Laplace Transform → time → frequency

$$x(t) \xrightarrow{L} X(s)$$

Bilateral

$s = j\omega$

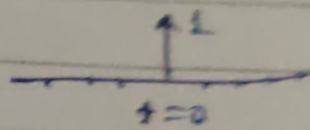
$$\boxed{X(s) = \int_{t=-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt}$$

Unilateral →

one side

$$\boxed{X(s) = \int_{0}^{\infty} x(t) \cdot e^{-st} \, dt}$$

$$\delta(t) \xrightarrow{L} 1$$



$$\Rightarrow \int \delta(t) = 1$$

$$\delta(t) \cdot e^{-st} \longrightarrow \delta(t) \cdot e^{-s \cdot 0}$$

$$\delta(t) \cdot e^{-st} \longrightarrow \delta(t)$$

$$\boxed{X(s) = 1}$$

$$\delta(t-1) \cdot e^{-st} \longrightarrow \underline{\delta(t-1)} \cdot e^{-s}$$

$t - 1 = 0$

$t = 1$

Convolution

$$\delta(t) * e^{-st} \longrightarrow e^{0} = 1$$

$$\delta(t) * x(t+3) \longrightarrow x(3)$$

$\delta(t) \longrightarrow 1$

$\int$ I. II

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$1/u(t) \xrightarrow{L} \frac{1}{s}$

$$u(t) \xrightarrow{L} = \int_{t=0}^{\infty} u(t) \cdot e^{-st} \cdot dt$$

$$= \int_{t=0}^{\infty} e^{-st} \cdot dt$$

$$= \left[\frac{e^{-st}}{-s} \right]_{t=0}^{\infty}$$

$$= -\frac{1}{s}\left[e^{-\infty} - e^{0} \right]$$

$$= -\frac{1}{s}\left[0 - 1\right] = \frac{1}{s}$$

$\Rightarrow t \xrightarrow{L} \frac{1}{s^2}$

$\Rightarrow t^2 \longrightarrow \frac{2}{s^3}$

$\Rightarrow t^n \longrightarrow \frac{\lfloor n}{s^{n+1}}$

Ex $t^4 \xrightarrow{L.T} \frac{\lfloor 4}{s^5} = \frac{4 \times 3 \times 2 \times 1}{s^5}$

$= \frac{24}{s^5}$

$\Rightarrow e^{-at} \longrightarrow \frac{1}{(s+a)}$

$\Rightarrow e^{+at} \longrightarrow \frac{1}{s-a}$

$\int_0^{\infty} e^{-at} \cdot e^{-st} dt$

$\int_0^{\infty} e^{-(a+s)t}$

$$t \longrightarrow \frac{1}{s^2}$$

$$t\cdot[t] \xrightarrow{LT} ? \quad \left\{ \boxed{t^2 = \frac{2}{s^3}} \right.$$

$$\Rightarrow x(t) \longrightarrow X(s)$$

$$\boxed{t\cdot x(t) \xrightarrow{LT} -\frac{d}{ds} X(s)}$$

$$t\cdot[t] \longrightarrow -\frac{d}{ds}\left[\frac{1}{s^2}\right] \qquad \left\{ \frac{d}{ds}\left[\frac{I}{II}\right] \quad \frac{II\cdot\frac{d}{ds}I + I\cdot\frac{d}{ds}II}{(II)^2} \right.$$

$$= -\frac{d}{ds}\left[s^{-2}\right]$$

$$= -\left[-\frac{2}{s^3}\right]$$

$$t\cdot[t] = \frac{2}{s^3}$$

$$\Rightarrow e^{-at} \xrightarrow{L} \frac{1}{s+a}$$

$$t\cdot e^{-at} \xrightarrow{L} -\frac{d}{ds}\left[\frac{1}{s+a}\right]$$

$$= -\left[\frac{(s+a)\cdot\frac{d}{ds}1 - 1\cdot\frac{d}{ds}(s+a)}{(s+a)^2}\right]$$

$$= -\left[\frac{0-1}{(s+a)^2}\right] = \frac{1}{(s+a)^2}$$

$$\boxed{x(t) \longrightarrow X(s)}$$

$$t^n\cdot x(t) \xrightarrow{LT} (-1)^n \frac{d^n}{ds^n} X(s)$$

(8) $L[\sin\omega t] \longrightarrow \frac{\omega}{s^2+\omega^2}$

$L[\sin at] \longrightarrow \frac{a}{s^2+a^2}$

(9) $L[\cos\omega t] \longrightarrow \frac{s}{s^2+\omega^2}$

(10) $e^{-at}\sin\omega t \longrightarrow$

$\sin\omega t \longrightarrow \frac{\omega}{s^2+\omega^2}$

$s \rightarrow (s+a)$

$= \frac{\omega}{(s+a)^2+\omega^2}$

(11) $e^{at}\cos\omega t \longrightarrow \frac{s}{s^2+\omega^2}$

$s \rightarrow (s-a)$

$\Longrightarrow \frac{(s-a)}{(s-a)^2+\omega^2}$

— × — × —

$$\boxed{\int_{-\infty}^{\infty} x(t)\,dt < \infty}$$

Laplace & Fourier Transform

$\int_{-\infty}^{\infty} e^{-at}\,dt$

$\left[\frac{e^{-at}}{-a}\right]_{-\infty}^{\infty} = e^{-\infty} - e^{+\infty}$

$\Downarrow$

∞

Laplace & Fourier T = Not find

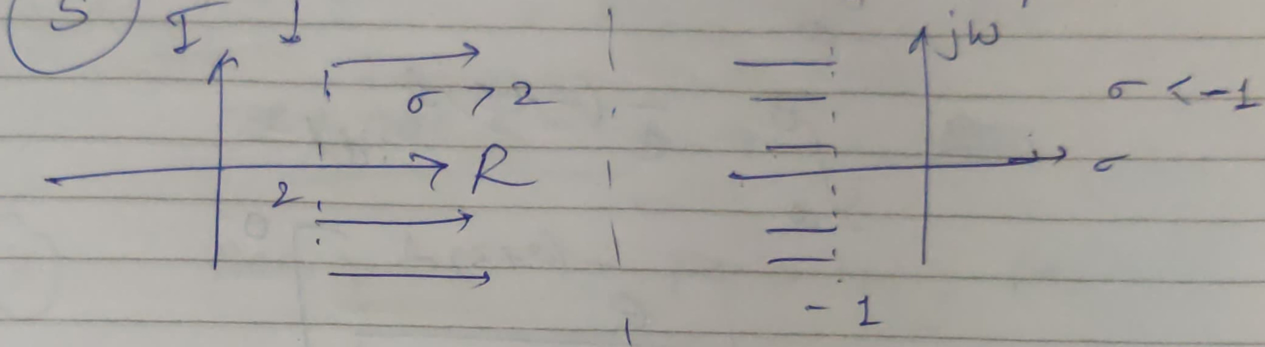
ROC ⇒ Region of Convergence

$$\begin{cases} u(t) & \sigma > \text{Constant} \\ u(-t) & \sigma < \text{Constant} \end{cases}$$

If $L[x_1 + x_2]$

ROC$_1$ ROC$_2$ → ROC

S



Common [ROC$_1$ & ROC$_2$] = 0

Laplace Can not be determine.

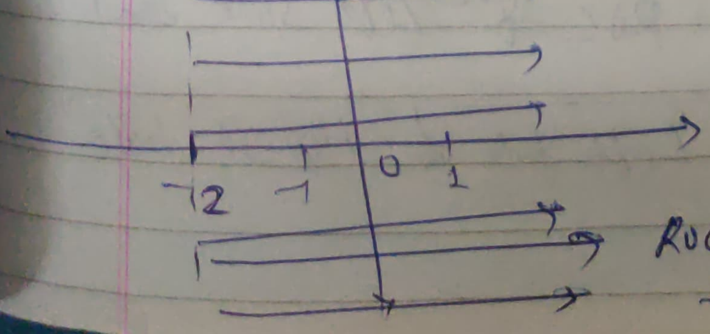
LT → S → Denominator > 0

→ σ = Only Real part

Ex

Q(1) $e^{-2t}u(t)$. Laplace & ROC

$$L[e^{-2t}u(t)] = \frac{1}{s+2}$$



$s + 2 > 0$

$s > -2$

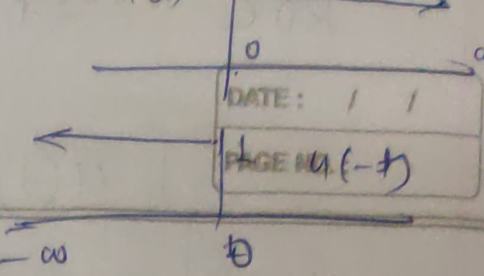
$\boxed{\sigma > -2}$

u(t)

↑

Right Side

ROC for Right Side Signal is Right Side of Highest pole



$$x(t) = e^{-2t}\, u[-t]$$

$$= \int_{-\infty}^{\infty} e^{-2t} \cdot e^{-st} \cdot u(-t) \cdot dt$$

↓ limit ($-\infty$ to 0)

$$= \int_{-\infty}^{0} e^{-2t} \cdot e^{-st}\, dt$$

$$= \int_{-\infty}^{0} e^{-(2+s)t} \cdot dt$$

$$= \left[\frac{e^{-(2+s)t}}{-(2+s)} \right]_{t=-\infty}^{0}$$

$$= \frac{1}{-(2+s)} \left[e^{0} - e^{-\infty} \right]$$

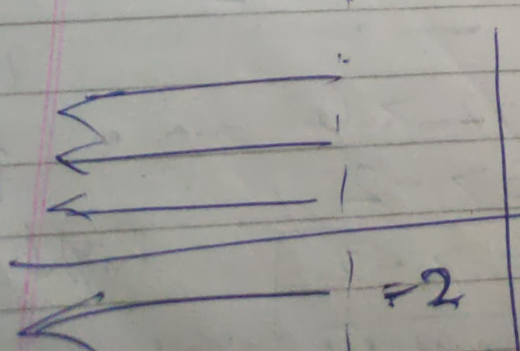
$$= \frac{1}{-(2+s)} \cdot [1 - 0]$$

$$= \frac{-1}{(2+s)} = \frac{-1}{(s+2)}$$

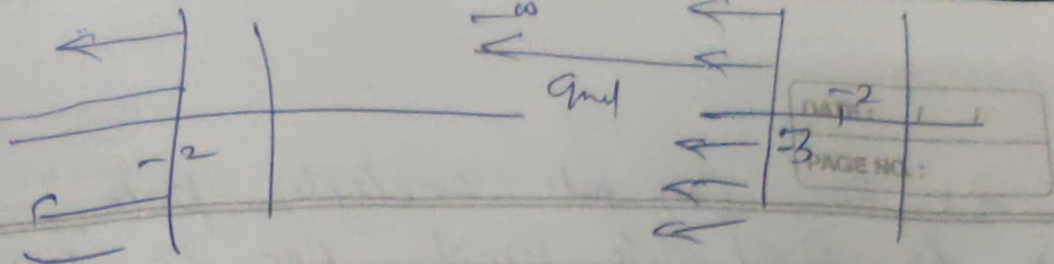
$\Rightarrow$ ROC $\quad s+2 < 0$

$\boxed{s < -2}$ Pole

$\left\{ \begin{array}{c} D = 0 \\ s+2 = 0 \\ s = -2 \\ \text{Pole} \end{array} \right\}$



ROC for $u(-t)$ Left side signal is Left side of "Highest pole"



ROC < -2 (highest left side) ROC < -3

ROC ⇒ common ⇒ $\boxed{ROC < -3}$

Q(3) ROC < -1 ; ROC < -3 ; ROC < -5

⇒ $\boxed{ROC < -5}$ ⇒ ~~ROC~~

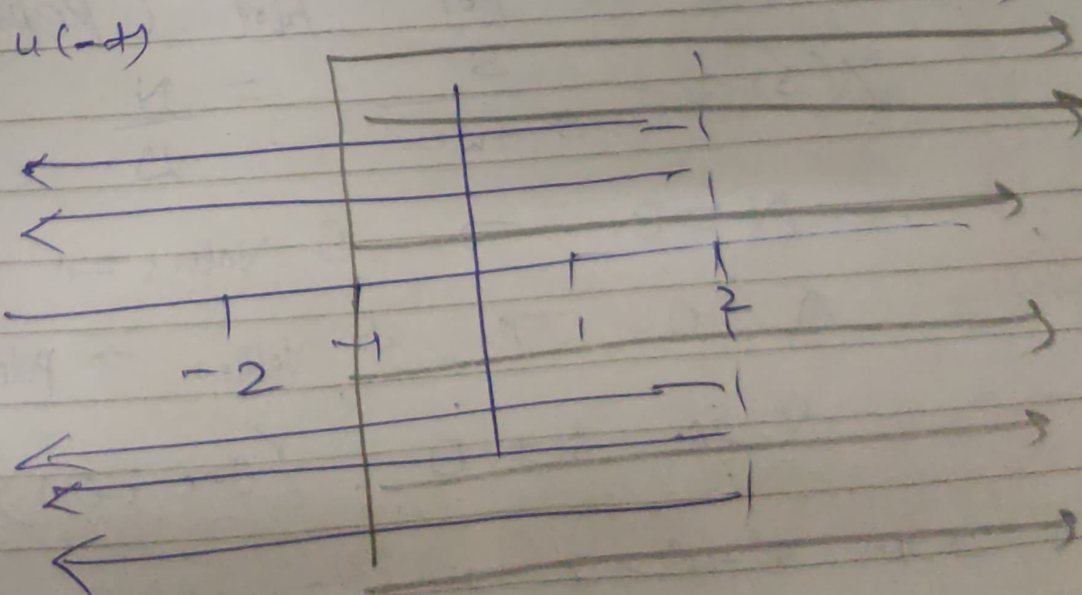
Q(4) ROC > -2 ; ROC > 4 ; ROC > 10

⇒ $\boxed{ROC > 10}$

$x_1(t) = e^{-2t} u(t)$ $x_2(t) = e^{4t} \cdot u(t)$; $x_3(t) = e^{10t} u(t)$

Q(5) ROC < +2 and ROC > -1

$-e^{2t} u(-t)$ $e^{-t} \cdot u(t)$



$\boxed{-1 < ROC < 2}$

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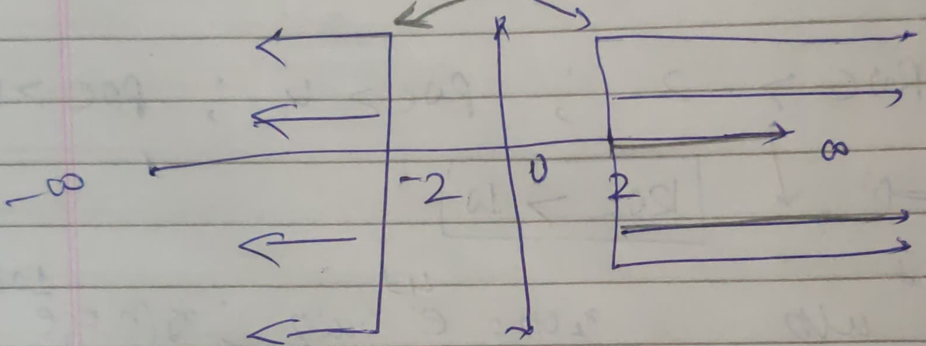
ROC ⇒ 1 ROC does not include pole.

- for Right side signal, ROC is Right of the Highest pole. {u(t)}

- for Left side signal (u(-t)), ROC is Left of the highest pole.

Q (6) $x_1(t) = e^{-2t} u(t)$ + $x_2(t) = -e^{2t} u(-t)$

ROC > 2 ROC < -2



ROC does not exist.

Laplace can not be determined.

Pole-zero ⇒ Plot (Representation of poles & zeros)

$$X(s) = \frac{s}{s^2+w^2} = \frac{N}{D}$$

$N = 0 \Rightarrow$ S values ⇒ zero's

$D = 0 \Rightarrow$ S values ⇒ poles

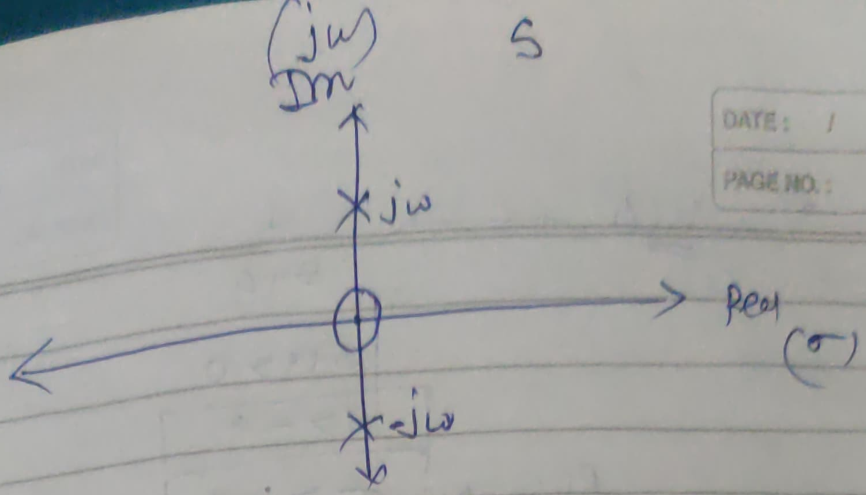
$N \Rightarrow s = 0$ {one zero} ⇒ ⊙ circle

$D \Rightarrow s^2 + w^2 = 0 \Rightarrow s^2 = -w^2$

$s = \sqrt{-w^2}$

$s = \pm jw$ ⊗ ; ⊗

Poles (Two) ⇒ $s_1 = jw$; $s_2 = -jw$



$$X(s) = \frac{(s+1)(s+2)}{(s+3)(s+4)(s^2+9)}$$

Zeros ⇒ $(s+1)(s+2) = 0$

$s_1 = -1$; $s_2 = -2$

(Two) ⇒

Poles ⇒ $D = 0$ ⇒ $(s+3)(s+4)(s^2+9) = 0$

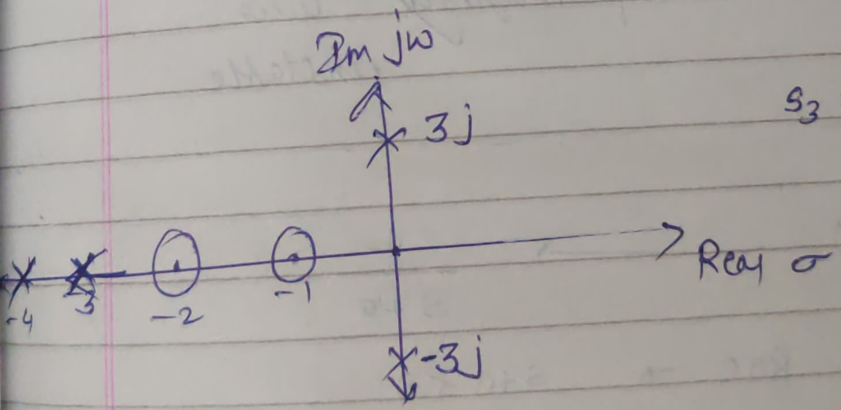
$s_1 = -3$; $s_2 = -4$; $s^2 + 9 = 0$

$s^2 = -9$

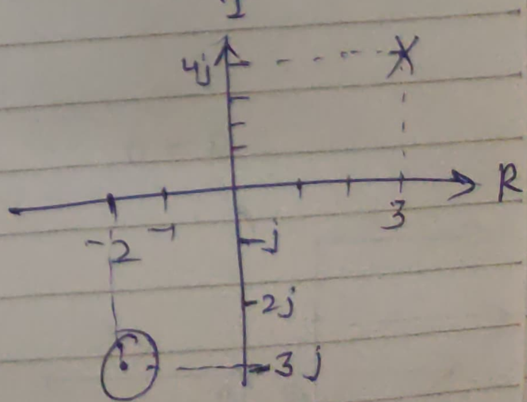
$s = \sqrt{-9}$

$s = \pm j3$

$s_3 = 3j$; $s_4 = -3j$



$$X(s) = \frac{s+(2+3j)}{[s-(3+4j)]}$$ →



$s + (2+3j) = 0$

Zero $s = -2 - 3j$

Poles $s - (3+4j) = 0$

$s = 3 + 4j$

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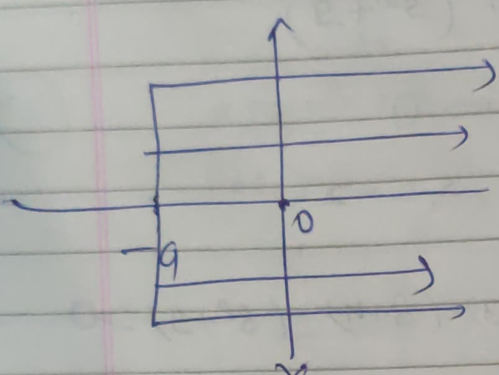
Q(1) $\bar{e}^{at}\, u(t) \longrightarrow \dfrac{1}{s+a}$

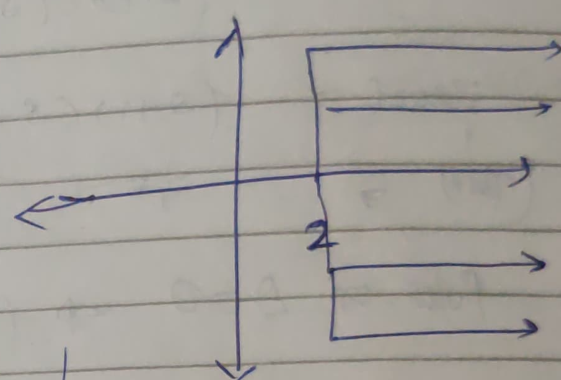
$s+a > 0$

$\boxed{s > -a}$

ROC ⇒ $\boxed{\sigma > -a}$

$a > 0$ | $a < 0$





ROC includes Imaginary axis
↓
"stable"

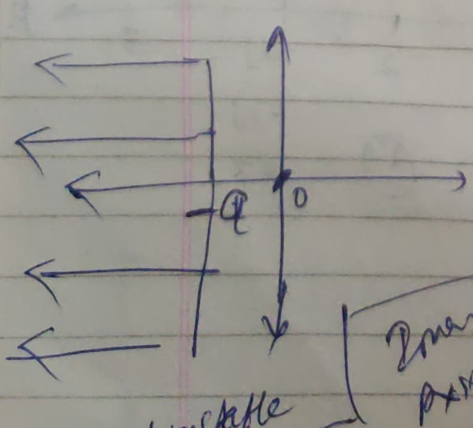
ROC does not include imaginary axis
unstable

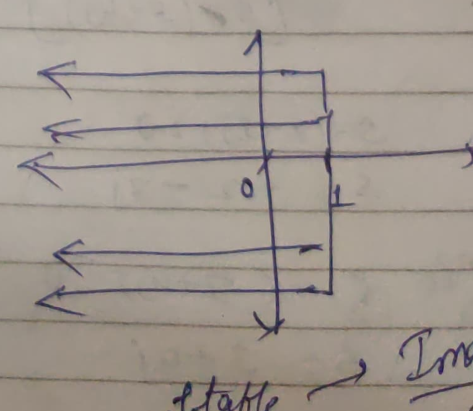
Q(2) $-e^{-at}\, u(-t) \longrightarrow \dfrac{1}{s+a}$

ROC → $s+a < 0$

$s < -a$

$a > 0$ | $a < 0$





unstable — Imag. axis does not include.

stable → Imag. axis cut

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Q③ $x(t) = e^{-b|t|}$

$$|t| = \begin{cases} t & ; t > 0 \\ 0 & ; t = 0 \\ -t & ; t < 0 \end{cases}$$

$$x(t) = \begin{cases} e^{-bt} & ; t > 0 \\ e^{bt} & ; t < 0 \end{cases}$$

$$L[x(t)] = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

$$= \int_{-\infty}^{0} x(t) \cdot e^{-st} \cdot dt + \int_{0}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

$$= \int_{-\infty}^{0} e^{bt} \cdot e^{-st} \cdot dt + \int_{0}^{\infty} e^{-bt} \cdot e^{-st} \cdot dt$$

$$= \int_{-\infty}^{0} e^{+(b-s) \cdot t} \cdot dt + \int_{0}^{\infty} e^{-(b+s) t} \, dt$$

$$= \left[\frac{e^{(b-s)t}}{(b-s)} \right]_{-\infty}^{0} + \frac{\left[e^{-(b+s) \cdot t} \right]_{0}^{\infty}}{-(b+s)}$$

$$= \frac{1}{(b-s)} \left[e^{0} - e^{-\infty} \right] + \frac{1}{-(s+b)} \left[e^{-\infty} - e^{0} \right]$$

$$= \frac{-1}{(s-b)} + \frac{(-1)(-1)}{(s+b)} = \frac{1}{s+b} - \frac{1}{s-b}$$

ROC $s+b > 0$, $s > -b$; $s-b < 0$, $s < b$

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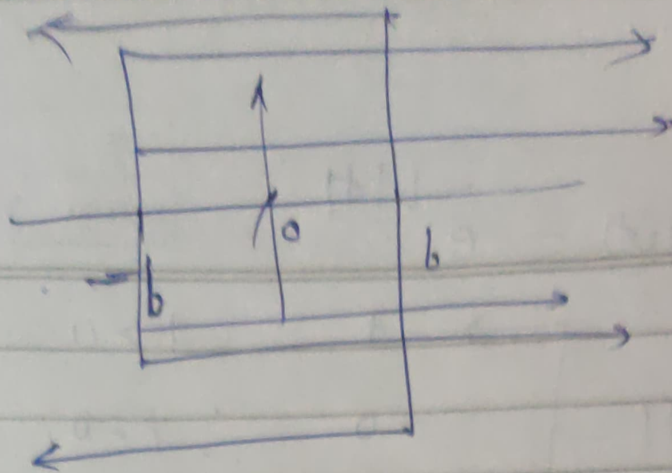


$\boxed{-b < ROC < b}$

$\boxed{-b < \sigma < b}$

$\boxed{-b < R_e(S) < b}$

b is positive

★★★

If b is negative

$b < 0$

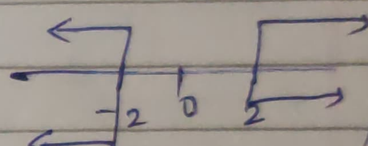
If $b = -2$

1st $S > -(-2)$

$S > 2$

and 2nd Roc

$S < -2$



Laplace Cannot be determine

Properties of Laplace Transform ⇒

(a) Linearity ⇒

$x_1(t) \longrightarrow X_1(s)$

$x_2(t) \longrightarrow X_2(s)$

$$a x_1(t) + b x_2(t) \xrightarrow{L} a \cdot X_1(s) + b X_2(s)$$

(b) Scaling ⇒ (Time)

$x(t) \longrightarrow X(s)$

$$x(at) \longrightarrow \frac{1}{|a|} X(s/a)$$

Ex

$$\sin t \xrightarrow{L} \frac{1}{s^2+1}$$

$$\sin(2t) \xrightarrow{L} \frac{1}{2} X(s/2) = \frac{1}{2}\left[\frac{1}{\left(\frac{s}{2}\right)^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{4}{s^2+4}\right] = \frac{2}{s^2+4}$$

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$$\sin(-2t) \longrightarrow \frac{1}{|-2|} \cdot X\left(\frac{s}{(-2)}\right)$$

$$= \frac{1}{2} \cdot \left[\frac{1}{\left(\frac{s}{-2}\right)^2 + 1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\frac{s^2}{4} + 1}\right]$$

$$= \frac{1}{2} \cdot \frac{4}{(s^2+4)} = \frac{2}{s^2+4}$$

(c) Time Shifting ⇒

$$x(t) \longrightarrow X(s)$$

$$x(t - t_0) \longrightarrow X(s) \cdot e^{-st_0}$$

$t - t_0 = 0$
$t = t_0$

Ex: $t^2 \xrightarrow{L} \frac{2}{s^3}$

$$(t-2)^2 \xrightarrow{L} X(s) \cdot e^{-s.2}$$

$$(t-2)^2 \xrightarrow{L} \frac{2}{s^3} e^{-2s}$$

$$t \xrightarrow{L} \frac{1}{s^2}$$

$$(t-2) \xrightarrow{L} X(s) \cdot e^{-s.2}$$

$$= \frac{1}{s^2} \cdot e^{-2s}$$

(d) Frequency Shifting ⇒

$$x(t) \xrightarrow{L} X(s)$$

$$L[e^{-at} x(t)] \longrightarrow X(s+a)$$

$$L[e^{at} x(t)] \longrightarrow X(s-a)$$

(d) Differenti

$$x(t) \xrightarrow{L} X(s)$$

$$\frac{d}{dt}x(t) \longrightarrow s \cdot X(s) - s^0 \cdot x(0)$$

$$\frac{d^2 x(t)}{dt^2} \longrightarrow s^2 X(s) - s^1 x(0) - s^0 x'(0)$$

$$\frac{d^3 x(t)}{dt^3} = s^3 X(s) - s^2 x(0) - s^1 x'(0) - s^0 x''(0)$$

(e) Integration ⇒ $x(t) \longrightarrow X(s)$

$$L\left[\int_{t=0}^{t} x(t)\right] = \frac{X(s)}{s} + \frac{\int_{-\infty}^{0} x(t) \cdot dt}{s}$$

Ex. $\sin t \xrightarrow{L} \frac{1}{s^2+1}$

$$L\left[\int_0^t \sin t\right] = ?$$

$$= \frac{X(s)}{s} = \frac{1}{s(s^2+1)} \quad \text{Ans}$$

(f) frequency Integration ⇒

$$x(t) \xrightarrow{L} X(s)$$

$$\frac{x(t)}{t} \xrightarrow{L} \int_{s=s}^{\infty} X(s) \cdot ds$$

(g) Time convolution → $x_1(t) * x_2(t)$

$$x_1(t) \xrightarrow{L} X_1(s)$$

$$x_2(t) \xrightarrow{L} X_2(s)$$

$$x_1(t) * x_2(t) \xrightarrow{L} X_1(s) \cdot X_2(s)$$

Frequency Convolution

(h) $x_1(t) \cdot x_2(t) \xrightarrow{L} X_1(s) * X_2(s)$

(I) Initial value theorem ⇒

$\lim_{t \to 0} x(t) \longrightarrow \boxed{\lim_{s \to \infty} sX(s)}$

Initial value

(J) final value theorem ⇒

$\lim_{t \to \infty} x(t) \xrightarrow{L} \lim_{s \to 0} sX(s)$

⇒ Application of Laplace ⇒

(i) to solve differential equation

↳ Transfer function

Ex (1) $\frac{d^2y(t)}{dt^2} + 3 \cdot \frac{dy(t)}{dt} + 2y(t) = x(t)$ — (1)

Transfer function $= \frac{L[O/p]}{L[I/p]}$ | Initial condition = 0

O/p → $y(t) \longrightarrow Y(s)$ ⇒ $\frac{Y(s)}{X(s)} = TF$

I/p → $x(t) \longrightarrow X(s)$

eq (1) Laplace

$L\left[\frac{d^2y(t)}{dt^2}\right] + L\left[3 \cdot \frac{dy(t)}{dt}\right] + 2L[y(t)] = L[x(t)]$

$s^2 Y(s) - s^1 y(0) - s^0 y'(0) + 3[sY(s) - s^0 y(0)] + 2Y(s) = X(s)$

⟹ Initial condition $y(0) = y'(0) = 0$

$s^2 Y(s) + 3sY(s) + 2Y(s) = X(s)$

$$Y(s)[s^2+3s+2] = X(s)$$

$$\boxed{TF = \frac{Y(s)}{X(s)} = \frac{1}{s^2+3s+2}}$$

$\Rightarrow$ find $y(t) = ?$ for impulse input

$x(t) = \delta(t)$

$X(s) = L[\delta(t)] = 1$

$$Y(s) = X(s) \cdot \frac{1}{s^2+3s+2}$$ ← factorization

$$Y(s) = \frac{1}{(s^2+3s+2)}$$

$$Y(s) = \frac{1}{s^2+2s+s+2} = \frac{1}{s(s+2)+1(s+2)}$$

$$Y(s) = \frac{1}{(s+2)(s+1)}$$

Partial fraction

$$Y(s) = \frac{A}{s+2} + \frac{B}{s+1}$$

$s+2=0 \Rightarrow s=-2 \Rightarrow \left.\frac{\cancel{(s+2)}}{(s+1)\cancel{(s+2)}}\right|_{s=-2}$

$$A = \left.\left(\frac{1}{s+1}\right)\right|_{s=-2} = \frac{1}{-2+1}$$

$$A = -1$$

$s+1=0 \Rightarrow s=-1$

$$B = \left.\frac{1}{(s+2)}\right|_{s=-1} = \frac{1}{-1+2} = \frac{1}{1} = 1$$

$$Y(s) = \frac{-1}{(s+2)} + \frac{1}{(s+1)}$$

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$$Y(s) = \frac{1}{s+1} - \frac{1}{(s+2)}$$

Laplace Inverse

$$\boxed{y(t) = e^{-1 \cdot t} - e^{-2t}}$$ $s \longrightarrow t$

$$TF = \frac{1}{(s^2+3s+2)}$$

Initial Value $\Rightarrow \lim_{s \to \infty} s \cdot TF.$

$$= \lim_{s \to \infty} s \cdot \frac{1}{(s^2+3s+2)}$$ $\left\{ \frac{\infty}{\infty} \neq, \frac{0}{0} \neq \right\}$

$$= \lim_{s \to \infty} \frac{s}{s^2\left[1 + \frac{3}{s} + \frac{2}{s^2}\right]}$$

$$= \lim_{s \to \infty} \frac{1}{\infty\left[1 + \frac{3}{\infty} + \frac{2}{\infty}\right]}$$

$$= \Rightarrow \frac{1}{\infty[1+0+0]} = \frac{1}{\infty} = 0$$

Final Value $= \lim_{s \to 0} s \cdot TF$

$$= \lim_{s \to 0} s \cdot \frac{1}{(s^2+3s+2)}$$

$$= \frac{0}{(0+0+2)} = \frac{0}{2} = 0$$

→ If initial Condition
$y(0) = y'(0) = 1$

$s^2 Y(s) - s - 1 + 3[s\,Y(s) - 1.1] + 2Y(s) = X(s)$

$s^2 Y(s) - s - 1 + 3s.Y(s) - 3 + 2Y(s) = X(s)$

$Y(s)[s^2 + 3s + 2] - s - 4 = X(s)$

$Y(s)[s^2 + 3s + 2] = X(s) + (s+4)$

O/p $\boxed{Y(s) = \dfrac{X(s)}{s^2+3s+2} + \dfrac{s+4}{s^2+3s+2}}$

When Input is unit impulse then find $y(t)$

$x(t) = \delta(t)$

$X(s) = 1$

$$Y(s) = \frac{1}{(s^2+3s+2)} + \frac{s+4}{(s^2+3s+2)}$$

LI

$$Y(s) = \frac{1}{(s+1)(s+2)} + \frac{s+4}{(s+1)(s+2)}$$

$$Y(s) = \frac{A}{(s+1)} + \frac{B}{(s+2)} + \frac{C}{(s+1)} + \frac{D}{(s+2)}$$

$$A = \left.\frac{1}{(s+2)}\right|_{s=-1} = \frac{1}{-1+2} = 1$$

$$B = \left.\frac{1}{s+1}\right|_{s=-2} = \frac{1}{1-2} = -1$$

$$C = \left.\frac{s+4}{s+2}\right|_{s=-1} = \frac{-1+4}{-1+2} = 3$$

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$s+2=0 \Rightarrow s=-2$

$$B = \left.\frac{s+4}{s+1}\right|_{s=-2} = \frac{-2+4}{-2+1} = \frac{2}{-1} = -2$$

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2} + \frac{3}{s+1} - \frac{2}{s+2}$$

$$Y(s) = \frac{4}{s+1} - \frac{3}{s+2}$$

Laplace Inverse

$$\Rightarrow \boxed{y(t) = 4 \cdot e^{-t} - 3 \cdot e^{-2t}}$$

3 c

$$x(t) = e^{2t}u(t) + e^{-3t}u(-t)$$

$$X(s) = \frac{1}{s-2} + \frac{(-1)}{s+3}$$

ROC $\Rightarrow$ $s-2>0$

$s>2$

$-(s+3)<0$

$(s+3)>0$

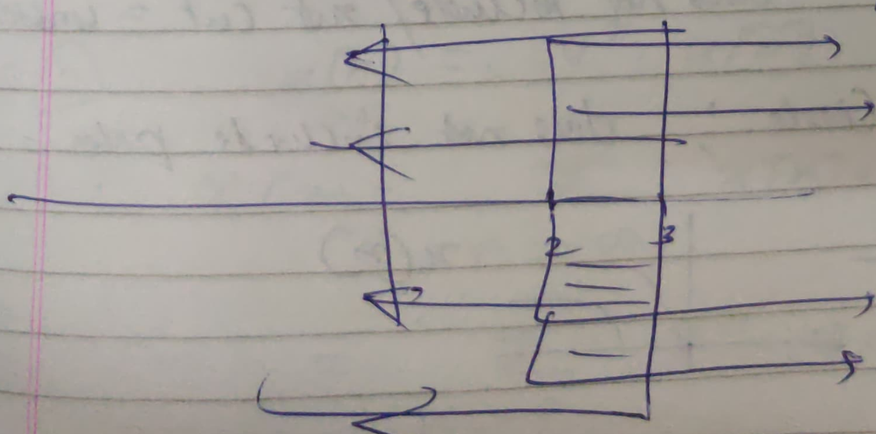
$s>-3$

$s+3<0$

$s<-3$

$s<(-3)\ (-1)$

$s<3$



2
3

$$\boxed{2 < s < 3}$$