)\} = F(z) \qquad R_+ < |z| < R_-$$

$$\mathcal{Z}_{II}\{f(-nT)\} = F(z^{-1}) \qquad \frac{1}{R_-} < |z| < \frac{1}{R_+}$$

5. Multiplication by nT

$$\mathcal{Z}_{II}\{f(nT)\} = F(z) \qquad R_+ < |z| < R_-$$

$$\mathcal{Z}_{II}\{nTf(nT)\} = -zT\frac{dF(z)}{dz} \qquad R_+ < |z| < R_-$$

6. Convolution

$$\mathcal{Z}_{II}\{f_1(nT) * f_2(nT)\} = F_1(z) F_2(z)$$

$$\text{ROC } F_1(z) \cup \text{ROC } F_2(z) \qquad \max(R_{+f_1}, R_{+f_2}) < |z| < \min(R_{-f_1}, R_{-f_2})$$

TABLE 2 Z-Transform Properties for Positive- and Negative-Time Sequences (continued)

7. Correlation

$$R_{f_1 f_2}(z) = \mathcal{Z}_{II}\{f_1(nT) \otimes f_2(nT)\} = F_1(z)F_2(z^{-1})$$

$$\text{ROC } F_1(z) \cup \text{ROC } F_2(z^{-1}) \qquad \max(R_{+f_1}, R_{+f_2}) < |z| < \min(R_{-f_1}, R_{-f_2})$$

8. Multiplication by e^{-anT}

$$\mathcal{Z}_{II}\{f(nT)\} = F(z) \qquad R_+ < |z| < R_-$$

$$\mathcal{Z}_{II}\{e^{-anT} f(nT)\} = F(e^{aT} z) \qquad \left|e^{-aT}\right| R_+ < |z| < \left|e^{-aT}\right| R_-$$

9. Frequency Translation

$$G(\omega) = \mathcal{Z}_{II}\{e^{j\omega_0 nT} f(nT)\} = G(z)|_{z=e^{j\omega T}} = F\left(e^{j(\omega-\omega_0)T}\right) = F(\omega - \omega_0)$$

ROC of $F(z)$ must include the unit circle

10. Product

$$\mathcal{Z}_{II}\{f(nT)\} = F(z) \qquad R_{+f} < |z| < R_{-f}$$

$$\mathcal{Z}_{II}\{h(nT)\} = H(z) \qquad R_{+h} < |z| < R_{-h}$$

$$\mathcal{Z}_{II}\{f(nT)h(nT)\} = G(z) = \frac{1}{2\pi j}\oint_C F(\tau)H\left(\frac{z}{\tau}\right)\frac{d\tau}{\tau}, \qquad R_{+f}R_{+h} < |z| < R_{-f}R_{-h}$$

$$\max\left(R_{+f}, \frac{|z|}{R_{-h}}\right) < |\tau| < \min\left(R_{-f}, \frac{|z|}{R_{+h}}\right)$$

counterclockwise integration

11. Parseval's Theorem

$$\mathcal{Z}_{II}\{f(nT)\} = F(z) \qquad R_{+f} < |z| < R_{-f}$$

$$\mathcal{Z}_{II}\{h(nT)\} = H(z) \qquad R_{+h} < |z| < R_{-h}$$

$$\sum_{n=-\infty}^{\infty} f(nT)h(nT) = \frac{1}{2\pi j}\oint_C F(z)H(z^{-1})\frac{dz}{z} \qquad R_{+f}R_{+h} < |z| = 1 < R_{-f}R_{-h}$$

$$\max\left(R_{+f}, \frac{1}{R_{-h}}\right) < |z| < \min\left(R_{-f}, \frac{1}{R_{+h}}\right)$$

counterclockwise integration

12. Complex Conjugate Signals

$$\mathcal{Z}_{II}\{f(nT)\} = F(z) \qquad R_{+f} < |z| < R_{-f}$$

$$\mathcal{Z}_{II}\{f^*(nT)\} = F^*(z^*) \qquad R_{+f} < |z| < R_{-f}$$

TABLE 3 Inverse Transforms of the Partial Fractions of $F(z)$

Partial Fraction Term	Inverse Transform Term If $F(z)$ Converges Absolutely for Some $\vert z\vert > \vert a\vert$
$\frac{z}{z-a}$	$a^k, \quad k \geq 0$
$\frac{z^2}{(z-a)^2}$	$(k+1)a^k, \quad k \geq 0$
$\frac{z^3}{(z-a)^3}$	$\frac{1}{2}(k+1)(k+2)a^k, \quad k \geq 0$
$\vdots$	$\vdots$
$\frac{z^n}{(z-a)^n}$	$\frac{1}{(n-1)!}(k+1)(k+2)\cdots(k+n-1)a^k, \quad k \geq 0$

Partial Fraction Term	Inverse Transform Term If $F(z)$ Converges Absolutely for Some $\vert z\vert < \vert a\vert$
$\frac{z}{z-a}$	$-a^k, \quad k \leq -1$
$\frac{z^2}{(z-a)^2}$	$-(k+1)a^k, \quad k \leq -1$
$\frac{z^3}{(z-a)^3}$	$-\frac{1}{2}(k+1)(k+2)a^k, \quad k \leq -1$
$\vdots$	$\vdots$
$\frac{z^n}{(z-a)^n}$	$-\frac{1}{(n-1)!}(k+1)(k+2)\cdots(k+n-1)a^k, \quad k \leq -1$

TABLE 4 Inverse Transforms of the Partial Fractions of $F_i(z)$[a]

Elementary Transform Term $F_i(z)$	Corresponding Time Sequence	
	(I) $F_i(z)$ converges for $\vert z\vert > R_c$	(II) $F_i(z)$ converges for $\vert z\vert < R_c$
1. $\frac{1}{z-a}$	$a^{k-1}\vert_{k\geq 1}$	$-a^{k-1}\vert_{k\leq 0}$
2. $\frac{z}{(z-a)^2}$	$ka^{k-1}\vert_{k\geq 1}$	$-ka^{k-1}\vert_{k\leq 0}$
3. $\frac{z(z+a)}{(z-a)^3}$	$k^2a^{k-1}\vert_{k\geq 1}$	$-k^2a^{k-1}\vert_{k\leq 0}$
4. $\frac{z(z^2+4az+a^2)}{(z-a)^4}$	$k^3a^{k-1}\vert_{k\geq 1}$	$-k^3a^{k-1}\vert_{k\leq 0}$

[a]The function must be a proper function

TABLE 5 Z-Transform Pairs[a]

Number	Discrete Time-Function $f(n), n \geq 0$	z-Transform $\mathcal{F}(z) = \mathcal{Z}[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n}$ $\quad |z| > R$
1	$u(n) = \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{otherwise} \end{cases}$	$\frac{z}{z-1}$
2	$e^{-\alpha n}$	$\frac{z}{z-e^{-\alpha}}$
3	n	$\frac{z}{(z-1)^2}$
4	n^2	$\frac{z(z+1)}{(z-1)^3}$
5	n^3	$\frac{z(z^2+4z+1)}{(z-1)^4}$
6	n^4	$\frac{z(z^3+11z^2+11z+1)}{(z-1)^5}$
7	n^5	$\frac{z(z^4+26z^3+66z^2+26z+1)}{(z-1)^6}$
8	n^k	$(-1)^k D^k \left(\frac{z}{z-1}\right); \; D = z\frac{d}{dz}$
9	$u(n-k)$	$\frac{z^{-k+1}}{z-1}$
10	$e^{-\alpha n} f(n)$	$\mathcal{F}(e^{\alpha} z)$
11	$n^{(2)} = n(n-1)$	$2\frac{z}{(z-1)^3}$
12	$n^{(3)} = n(n-1)(n-2)$	$3!\frac{z}{(z-1)^4}$
13	$n^{(k)} = n(n-1)(n-2)\ldots(n-k+1)$	$k!\frac{z}{(z-1)^{k+1}}$
14	$n^{[k]} f(n), n^{[k]} = n(n+1)(n+2)\ldots(n+k-1)$	$(-1)^k z^k \frac{d^k}{dz^k}[\mathcal{F}(z)]$
15	$(-1)^k n(n-1)(n-2)\ldots(n-k+1) f_{n-k+1}{}^{\dagger}$	$z\mathcal{F}^{(k)}(z), \; \mathcal{F}^{(k)}(z) = \frac{d^k}{dz^k}\mathcal{F}(z)$
16	$-(n-1) f_{n-1}$	$\mathcal{F}^{(1)}(z)$
17	$(-1)^k (n-1)(n-2)\ldots(n-k) f_{n-k}$	$\mathcal{F}^{(k)}(z)$
18	$nf(n)$	$-z\mathcal{F}^{(1)}(z)$
19	$n^2 f(n)$	$z^2\mathcal{F}^{(2)}(z) + z\mathcal{F}^{(1)}(z)$

[a] Source: E. I. Jury, *Theory and Application of the Z-Transform Method*, New York, John Wiley & Sons, Inc., 1964. With permission.

† It may be noted that f_n is the same as $f(n)$

TABLE 5 Z-Transform Pairs[a] (continued)

Number	Discrete Time-Function $f(n), n \geq 0$	z-Transform $\mathcal{F}(z) = \mathcal{Z}[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n}$ $\quad |z| > R$
20	$n^3 f(n)$	$-z^3\mathcal{F}^{(3)}(z) - 3z^2\mathcal{F}^{(2)}(z) - z\mathcal{F}^{(1)}(z)$
21	$\dfrac{c^n}{n!}$	$e^{c/z}$
22	$\dfrac{(\ln c)^n}{n!}$	$c^{1/z}$
23	$\binom{k}{n} c^n a^{k-n},\ \binom{k}{n} = \dfrac{k!}{(k-n)!n!},\ n \leq k$	$\dfrac{(az+c)^k}{z^k}$
24	$\binom{n+k}{k} c^n$	$\dfrac{z^{k+1}}{(z-c)^{k+1}}$
25	$\dfrac{c^n}{n!},\quad (n = 1, 3, 5, 7, \ldots)$	$\sinh\left(\dfrac{c}{z}\right)$
26	$\dfrac{c^n}{n!},\quad (n = 0, 2, 4, 6, \ldots)$	$\cosh\left(\dfrac{c}{z}\right)$
27	$\sin(\alpha n)$	$\dfrac{z \sin\alpha}{z^2 - 2z\cos\alpha + 1}$
28	$\cos(\alpha n)$	$\dfrac{z(z - \cos\alpha)}{z^2 - 2z\cos\alpha + 1}$
29	$\sin(\alpha n + \psi)$	$\dfrac{z^2 \sin\psi + z\sin(\alpha - \psi)}{z^2 - 2z\cos\alpha + 1}$
30	$\cosh(\alpha n)$	$\dfrac{z(z - \cosh\alpha)}{z^2 - 2z\cosh\alpha + 1}$
31	$\sinh(\alpha n)$	$\dfrac{z\sinh\alpha}{z^2 - 2z\cosh\alpha + 1}$
32	$\dfrac{1}{n},\quad n > 0$	$\ln\dfrac{z}{z-1}$
33	$\dfrac{1 - e^{-\alpha n}}{n}$	$\alpha + \ln\dfrac{z - e^{-\alpha}}{z-1},\quad \alpha > 0$
34	$\dfrac{\sin\alpha n}{n}$	$\alpha + \tan^{-1}\dfrac{\sin\alpha}{z - \cos\alpha},\quad \alpha > 0$
35	$\dfrac{\cos\alpha n}{n},\quad n > 0$	$\ln\dfrac{z}{\sqrt{z^2 - 2z\cos\alpha + 1}}$
36	$\dfrac{(n+1)(n+2)\ldots(n+k-1)}{(k-1)!}$	$\left(1 - \dfrac{1}{z}\right)^{-k},\quad k = 2, 3, \ldots$
37	$\sum_{m=1}^{n} \dfrac{1}{m}$	$\dfrac{z}{z-1}\ln\dfrac{z}{z-1}$
38	$\sum_{m=0}^{n-1} \dfrac{1}{m!}$	$\dfrac{e^{1/z}}{z-1}$

TABLE 5 Z-Transform Pairs[a] (continued)

Number	Discrete Time-Function $f(n), n \geq 0$	z-Transform $\mathcal{F}(z) = \mathcal{Z}[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n}$ $\quad |z| > R$
39	$\dfrac{(-1)^{(n-p)/2}}{2^n \left(\frac{n-p}{2}\right)! \left(\frac{n+p}{2}\right)!}$, for $n \geq p$ and $n - p =$ even; $= 0$, for $n < p$ or $n - p =$ odd	$J_p(z^{-1})$
40	$\begin{cases} \binom{\alpha}{n/k} b^{n/k}, & n = mk, \quad (m = 0, 1, 2, \ldots) \\ = 0 & n \neq mk \end{cases}$	$\left(\dfrac{z^k + b}{z^k}\right)^{\alpha}$
41	$a^n P_n(x) = \dfrac{a^n}{2^n n!} \left(\dfrac{d}{dx}\right)^n (x^2 - 1)^n$	$\dfrac{z}{\sqrt{z^2 - 2xaz + a^2}}$
42	$a^n T_n(x) = a^n \cos(n \cos^{-1} x)$	$\dfrac{z(z - ax)}{z^2 - 2xaz + a^2}$
43	$\dfrac{L_n(x)}{n!} = \sum_{r=0}^{\infty} \binom{n}{r} \dfrac{(-x)^r}{r!}$	$\dfrac{z}{z-1} e^{-x/(z-1)}$
44	$\dfrac{H_n(x)}{n!} = \sum_{k=0}^{[n/2]} \dfrac{(-1)^{n-k} x^{n-2k}}{k(n-2k)! 2^k}$	$e^{-x/z - 1/2z^2}$
45	$a^n P_n^m(x) = a^n (1 - x^2)^{m/2} \left(\dfrac{d}{dx}\right)^m P_n(x)$, $m =$ integer	$\dfrac{(2m)!}{2^m m!} \dfrac{z^{m+1}(1 - x^2)^{m/2} a^m}{(z^2 - 2xaz + a^2)^{m+1/2}}$
46	$\dfrac{L_n^m(x)}{n!} = \left(\dfrac{d}{dx}\right)^m \dfrac{L_n(x)}{n!}$, $\quad m =$ integer	$\dfrac{(-1)^m z}{(z-1)^{m+1}} e^{-x/(z-1)}$
47	$-\dfrac{1}{n} \mathcal{Z}^{-1} \left[z \dfrac{\mathcal{F}'(z)}{\mathcal{F}(z)} - \dfrac{\mathcal{G}'(z)}{\mathcal{G}(z)} \right]$, where $\mathcal{F}(z)$ and $\mathcal{G}(z)$ are rational polynomials in z of the same order	$\ln \dfrac{\mathcal{F}(z)}{\mathcal{G}(z)}$
48	$\dfrac{1}{m(m+1)(m+2)\ldots(m+n)}$	$(m-1)! z^m \left[e^{1/z} - \sum_{k=0}^{m-1} \dfrac{1}{k! z^k} \right]$
49	$\dfrac{\sin(\alpha n)}{n!}$	$e^{\cos\alpha/z} \cdot \sin\left(\dfrac{\sin\alpha}{z}\right)$
50	$\dfrac{\cos(\alpha n)}{n!}$	$e^{\cos\alpha/z} \cdot \cos\left(\dfrac{\sin\alpha}{z}\right)$
51	$\sum_{k=0}^{n} f_k g_{n-k}$	$\mathcal{F}(z)\mathcal{G}(z)$
52	$\sum_{k=0}^{n} k f_k g_{n-k}$	$-\mathcal{F}^{(1)}(z)\mathcal{G}(z)$, $\mathcal{F}^{(1)}(z) = \dfrac{d\mathcal{F}(z)}{dz}$
53	$\sum_{k=0}^{n} k^2 f_k g_{n-k}$	$\mathcal{F}^{(2)}(z)\mathcal{G}(z)$
54	$\dfrac{\alpha^n + (-\alpha)^n}{2\alpha^2}$	$\dfrac{1}{\alpha^2} \quad \dfrac{z^2}{z^2 - \alpha^2}$
55	$\dfrac{\alpha^n - \beta^n}{\alpha - \beta}$	$\dfrac{z}{(z - \alpha)(z - \beta)}$

TABLE 5 Z-Transform Pairs[a] (continued)

Number	Discrete Time-Function $f(n), n \geq 0$	z-Transform $\mathcal{F}(z) = \mathcal{Z}[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n}$ $\quad |z| > R$
56	$(n+k)^{(k)}$	$k!z^k \dfrac{z}{(z-1)^{k+1}}$
57	$(n-k)^{(k)}$	$k!z^{-k} \dfrac{z}{(z-1)^{k+1}}$
58	$\dfrac{(n \mp k)^{(m)}}{m!} e^{\alpha(n-k)}$	$\dfrac{z^{1\mp k} e^{m\alpha}}{(z-e^{\alpha})^{m+1}}$
59	$\dfrac{1}{n} \sin \dfrac{\pi}{2} n$	$\dfrac{\pi}{2} + \tan^{-1} \dfrac{1}{z}$
60	$\dfrac{\cos\alpha(2n-1)}{2n-1}, \quad n > 0$	$\dfrac{1}{4\sqrt{z}} \ln \dfrac{z + 2\sqrt{z}\cos\alpha + 1}{z - 2\sqrt{z}\cos\alpha + 1}$
61	$\dfrac{\gamma^n}{(\gamma-1)^2} + \dfrac{n}{1-\gamma} - \dfrac{1}{(1-\gamma)^2}$	$\dfrac{z}{(z-\gamma)(z-1)^2}$
62	$\dfrac{\gamma+a_0}{(\gamma-1)^2}\gamma^n + \dfrac{1+a_0}{1-\gamma}n + \left(\dfrac{1}{1-\gamma} - \dfrac{a_0+1}{(1-\gamma)^2}\right)$	$\dfrac{z(z+a_0)}{(z-\gamma)(z-1)^2}$
63	$a^n \cos \pi n$	$\dfrac{z}{z+a}$
64	$e^{-\alpha n} \cos an$	$\dfrac{z(z - e^{-\alpha}\cos a)}{z^2 - 2ze^{-\alpha}\cos a + e^{-2\alpha}}$
65	$e^{-\alpha n} \sinh(an + \psi)$	$\dfrac{z^2 \sinh\psi + ze^{-\alpha}\sinh(a-\psi)}{z^2 - 2ze^{-\alpha}\cosh a + e^{-2\alpha}}$
66	$\dfrac{\gamma^n}{(\gamma-\alpha)^2+\beta^2} + \dfrac{(\alpha^2+\beta^2)^{n/2}\sin(n\theta+\psi)}{\beta[(\alpha-\gamma)^2+\beta^2]^{1/2}}$ $\theta = \tan^{-1}\frac{\beta}{\alpha}$ $\psi = \tan^{-1}\frac{\beta}{\alpha-\gamma}$	$\dfrac{z}{(z-\gamma)[(z-\alpha)^2+\beta^2]}$
67	$\dfrac{n\gamma^{n-1}}{(\gamma-1)^3} - \dfrac{3\gamma^n}{(\gamma-1)^4}$ $+ \dfrac{1}{2}\left[\dfrac{n(n-1)}{(1-\gamma)^2} - \dfrac{4n}{(1-\gamma)^3} + \dfrac{6}{(1-\gamma)^4}\right]$	$\dfrac{z}{(z-\gamma)^2(z-1)^3}$
68	$\sum_{v=0}^{k}(-1)^v \binom{k}{v} \dfrac{(n+k-v)^{(k)}}{k!} e^{\alpha(n-v)}$	$\dfrac{z(z-1)^k}{(z-e^{\alpha})^{k+1}}$
69	$\dfrac{f(n)}{n}$	$\int_z^{\infty} p^{-1}\mathcal{F}(p)\,dp + \lim_{n\to 0} \dfrac{f(n)}{n}$
70	$\dfrac{f_{n+2}}{n+1}, \quad f_0 = 0, \; f_1 = 0$	$z\int_z^{\infty} \mathcal{F}(p)\,dp$

TABLE 5 Z-Transform Pairs[a] (continued)

Number	Discrete Time-Function $f(n), n \ge 0$	z-Transform $\mathcal{F}(z) = \mathcal{Z}[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n}$ $\quad |z| > R$
71	$\frac{1+a_0}{(1-\gamma)[(1-\alpha)^2+\beta^2]} + \frac{(\gamma+a_0)\gamma^n}{(\gamma-1)[(\gamma-\alpha)^2+\beta^2]} + \frac{[\alpha^2+\beta^2]^{n/2}[(a_0+\alpha)^2+\beta^2]^{1/2}}{\beta[(\alpha-1)^2+\beta^2]^{1/2}[(\alpha-\gamma)^2+\beta^2]^{1/2}}, \times \sin(n\theta+\psi+\lambda)$ $\psi = \psi_1+\psi_2,\ \psi_1 = -\tan^{-1}\frac{\beta}{\alpha-1},\ \theta = \tan^{-1}\frac{\beta}{\alpha}$ $\lambda = \tan^{-1}\frac{\beta}{a_0+\alpha},\ \psi_2 = -\tan^{-1}\frac{\beta}{\alpha-\gamma}$	$\frac{z(z+a_0)}{(z-1)(z-\gamma)[(z-\alpha)^2+\beta^2]}$
72	$(n+1)e^{\alpha n} - 2ne^{\alpha(n+1)} + e^{\alpha(n-2)}(n-1)$	$\left(\frac{z-1}{z-e^{\alpha}}\right)^2$
73	$(-1)^n\frac{\cos\alpha n}{n}, \quad n>0$	$\ln\frac{z}{\sqrt{z^2+2z\cos\alpha+1}}$
74	$\frac{(n+k)!}{n!}f_{n+k}, \quad f_n = 0, \text{ for } 0 \le n < k$	$(-1)^k z^{2k}\frac{d^k}{dz^k}[\mathcal{F}(z)]$
75	$\frac{f(n)}{n+h}, \quad h>0$	$z^h\int_z^{\infty} p^{-(1+h)}\mathcal{F}(p)\,dp$
76	$-na^n\cos\frac{\pi}{2}n$	$\frac{2a^2z^2}{(z^2+a^2)^2}$
77	$na^n\frac{1+\cos\pi n}{2}$	$\frac{2a^2z^2}{(z^2-a^2)^2}$
78	$a^n\sin\frac{\pi}{4}n\cdot\frac{1+\cos\pi n}{2}$	$\frac{a^2z^2}{z^4+a^4}$
79	$a^n\left(\frac{1+\cos\pi n}{2} - \cos\frac{\pi}{2}n\right)$	$\frac{2a^2z^2}{z^4-a^4}$
80	$\frac{P_n(x)}{n!}$	$e^{xz^{-1}}J_0(\sqrt{1-x^2}z^{-1})$
81	$\frac{P_n^{(m)}(x)}{(n+m)!}, m>0, P_n^m = 0, \text{ for } n<m$	$(-1)^m e^{xz^{-1}}J_m(\sqrt{1-x^2}z^{-1})$
82	$\frac{1}{(n+\alpha)^{\beta}}, \quad \alpha>0, \mathrm{Re}\,\beta>0$	$\Phi(z^{-1},\alpha,\beta)$, where $\Phi(1,\beta,\alpha) = \zeta(\beta,\alpha)$ = generalized Rieman-Zeta function
83	$a^n\left(\frac{1+\cos\pi n}{2} + \cos\frac{\pi}{2}n\right)$	$\frac{2z^4}{z^4-a^4}$

TABLE 5 Z-Transform Pairs[a] (continued)

Number	Discrete Time-Function $f(n), n \geq 0$	z-Transform $\mathcal{F}(z) = \mathcal{Z}[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n}$ $\quad |z| > R$
84	$\frac{c^n}{n}, \quad (n = 1, 2, 3, 4, \ldots)$	$\ln z - \ln(z - c)$
85	$\frac{c^n}{n}, \quad n = 2, 4, 6, 8, \ldots$	$\ln z - \frac{1}{2}\ln(z^2 - c^2)$
86	$n^2 c^n$	$\frac{cz(z+c)}{(z-c)^3}$
87	$n^3 c^n$	$\frac{cz(z^2 + 4cz + c^2)}{(z-c)^4}$
88	$n^k c^n$	$-\frac{d\mathcal{F}(z/c)}{dz}, \qquad \mathcal{F}(\mathcal{Z}) = \mathcal{Z}[n^{k-1}]$
89	$-\cos\frac{\pi}{2}n \sum_{i=0}^{(n-2)/4} \binom{n/2}{2i+1} a^{n-2-4i}(a^4 - b^4)^i$	$\frac{z^2}{z^4 + 2a^2z^2 + b^4}$
90	$n^k f(n), \quad k > 0$ and integer	$-z\frac{d}{dz}\mathcal{F}_1(z), \mathcal{F}_1(\mathcal{Z}) = \mathcal{Z}[n^{k-1}f(n)]$
91	$\frac{(n-1)(n-2)(n-3)\ldots(n-k+1)}{(k-1)!}a^{n-k}$	$\frac{1}{(z-a)^k}$
92	$\frac{k(k-1)(k-2)\ldots(k-n+1)}{n!}$	$\left(1 + \frac{1}{z}\right)^k$
93	$na^n \cos bn$	$\frac{[(z/a)^3 + z/a]\cos b - 2(z/a)^2}{[(z/a)^2 - 2(z/a)\cos b + 1]^2}$
94	$na^n \sin bn$	$\frac{(z/a)^3 \sin b - (z/a)\sin b}{[(z/a)^2 - 2(z/a)\cos b + 1]^2}$
95	$\frac{na^n}{(n+1)(n+2)}$	$\frac{z(a-2z)}{a^2}\ln\left(1 - \frac{a}{z}\right) - \frac{2}{a}z$
96	$\frac{(-a)^n}{(n+1)(2n+1)}$	$2\sqrt{z/a}\tan^{-1}\sqrt{a/z} - \frac{z}{a}\ln\left(1 + \frac{a}{z}\right)$
97	$\frac{a^n \sin \alpha n}{n+1}$	$\frac{z\cos\alpha}{a}\tan^{-1}\frac{a\sin\alpha}{z - a\cos\alpha} + \frac{z\sin\alpha}{2a}\ln\frac{z^2 - 2az\cos\alpha + a^2}{z^2}$
98	$\frac{a^n \cos(\pi/2)n \sin\alpha(n+1)}{n+1}$	$\frac{z}{4a}\ln\frac{z^2 + 2az\sin\alpha + a^2}{z^2 - 2az\sin\alpha + a^2}$
99	$\frac{1}{(2n)!}$	$\cosh(z^{-1/2})$
100	$\binom{-\frac{1}{2}}{n}(-a)^n$	$\sqrt{z/(z-a)}$
101	$\binom{-\frac{1}{2}}{\frac{n}{2}}a^n \cos\frac{\pi}{2}n$	$\frac{z}{\sqrt{z^2 - a^2}}$

TABLE 5 Z-Transform Pairs[a] (continued)

Number	Discrete Time-Function $f(n), n \geq 0$	z-Transform $\mathcal{F}(z) = \mathcal{Z}[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n}$ $\quad |z| > R$
102	$\dfrac{B_n(x)}{n!}$ $\quad B_n(x)$ are Bernoulli polynomials	$\dfrac{e^{x/z}}{z(e^{1/z} - 1)}$
103	$W_n(x) \doteq$ Tchebycheff polynomials of the second kind	$\dfrac{z^2}{z^2 - 2xz + 1}$
104	$\left\lvert \sin \dfrac{n\pi}{m} \right\rvert, \quad m = 1, 2, \ldots$	$\dfrac{z \sin \pi/m}{z^2 - 2z \cos \pi/m + 1} \dfrac{1 + z^{-m}}{1 - z^{-m}}$
105	$Q_n(x) = \sin(n \cos^{-1} x)$	$\dfrac{z}{z^2 - 2xz + 1}$