Fourier Transform

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- Fourier Integral
- Fourier Transform
- Properties of Fourier Transform
- Convolution
- Parseval's Theorem

Continuous-Time Fourier Transform

Introduction

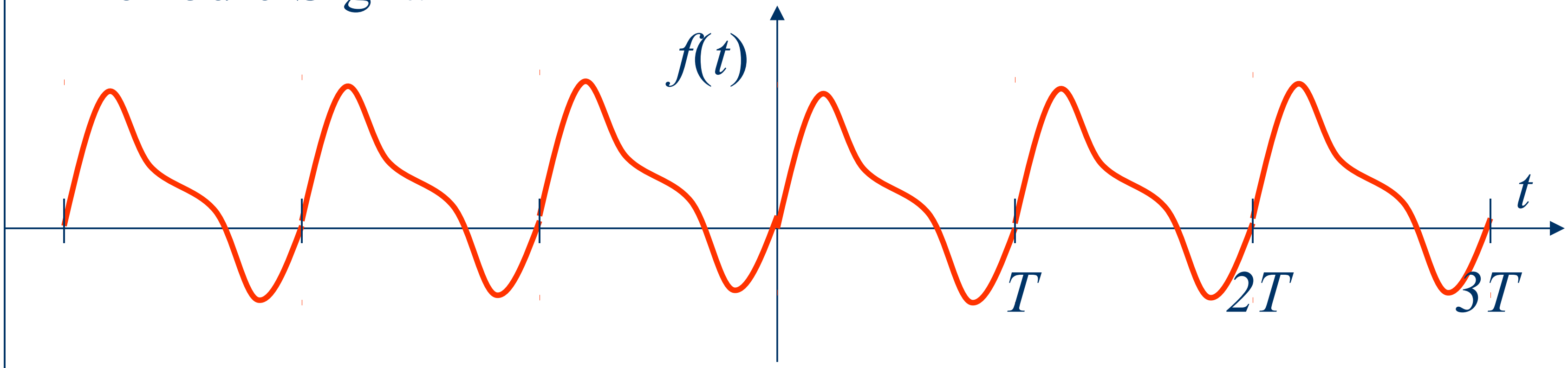
The Topic

	Continuous Time	Discrete Time
Periodic	Fourier Series	Discrete Fourier Transform
Aperiodic	Continuous Fourier Transform	Fourier Transform

Review of Fourier Series

- Deal with continuous-time periodic signals.
- Discrete frequency spectra.

A Periodic Signal



Two Forms for Fourier Series

Sinusoidal Form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2\pi nt}{T} + \sum_{n=1}^{\infty} b_n \sin \frac{2\pi nt}{T}$$

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t dt$$

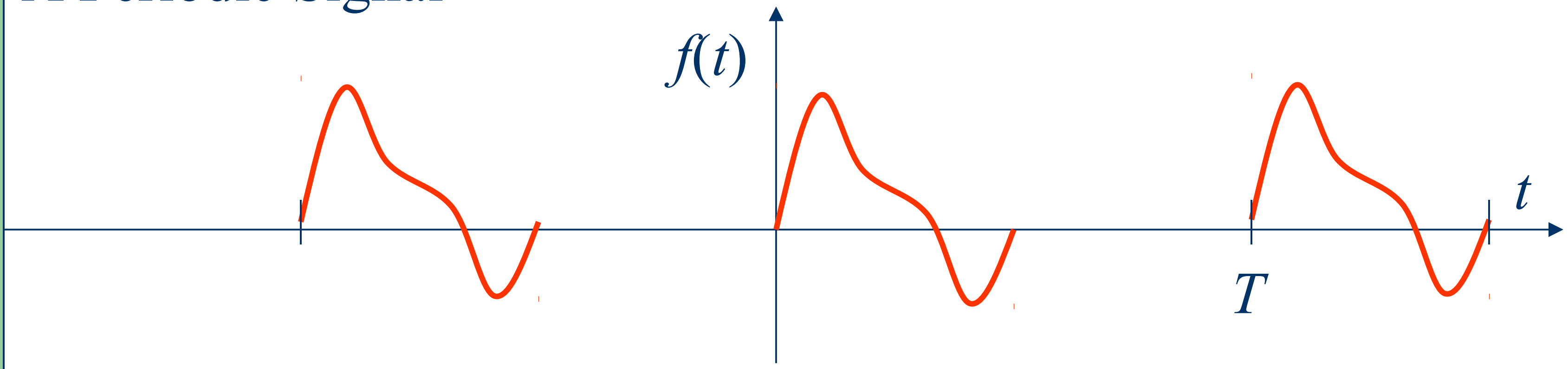
Complex Form:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t} dt$$

How to Deal with Aperiodic Signal?

A Periodic Signal



If $T \rightarrow \infty$, what happens?

Continuous-Time Fourier Transform

Fourier Integral

Fourier Integral

$$f_T(t)=\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \qquad c_n=\frac{1}{T}\int_{-T/2}^{T/2} f_T(t)e^{-jn\omega_0 t}dt$$

$$=\sum_{n=-\infty}^{\infty}\left[\frac{1}{T}\int_{-T/2}^{T/2} f_T(\tau)e^{-jn\omega_0 \tau}d\tau\right]e^{jn\omega_0 t}$$

$$\omega_0=\frac{2\pi}{T} \Rightarrow \frac{1}{T}=\frac{\omega_0}{2\pi}$$

$$=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left[\int_{-T/2}^{T/2} f_T(\tau)e^{-jn\omega_0 \tau}d\tau\right]\omega_0 e^{jn\omega_0 t}$$

Let $\Delta\omega=\omega_0=\frac{2\pi}{T}$

$$=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left[\int_{-T/2}^{T/2} f_T(\tau)e^{-jn\omega_0 \tau}d\tau\right]e^{jn\omega_0 t}\Delta\omega$$

$$T\to\infty \Rightarrow d\omega=\Delta\omega\approx 0$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f_T(\tau)e^{-j\omega \tau}d\tau\right]e^{j\omega t}d\omega$$

Fourier Integral

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\left[\int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}d\tau\right]}_{F(j\omega)}e^{j\omega t}d\omega$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$$ Synthesis

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$ Analysis

Fourier Series vs. Fourier Integral

Fourier Series:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

Period Function

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_0 t} dt$$

Discrete Spectra

Fourier Integral:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega$$

Non-Period Function

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

Continuous Spectra

Continuous-Time Fourier Transform

Fourier Transform

Fourier Transform Pair

Inverse Fourier Transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega$$

Synthesis

Fourier Transform:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

Analysis

Existence of the Fourier Transform

Sufficient Condition:

$f(t)$ is absolutely integrable, i.e.,

$$\int_{-\infty}^{\infty} |f(t)|\,dt < \infty$$

Continuous Spectra

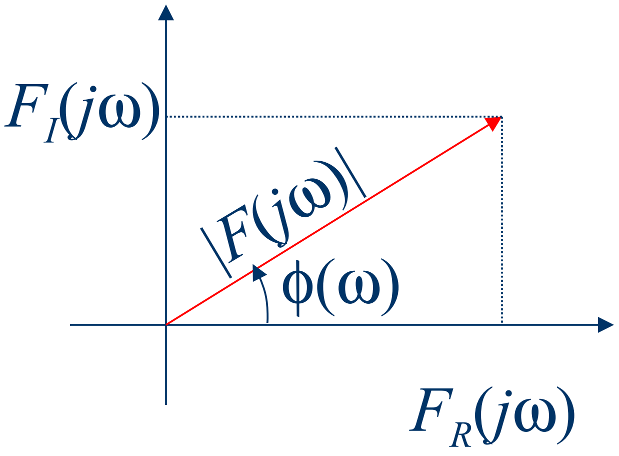
$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$F(j\omega) = F_R(j\omega) + jF_I(j\omega)$$

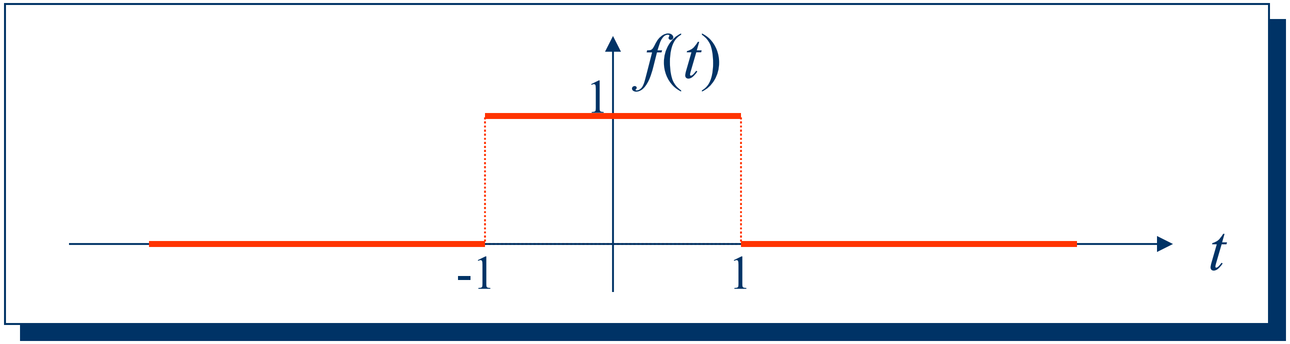
$$= |F(j\omega)|e^{j\phi(\omega)}$$

Magnitude: $|F(j\omega)|$

Phase: $\phi(\omega)$



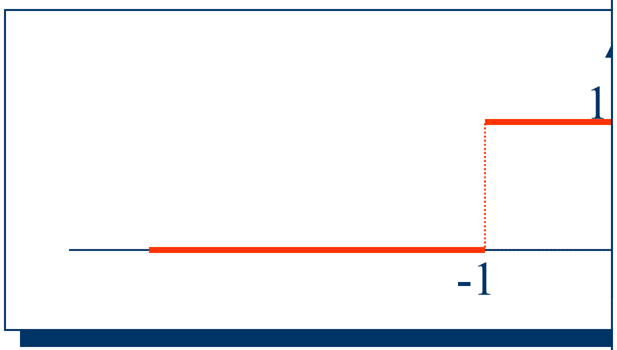
Example

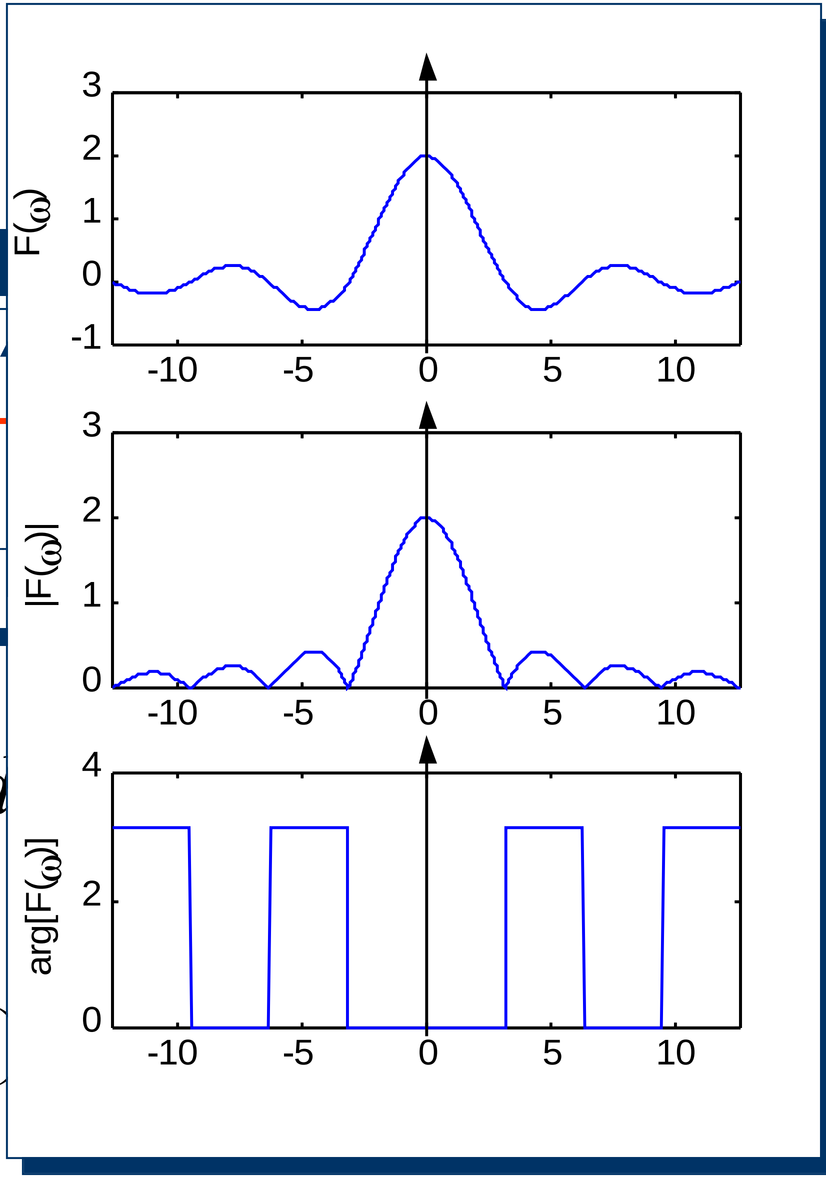


$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{-1}^{1} e^{-j\omega t} dt = \frac{1}{-j\omega} e^{-j\omega t} \Big|_{-1}^{1}$$

$$= \frac{j}{\omega}\left(e^{-j\omega} - e^{j\omega}\right) = \frac{2\sin\omega}{\omega}$$

Example

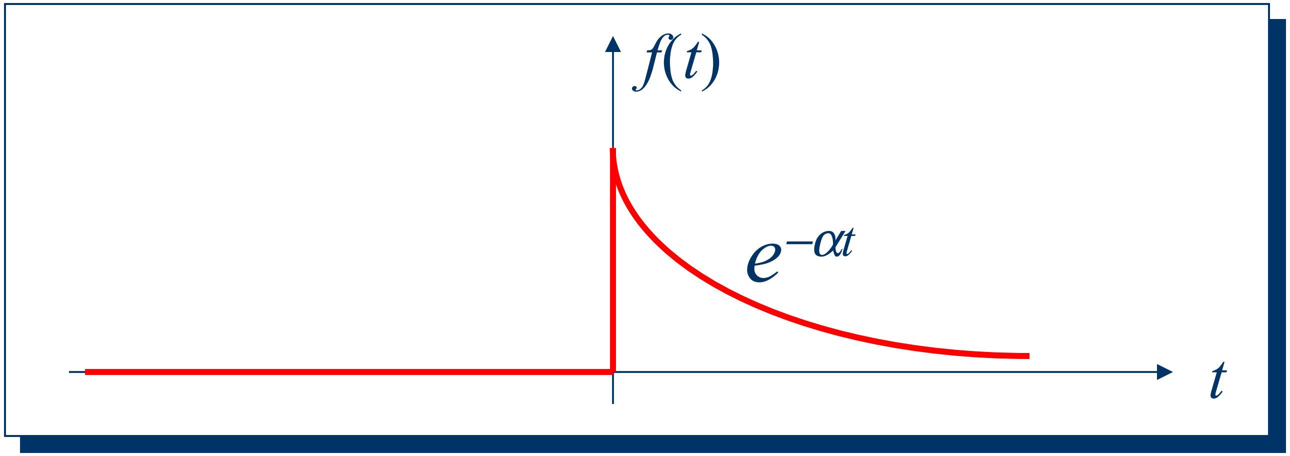




$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} d \quad \ldots \quad t \Big|_{-1}^{1}$$

$$= \frac{j}{\omega}(e^{-j\omega} - e^{j\omega}$$

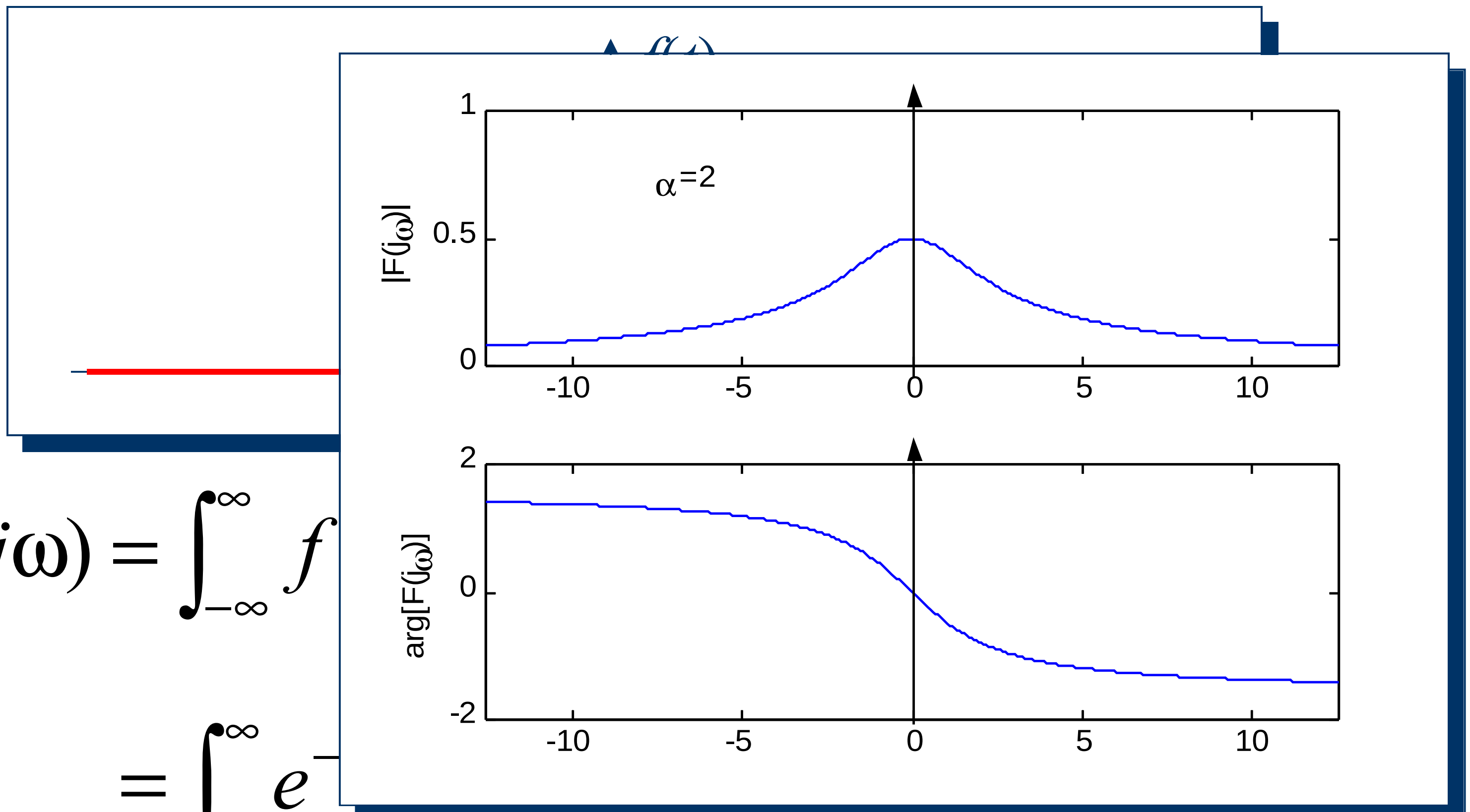
Example



$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{0}^{\infty} e^{-\alpha t} e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-(\alpha + j\omega)t} dt = \frac{1}{\alpha + j\omega}$$

Example



$$F(j\omega) = \int_{-\infty}^{\infty} f \ldots$$

$$= \int_{0}^{\infty} e^{-\ldots}$$

$$\alpha + j\omega$$

Continuous-Time Fourier Transform

Properties of Fourier Transform

Notation

$$\mathcal{F}[f(t)] = F(j\omega)$$

$$\mathcal{F}^{-1}[F(j\omega)] = f(t)$$

$$f(t) \xleftrightarrow{\mathcal{F}} F(j\omega)$$

Linearity

$$a_1 f_1(t) + a_2 f_2(t) \stackrel{\mathcal{F}}{\longleftrightarrow} a_1 F_1(j\omega) + a_2 F_2(j\omega)$$

!Home Work !

Time Scaling

$$f(at) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} F\left(j\frac{\omega}{a}\right)$$

!Home Work !

Time Reversal

$$f(-t) \stackrel{\mathcal{F}}{\longleftrightarrow} F(-j\omega)$$

Pf)

$$\mathcal{F}[f(-t)] = \int_{-\infty}^{\infty} f(-t)e^{-j\omega t}dt \;= \int_{t=-\infty}^{t=\infty} f(-t)e^{-j\omega t}dt$$

$$= \int_{-t=-\infty}^{-t=\infty} f(t)e^{j\omega t}d(-t) \;= \int_{-t=-\infty}^{-t=\infty} f(t)e^{j\omega t}d(-t)$$

$$= -\int_{t=\infty}^{t=-\infty} f(t)e^{j\omega t}dt \;= \int_{t=-\infty}^{t=\infty} f(t)e^{j\omega t}dt$$

$$= \int_{-\infty}^{\infty} f(t)e^{j\omega t}dt \;= F(-j\omega)$$

Time Shifting

$$f(t-t_0) \xleftrightarrow{\mathcal{F}} F(j\omega)e^{-j\omega t_0}$$

Pf)

$$\mathcal{F}[f(t-t_0)] = \int_{-\infty}^{\infty} f(t-t_0)e^{-j\omega t}dt = \int_{t=-\infty}^{t=\infty} f(t-t_0)e^{-j\omega t}dt$$

$$= \int_{t+t_0=-\infty}^{t+t_0=\infty} f(t)e^{-j\omega(t+t_0)}d(t+t_0)$$

$$= e^{-j\omega t_0}\int_{t=-\infty}^{t=\infty} f(t)e^{-j\omega t}dt$$

$$= e^{-j\omega t_0}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = F(j\omega)e^{-j\omega_0 t}$$

Frequency Shifting (Modulation)

$$f(t)e^{j\omega_0 t} \longleftrightarrow^{\mathcal{F}} F[j(\omega-\omega_0)]$$

Pf)

$$\mathcal{F}[f(t)e^{j\omega_0 t}] = \int_{-\infty}^{\infty} f(t)e^{j\omega_0 t}e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_0)t}dt$$

$$= F[j(\omega-\omega_0)]$$

Symmetry Property

$$\mathcal{F}[F(jt)] = 2\pi f(-\omega)$$

Proof

$$2\pi f(t) = \int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$$

$$2\pi f(-t) = \int_{-\infty}^{\infty} F(j\omega)e^{-j\omega t}d\omega$$

Interchange symbols ω and *t*

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(jt)e^{-j\omega t}dt \quad = \mathcal{F}[F(jt)]$$

Fourier Transform for Real Functions

If $f(t)$ is a real function, and $F(j\omega) = F_R(j\omega) + jF_I(j\omega)$

⇨ $F(-j\omega) = F^*(j\omega)$

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$F^*(j\omega) = \int_{-\infty}^{\infty} f(t)e^{j\omega t}dt = F(-j\omega)$$

Fourier Transform for Real Functions

If $f(t)$ is a real function, and $F(j\omega) = F_R(j\omega) + jF_I(j\omega)$

- $F(-j\omega) = F^*(j\omega)$

- $F_R(j\omega)$ is even, and $F_I(j\omega)$ is odd.

$$\underbrace{F_R(-j\omega) = F_R(j\omega)} \quad \underbrace{F_I(-j\omega) = -F_I(j\omega)}$$

- *Magnitude spectrum* $|F(j\omega)|$ is even, and *phase spectrum* $\phi(\omega)$ is odd.

Fourier Transform for Real Functions

If $f(t)$ is real and even

➡ $F(j\omega)$ is real ✓

Pf)

Even ➡ $f(t) = f(-t)$

➡ $F(j\omega) = F(-j\omega)$

Real ➡ $F(-j\omega) = F^*(j\omega)$

➡ $F(j\omega) = F^*(j\omega)$

If $f(t)$ is real and odd

➡ $F(j\omega)$ is pure imaginary ✓

Pf)

Odd ➡ $f(t) = -f(-t)$

➡ $F(j\omega) = -F(-j\omega)$

Real ➡ $F(-j\omega) = F^*(j\omega)$

➡ $F(j\omega) = -F^*(j\omega)$

Example:

$$\mathcal{F}[f(t)] = F(j\omega) \qquad \mathcal{F}[f(t)\cos\omega_0 t] = ?$$

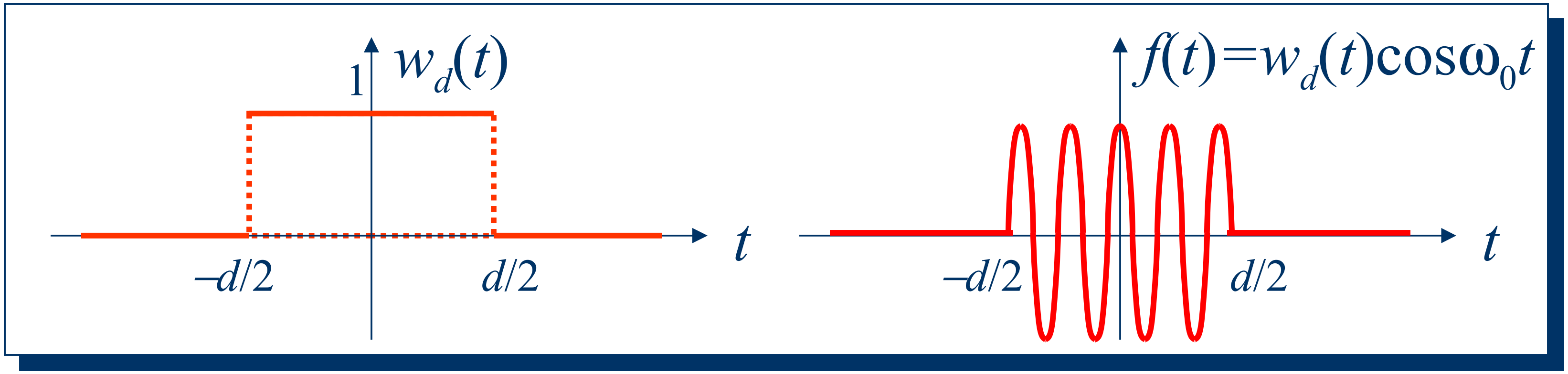
Sol)

$$f(t)\cos\omega_0 t = \frac{1}{2} f(t)(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

$$\mathcal{F}[f(t)\cos\omega_0 t] = \frac{1}{2}\mathcal{F}[f(t)e^{j\omega_0 t}] + \frac{1}{2}\mathcal{F}[f(t)e^{-j\omega_0 t}]$$

$$= \frac{1}{2}F[j(\omega-\omega_0)] + \frac{1}{2}F[j(\omega+\omega_0)]$$

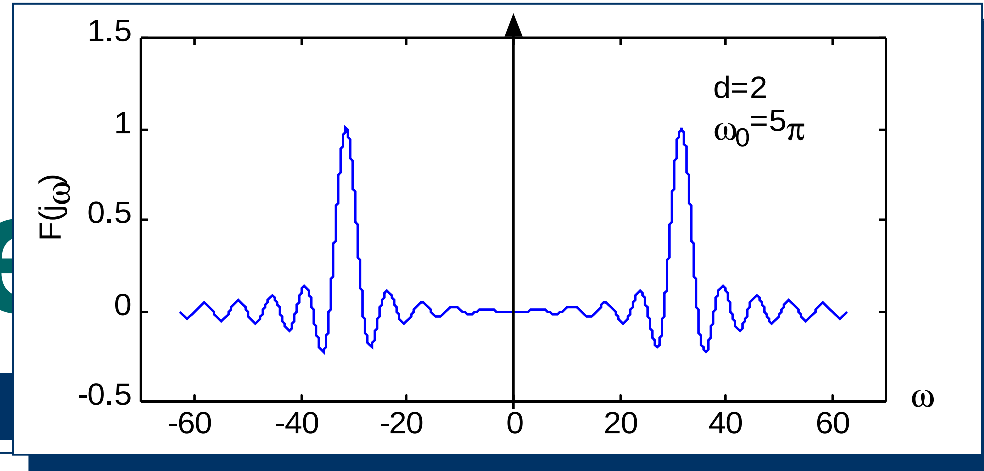
Example:

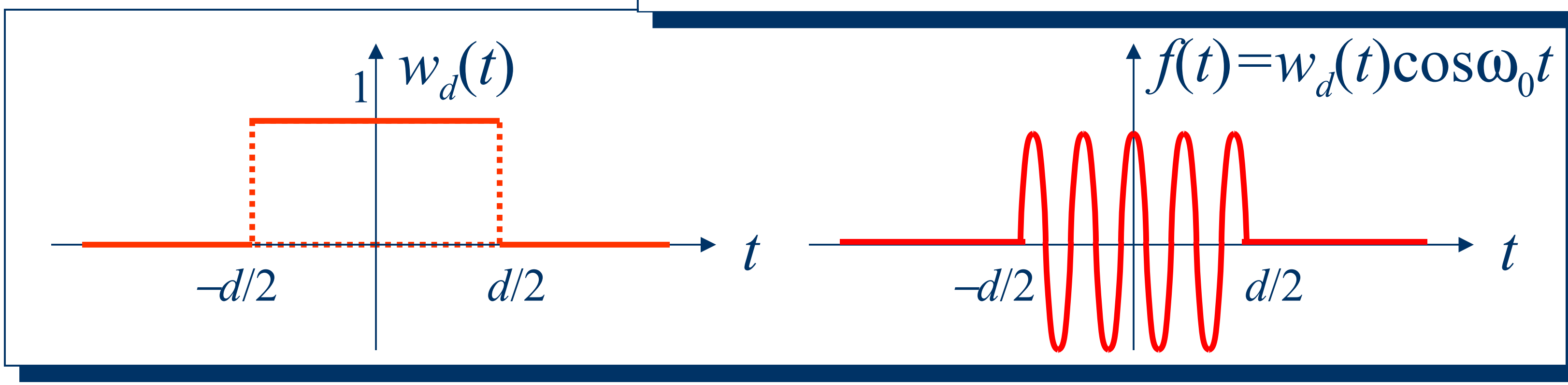


$$W_d(j\omega) = \mathcal{F}[w_d(t)] = \int_{-d/2}^{d/2} e^{-j\omega t}\,dt = \frac{2}{\omega}\sin\left(\frac{\omega d}{2}\right)$$

$$F(j\omega) = \mathcal{F}[w_d(t)\cos\omega_0 t] = \frac{\sin\frac{d}{2}(\omega-\omega_0)}{\omega-\omega_0} + \frac{\sin\frac{d}{2}(\omega+\omega_0)}{\omega+\omega_0}$$

Example

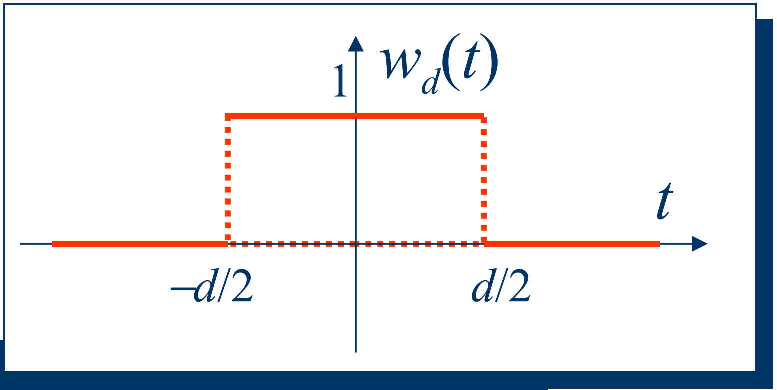




$$W_d(j\omega)=\mathcal{F}[w_d(t)]=\int_{-d/2}^{d/2} e^{-j\omega t}\,dt=\frac{2}{\omega}\sin\left(\frac{\omega d}{2}\right)$$

$$F(j\omega)=\mathcal{F}[w_d(t)\cos\omega_0 t]=\frac{\sin\frac{d}{2}(\omega-\omega_0)}{\omega-\omega_0}+\frac{\sin\frac{d}{2}(\omega+\omega_0)}{\omega+\omega_0}$$

Example:



$$f(t) = \frac{\sin at}{\pi t} \qquad F(j\omega) = ?$$

Sol)

$$W_d(j\omega) = \frac{2}{\omega}\sin\left(\frac{\omega d}{2}\right)$$

$$\Rightarrow \mathcal{F}[W_d(jt)] = \mathcal{F}\left[\frac{2}{t}\sin\left(\frac{td}{2}\right)\right] = 2\pi w_d(-\omega)$$

$$\Rightarrow \mathcal{F}[f(t)] = \mathcal{F}\left[\frac{\sin at}{\pi t}\right] = w_{2a}(-\omega) = \begin{cases} 0 & \omega <|a| \\ 1 & \omega >|a| \end{cases}$$

Answer is just opposite to as expected

Fourier Transform of *f'(t)*

$$f(t) \xleftrightarrow{\mathcal{F}} F(j\omega) \text{ and } \lim_{t\to\pm\infty} f(t) = 0$$

$$\Rightarrow \quad f'(t) \xleftrightarrow{\mathcal{F}} j\omega F(j\omega)$$

Pf)

$$\mathcal{F}[f'(t)] = \int_{-\infty}^{\infty} f'(t) e^{-j\omega t} dt$$

$$= f(t)e^{-j\omega t}\Big|_{-\infty}^{\infty} + j\omega \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$= j\omega F(j\omega)$$

Fourier Transform of $f^{(n)}(t)$

$$f(t) \xleftrightarrow{\mathcal{F}} F(j\omega) \text{ and } \lim_{t\to\pm\infty} f(t) = 0$$

$$\Rightarrow \quad f^{(n)}(t) \xleftrightarrow{\mathcal{F}} (j\omega)^n F(j\omega)$$

!Home Work!

Fourier Transform of $f^{(n)}(t)$

$$f(t) \xleftrightarrow{\mathcal{F}} F(j\omega) \text{ and } \lim_{t \to \pm\infty} f(t) = 0$$

$$\Rightarrow \quad f^{(n)}(t) \xleftrightarrow{\mathcal{F}} (j\omega)^n F(j\omega)$$

!Home Work!

Fourier Transform of Integral

$$f(t) \xleftrightarrow{\mathcal{F}} F(j\omega) \text{ and } \int_{-\infty}^{\infty} f(t)dt = F(0) = 0$$

$$\Rightarrow \quad \mathcal{F}\left[\int_{-\infty}^{t} f(x)dx\right] = \frac{1}{j\omega}F(j\omega)$$

Let $\phi(t) = \int_{-\infty}^{t} f(x)dx \quad \Rightarrow \quad \lim_{t\to\infty}\phi(t) = 0$

$$\mathcal{F}[\phi'(t)] = \mathcal{F}[f(t)] = F(j\omega) = j\omega\Phi(j\omega)$$

$$\Phi(j\omega) = \frac{1}{j\omega}F(j\omega)$$

The Derivative of Fourier Transform

$$\mathcal{F}[-jtf(t)] \overset{\mathcal{F}}{\longleftrightarrow} \frac{dF(j\omega)}{d\omega}$$

Pf)

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$\frac{dF(j\omega)}{d\omega} = \frac{d}{d\omega} \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} f(t) \frac{\partial}{\partial\omega} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} [-jtf(t)] e^{-j\omega t} dt = \mathcal{F}[-jtf(t)]$$

!Thank You!