# Jaipur Engineering College & Research Centre, Jaipur

## Session 2020-21

## Notes – Unit VII

## Electromagnetic Fields (3EE4-08)


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# Vision and Mission of Institute

## Vision of institute

To become a renowned centre of outcome based learning, and work towards, professional, cultural and social enrichment of the lives of individuals and communities.

## Mission of institute

**M1**.Focus on evaluation of learning outcomes and motivate students to inculcate research aptitude by project based learning.

**M2**.Identify, based on informed perception of Indian, regional and global needs, the areas of focus and provide platform to gain knowledge and solutions.

**M3**.Offer opportunities for interaction between academia and industry.

**M4**.Develop human potential to its fullest extent so that intellectually capable and imaginatively gifted leaders may emerge in a range of professions

# Vision and Mission of Electrical Engineering Department

## Vision of department

The Electrical Engineering department strives to recognized globally for outcome based technical knowledge and produce quality human being who can manage the advance technologies and contribute to society.

## Mission Of department

**M1**. To impart quality technical knowledge to the learners to make them globally competitive Electrical Engineers.

**M2**. To provide the learners ethical guidelines along with excellent academic environment for a long productive career.

**M3**. To promote industry- institute relationship.

# Syllabus of Electromagnetic fields

**unit 1- Review of Vector Calculus**

Vector algebra- addition, subtraction, components of vectors, scalar and vector multiplications, triple products, three orthogonal coordinate systems (rectangular, cylindrical and spherical). Vector calculus differentiation, partial differentiation, integration, vector operator del,gradient, divergence and curl; integral theorems of vectors. Conversion of a vector from one coordinate system to another.

**Unit 2- Static Electric Field**

Coulomb's law, Electric field intensity, Electrical field due to point charges. Line, Surface and Volume charge distributions. Gauss law and its applications. Absolute Electric potential, Potential difference, Calculation of potential differences for different configurations. Electric dipole, Electrostatic Energy and Energy density.

**Unit 3- Conductors, Dielectrics and Capacitance**

Current and current density, Ohms Law in Point form, Continuity of current, Boundary conditions of perfect dielectric materials. Permittivity of dielectric materials, Capacitance, Capacitance of a two wire line, Poisson's equation, Laplace's equation, Solution of Laplace and Poisson's equation, Application of Laplace's and Poisson's equations.

**unit 4- Static Magnetic Fields**

Biot-Savart Law, Ampere Law, Magnetic flux and magnetic flux density, Scalar and Vector Magnetic potentials. Steady magnetic fields produced by current carrying conductors.

**Unit5- Magnetic Forces, Materials and Inductance**

Force on a moving charge, Force on a differential current element, Force between differential current elements, Nature of magnetic materials, Magnetization and permeability, Magnetic boundary conditions, Magnetic circuits, inductances and mutual inductances.

**Unit 6- Time Varying Fields and Maxwell's Equations**

Faraday's law for Electromagnetic induction, Displacement current, Point form of Maxwell's equation, Integral form of Maxwell's equations, Motional Electromotive forces. Boundary Conditions

**Unit 7- Electromagnetic Waves**

Derivation of Wave Equation, Uniform Plane Waves, Maxwell's equation in Phasor form, Wave equation in Phasor form, Plane waves in free space and in a homogenous material. Wave equation for a conducting medium, Plane waves in lossy dielectrics, Propagation in good conductors, Skin effect. Poynting theorem.

# Course outcomes for Electromagnetic fields

**CO1-** Acquire basic understanding of vectors , their representation and conversion in different coordinate systems.

**CO2-** Able to compute the force, fields & energy of the electrostatic & magneto static fields. Able to analyze the materials, conductors, dielectrics, inductances and capacitances.

**CO3-** Understand the concept of time varying field and able to solve electromagnetic relation using Maxwell equations. Also able to analyze the electromagnetic waves.

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## Electromagnetic waves ⇒

### Time varying Potential or Retarted Potential or Derivation of wave equation ⇒

For static EM fields, electric scalar Potential as

$$V = \int_V \frac{\rho_v \, dV}{4\pi\epsilon R} \quad — (1)$$

and magnetic vector Potential as

$$\bar{A} = \int_V \frac{\mu \bar{J} \, dV}{4\pi R} \quad — (2)$$

Recall that $\bar{A}$ was defined from $\nabla \cdot \bar{B} = 0$, which still holds for time varying fields.

hence $$\boxed{\bar{B} = \nabla \times \bar{A}} \quad — (3)$$

hold for time varying situations.

from faraday's law - $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$

So,

$$\nabla \times \bar{E} = -\frac{\partial}{\partial t}(\nabla \times \bar{A}) \quad — (4)(a)$$

or

$$\nabla \times \left(E + \frac{\partial A}{\partial t}\right) = 0 \quad — (4)(b)$$

since the curl of gradient of a scalar field is identically zero.

$$E + \frac{\partial A}{\partial t} = -\nabla V$$

$$\boxed{\bar{E} = -\nabla V - \frac{\partial \bar{A}}{\partial t}} \quad — (5)$$

from eq. (3) and (5), we can determine the vector fields $\bar{B}$ & $\bar{E}$, provided the potentials $\bar{A}$ & $V$ are known.

we know that $\nabla \cdot \bar{D} = \rho_v$ is valid for the varying condition. By taking the divergence of eq. (5)

$$\nabla \cdot \bar{E} = \frac{\rho_v}{\epsilon} = -\nabla^2 V - \frac{\partial}{\partial t}(\bar{\nabla} \cdot \bar{A})$$

$$\nabla^2 V + \frac{\partial}{\partial t}(\bar{\nabla} \cdot \bar{A}) = -\frac{\rho_v}{\epsilon} \quad -(6)$$

and $\nabla \times \bar{H} = \bar{J} + \frac{\partial D}{\partial t}$, By taking the curl of eq. (3)

$$\nabla \times \nabla \times \bar{A} = \mu J + \epsilon \mu \frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial A}{\partial t}\right)$$

$$= \mu \bar{J} - \mu\epsilon \nabla\left(\frac{\partial V}{\partial t}\right) - \mu\epsilon \frac{\partial^2 \bar{A}}{\partial t^2} \quad -(7)$$

and by vector identity -

$$\nabla \times \nabla \times A = \nabla(\nabla \cdot \bar{A}) - \nabla^2 A$$

$$\nabla^2 A - \nabla(\nabla \cdot \bar{A}) = -\mu \bar{J} + \mu\epsilon \nabla\left(\frac{\partial V}{\partial t}\right) + \mu\epsilon \frac{\partial^2 A}{\partial t^2} \quad -(8)$$

→ A vector field is uniquely defined when it's curl and divergence are specified. The curl of a $\bar{A}$ has been given by $\bar{B} = \nabla \times \bar{A}$; for reason that will be obvious shortly, we may choose the divergence of $\bar{A}$ as

$$\boxed{\bar{\nabla} \cdot \bar{A} = -\mu\epsilon \frac{\partial V}{\partial t}} \quad -(9)$$

This choice relates A & V and it is called Lorentz condition for potentials.

By imposing the Lorentz condition in eq. (6) and (8) -

$$\boxed{\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\epsilon}} \quad \text{— (10)}$$

and

$$\boxed{\nabla^2 \bar{A} - \mu\epsilon \frac{\partial^2 \bar{A}}{\partial t^2} = -\mu \bar{J}} \quad \text{— (11)}$$

which are the wave equations.

and the solution to eq. (10) & (11) are

$$V = \int_v \frac{[\rho_v]\, dv}{4\pi\epsilon R} \quad \text{— (12)}$$

and

$$\bar{A} = \int_v \frac{\mu [J]\, dv}{4\pi R} \quad \text{— (13)}$$

The term $[\rho_v]$ and $[J]$ means that the time $t$ in $\rho_v(x, y, z, t)$ and $[J(x, y, z, t)]$ is replaced by retarded time $t'$ given by

$$t' = t - \frac{R}{u} \quad \text{— (14)}$$

where $\bar{R} = |\bar{r} - \bar{r}'|$ is the distance b/w the source point $\bar{r}'$ and observation point $\bar{r}$ and

$$u = \frac{1}{\sqrt{\mu\epsilon}} \quad \text{— (15)}$$

is velocity of wave propagation. In free space $u = c = 3 \times 10^8$ m/s is the speed of light in vacuum. Potentials $V$ and $\bar{A}$ in eq. (12) & (13) are respectively, called the retarded electric ~~potential~~ scalar

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Potential and retarted magnetic vector potential.

So far, our time dependence of EM fields has been arbitrary. To be specific, we shall assume that the fields are *time harmonic*.

A **time-harmonic field** is one that varies periodically or sinusoidally with time.

Not only is sinusoidal analysis of practical value, it can be extended to most waveforms by Fourier analysis. Sinusoids are easily expressed in phasors, which are more convenient to work with. Before applying phasors to EM fields, it is worthwhile to have a brief review of the concept of phasor.

A *phasor* is a complex number that contains the amplitude and the phase of a sinusoidal oscillation. As a complex number, a phasor $z$ can be represented as

$$z = x + jy = r\angle\phi \tag{8.57}$$

[5] For example, see D. K. Cheng, *Fundamental of Engineering Electromagnetics*. Reading, MA: Addison-Wesley, 1993, pp. 253–254.

or

$$z = r\,e^{j\phi} = r\,(\cos\phi + j\sin\phi) \tag{8.58}$$

where $j = \sqrt{-1}$, $x$ is the real part of $z$, $y$ is the imaginary part of $z$, $r$ is the magnitude of $z$, given by

$$r = |z| = \sqrt{x^2 + y^2} \tag{8.59}$$

and $\phi$ is the phase of $z$, given by

$$\phi = \tan^{-1}\frac{y}{x} \tag{8.60}$$

Here $x$, $y$, $z$, $r$, and $\phi$ should not be mistaken as the coordinate variables, although they look similar (different letters could have been used but it is hard to find better ones). The phasor $z$ can be represented in *rectangular form* as $z = x + jy$ or in *polar form* as $z = r\,\underline{/\phi} = r\,e^{j\phi}$. The two forms of representing $z$ are related in eqs. (8.57) to (8.60) and illustrated in Figure 8.12. Addition and subtraction of phasors are better performed in rectangular form; multiplication and division are better done in polar form.

Given complex numbers

$$z = x + jy = r\,\underline{/\phi}, \qquad z_1 = x_1 + jy_1 = r_1\,\underline{/\phi_1}, \qquad \text{and} \qquad z_2 = x_2 + jy_2 = r_2\,\underline{/\phi_2}$$

the following basic properties should be noted.

addition:

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2) \tag{8.61a}$$

subtraction:

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2) \tag{8.61b}$$

multiplication:

$$z_1 z_2 = r_1 r_2\,\underline{/\phi_1 + \phi_2} \tag{8.61c}$$

division:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,\underline{/\phi_1 - \phi_2} \tag{8.61d}$$

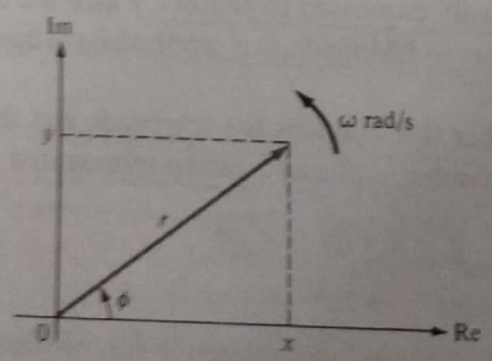


**Figure 8.12** Representation of a phasor $z = x + jy = r\,\underline{/\phi}$.

Square root:

$$\sqrt{z} = \sqrt{r}\,\underline{/\phi/2} \tag{8.61e}$$

Complex conjugate:

$$z^* = x - jy = r\underline{/-\phi} = re^{-j\phi} \tag{8.61f}$$

Other properties of complex numbers can be found in Appendix B.2.

To introduce the time element, we let

$$\phi = \omega t + \theta \tag{8.62}$$

where $\theta$ may be a function of time or space coordinates or a constant. The real (Re) and imaginary (Im) parts of

$$re^{j\phi} = re^{j\theta}e^{j\omega t} \tag{8.63}$$

are, respectively, given by

$$\mathrm{Re}(re^{j\phi}) = r\cos(\omega t + \theta) \tag{8.64a}$$

and

$$\mathrm{Im}(re^{j\phi}) = r\sin(\omega t + \theta) \tag{8.64b}$$

Thus a sinusoidal current $I(t) = I_o\cos(\omega t + \theta)$, for example, equals the real part of $I_o e^{j\theta}e^{j\omega t}$. The current $I'(t) = I_o\sin(\omega t + \theta)$, which is the imaginary part of $I_o e^{j\theta}e^{j\omega t}$, can also be represented as the real part of $I_o e^{j\theta}e^{j\omega t}e^{-j90°}$ because $\sin\alpha = \cos(\alpha - 90°)$. However, in performing our mathematical operations, we must be consistent in our use of either the real part or the imaginary part of a quantity but not both at the same time.

The complex term $I_o e^{j\theta}$, which results from dropping the time factor $e^{j\omega t}$ in $I(t)$, is called the *phasor* current, denoted by $I_s$; that is,

$$I_s = I_o e^{j\theta} = I_o\underline{/\theta} \tag{8.65}$$

where the subscript $s$ denotes the phasor form of $I(t)$. Thus $I(t) = I_o\cos(\omega t + \theta)$, the *instantaneous form*, can be expressed as

$$I(t) = \mathrm{Re}(I_s e^{j\omega t}) \tag{8.66}$$

In general, a phasor could be scalar or vector. If a vector $\mathbf{A}(x, y, z, t)$ is a time-harmonic field, the *phasor form* of $\mathbf{A}$ is $\mathbf{A}_s(x, y, z)$; the two quantities are related as

$$\boxed{\mathbf{A} = \mathrm{Re}(\mathbf{A}_s e^{j\omega t})} \tag{8.67}$$

For example, if $\mathbf{A} = A_o\cos(\omega t - \beta x)\,\mathbf{a}_y$, we can write $\mathbf{A}$ as

$$\mathbf{A} = \mathrm{Re}(A_o e^{-j\beta x}\mathbf{a}_y e^{j\omega t}) \tag{8.68}$$

Comparing this with eq. (8.67) indicates that the phasor form of $\mathbf{A}$ is

$$\mathbf{A}_s = A_o e^{-j\beta x}\mathbf{a}_y \tag{8.69}$$

Notice from eq. (8.67) that

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{\partial}{\partial t}\mathrm{Re}(\mathbf{A}_s e^{j\omega t}) = \mathrm{Re}(j\omega \mathbf{A}_s e^{j\omega t}) \tag{8.70}$$

showing that taking the time derivative of the instantaneous quantity is equivalent to multiplying its phasor form by $j\omega$. That is,

$$\frac{\partial \mathbf{A}}{\partial t} \rightarrow j\omega \mathbf{A}_s \tag{8.71}$$

Similarly,

$$\int \mathbf{A}\, \partial t \rightarrow \frac{\mathbf{A}_s}{j\omega} \tag{8.72}$$

Note that the real part is chosen in eq. (8.67) as in circuit analysis; the imaginary part could equally have been chosen. Also notice the basic difference between the instantaneous form $\mathbf{A}(x, y, z, t)$ and its phasor form $\mathbf{A}_s(x, y, z)$: the former is time dependent and real, whereas the latter is time invariant and generally complex. It is easier to work with $\mathbf{A}_s$ and obtain $\mathbf{A}$ from $\mathbf{A}_s$ whenever necessary by using eq. (8.67).

We shall now apply the phasor concept to time-varying EM fields. The field quantities $\mathbf{E}(x, y, z, t)$, $\mathbf{D}(x, y, z, t)$, $\mathbf{H}(x, y, z, t)$, $\mathbf{B}(x, y, z, t)$, $\mathbf{J}(x, y, z, t)$, and $\rho_v(x, y, z, t)$ and their derivatives can be expressed in phasor form by using eqs. (8.67) and (8.71). In phasor form, Maxwell's equations for time-harmonic EM fields in a linear, isotropic, and homogeneous medium are presented in Table 8.2. From Table 8.2, note that the time factor $e^{j\omega t}$ disappears because it is associated with every term and therefore factors out, resulting in time-independent equations. Herein lies the justification for using phasors: the time factor can be suppressed in our analysis of time-harmonic fields and inserted when necessary. Also note that in Table 8.2, the time factor $e^{j\omega t}$ has been assumed. It is equally possible to have assumed the time factor $e^{-j\omega t}$, in which case we would need to replace every $j$ in Table 8.2 with $-j$.

**Table 8.2** Time-Harmonic Maxwell's Equations Assuming Time Factor $e^{j\omega t}$

Phasor form of maxwell's equations

| Point Form | Integral Form |
|---|---|
| $\nabla \cdot \mathbf{D}_s = \rho_{vs}$ | $\oint \mathbf{D}_s \cdot d\mathbf{S} = \int \rho_{vs}\, dv$ |
| $\nabla \cdot \mathbf{B}_s = 0$ | $\oint \mathbf{B}_s \cdot d\mathbf{S} = 0$ |
| $\nabla \times \mathbf{E}_s = -j\omega \mathbf{B}_s$ | $\oint \mathbf{E}_s \cdot d\mathbf{l} = -j\omega \int \mathbf{B}_s \cdot d\mathbf{S}$ |
| $\nabla \times \mathbf{H}_s = \mathbf{J}_s + j\omega \mathbf{D}_s$ | $\oint \mathbf{H}_s \cdot d\mathbf{l} = \int (\mathbf{J}_s + j\omega \mathbf{D}_s) \cdot d\mathbf{S}$ |

→ In general, waves are means of transporting energy or information

→ Typical examples of EM waves —
Radio waves, TV signals, radar beams and light rays.

→ Generally, EM wave motion in following media:-

1) Free space ($\sigma = 0$, $\epsilon = \epsilon_0$, $\mu = \mu_0$)
2) Lossless dielectrics ($\sigma = 0$, $\epsilon = \epsilon_0 \epsilon_r$, $\mu = \mu_0 \mu_r$, or $\sigma << \omega\epsilon$)
3) Lossy dielectrics ($\sigma \neq 0$, $\epsilon = \epsilon_r \epsilon_0$, $\mu = \mu_r \mu_0$)
4) Good conductors ($\sigma \simeq \infty$, $\epsilon = \epsilon_0$, $\mu = \mu_r \mu_0$ or $\sigma >> \omega\epsilon$)

where $\omega$ = angular frequency of wave.

- <u>note</u> → Case 3, for lossy dielectrics, is the most general case and will be considered first. Once this case has been solved, we simply derive the other cases (1, 2, and 4) from it as special cases by changing the values of $\sigma$, $\epsilon$ and $\mu$.

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# wave Propagation in lossy dielectrics ⇒

A lossy dielectric is a medium in which an EM wave, as it propagates, loses power owing to imperfect dielectric.

→ A lossy dielectric is partially conducting medium with $\sigma \neq 0$, as distinct from a lossless dielectric (perfect dielectric) with $\sigma = 0$

Consider a linear, isotropic, homogeneous, lossy dielectric medium that is charge free ($\rho_v = 0$). Assuming and suppressing the time factor $e^{j\omega t}$, Maxwell's equations becomes

$$\nabla \cdot E_s = 0 \quad — (1)$$

$$\nabla \cdot H_s = 0 \quad — (2)$$

$$\nabla \times \bar{E}_s = -j\omega\mu H_s \quad — (3)$$

$$\nabla \times \bar{H}_s = (\sigma + j\omega\epsilon) E_s \quad — (4)$$

taking curl of eq. (3), gives

$$\nabla \times \nabla \times E_s = -j\omega\mu (\nabla \times H_s) \quad — (5)$$

Applying the vector identity

$$\nabla \times (\nabla \times \bar{A}) = \nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A} \quad — (6)$$

to the left hand side of eq. (5) and put the value from eq. (3) and (4), we obtain

$$\nabla(\underbrace{\nabla \cdot \bar{E}_s}_{0}) - \nabla^2 E_s = -j\omega\mu(\sigma + j\omega\epsilon) E_s$$

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or $$\boxed{\nabla^2 E_s - \gamma^2 \bar{E}_s = 0} \quad \text{— (7)}$$

where $$\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon) \quad \text{— (8)}$$

$\gamma$ = Propagation Constant of the medium

By similar Procedure —

$$\boxed{\nabla^2 \bar{H}_s - \gamma^2 \bar{H}_s = 0} \quad \text{— (9)}$$

eq. (7) & (9) are known as homogeneous vector Helmholtz's equations or simply vector wave equations.

since $\gamma$ is a complex quantity, we may let

$$\boxed{\gamma = \alpha + j\beta} \quad \text{— (10)}$$

we obtain $\alpha$ and $\beta$ from eq. (8) and (10)

$$-\mathrm{Re}\,\gamma^2 = \beta^2 - \alpha^2 = \omega^2\mu\epsilon \quad \text{— (11)}$$

and $$|\gamma^2| = \beta^2 + \alpha^2 = \omega\mu\sqrt{\sigma^2 + \omega^2\epsilon^2} \quad \text{— (12)}$$

from eq. (11) and (12), we obtain

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\epsilon}\right]^2} - 1\right]} \quad \text{— (13)}$$

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\epsilon}\right]^2} + 1\right]} \quad \text{— (14)}$$

Without loss of generality, if we assume that the wave propagates along $+\bar{a}_z$ and that $\bar{E}_s$ has only an $x$-component, then

$$\bar{E}_s = E_{xs}(z)\,\bar{a}_x \quad \text{— (15)}$$

substituting this into eq. (7)

$$(\nabla^2 - \gamma^2)\,E_{xs}(z) = 0 \quad \text{— (16)}$$

Hence

$$\underbrace{\frac{\partial^2 E_{xs}(z)}{\partial x^2}}_{0} + \underbrace{\frac{\partial^2 E_{xs}(z)}{\partial y^2}}_{0} + \frac{\partial^2 E_{xs}(z)}{\partial z^2} - \gamma^2 E_{xs}(z) = 0$$

or

$$\left[\frac{d^2}{dz^2} - \gamma^2\right] E_{xs}(z) = 0 \quad \text{— (17)}$$

This is a scalar wave equation, a linear homogeneous differential equation, with solution

$$E_{xs}(z) = E_0 e^{-\gamma z} + E_0' e^{\gamma z} \quad \text{— (18)}$$

where $E_0$ and $E_0'$ are constants.

The field must be finite at infinity requires that $E_0' = 0$

inserting the time factor $e^{j\omega t}$ into eq. (18) and using eq. (10), we obtain

$$E(z,t) = \mathrm{Re}\left[E_{xs}(z)\,e^{j\omega t}\,\bar{a}_x\right] = \mathrm{Re}\left(E_0 e^{-\alpha z} e^{j(\omega t - \beta z)}\,\bar{a}_x\right)$$

or

$$\boxed{\bar{E}(z,t) = E_0\, e^{-\alpha z} \cos(\omega t - \beta z)\,\bar{a}_x} \quad \text{— (19)}$$

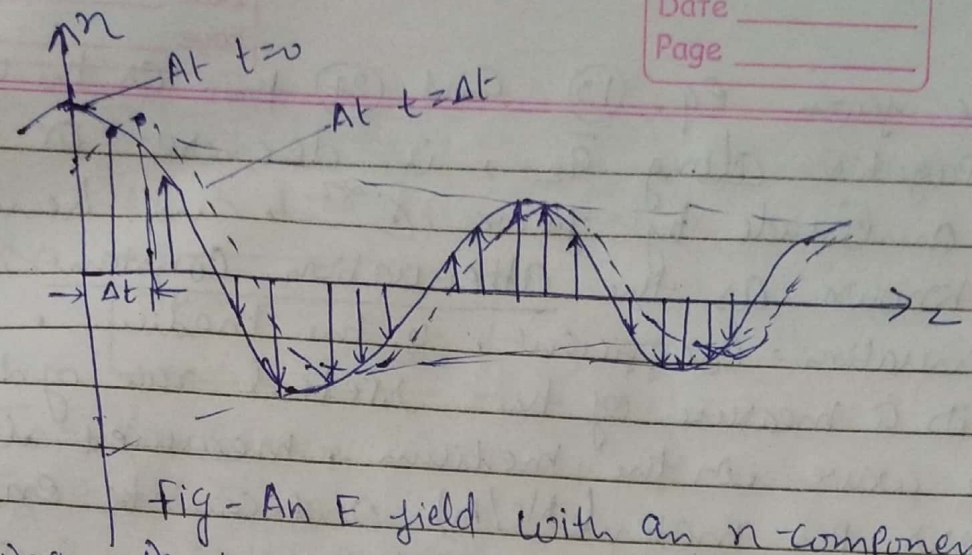


Fig - An E field with an x-component travelling in the +z direction at times t=0 and t=Δt arrows indicates the instantaneous values of $\bar{E}$.

→ Similarly we can find $\bar{H}(z,t)$

$$\bar{H}(z,t) = Re\left(H_0 e^{-\alpha z} e^{j(\omega t - \beta z)} \bar{a}_y\right) \quad -(20)$$

where $H_0 = \dfrac{E_0}{\eta}$ — (21)

and $\eta$ is a complex quantity known as intrinsic impedance, in ohms, of the medium

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} = |\eta| \angle \theta_n = |\eta| e^{j\theta_n} \quad -(22)$$

with

$$\boxed{|\eta| = \frac{\sqrt{\mu/\epsilon}}{\left[1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/4}}, \quad \tan 2\theta_n = \frac{\sigma}{\omega\epsilon}} \quad -(23)$$

where $0 \le \theta_n \le 45°$

Put eq. (21), (22) into eq. (20), gives

$$\bar{H} = Re\left[\frac{E_0}{|\eta| e^{j\theta_n}} e^{-\alpha z} e^{j(\omega t - \beta z)} \bar{a}_y\right]$$

or

$$\bar{H} = \frac{E_0}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_n) \bar{a}_y] \quad -(24)$$

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→ notice from eq. (19) and (24) that as the wave propagates along $\bar{a}_z$, it decreses or attenuates in amplitude by factor $e^{-\alpha z}$, and hence $\alpha$ is known as the <u>attenuation constant</u>, or attenuation coefficient, of the medium.
it is a measure of the spetial rate of decay of the wave in the medium, measured in nepers per meter (Np/m) and can be expressed in decibles per meter (dB/m)

$$1\,Np = 20 \log_{10} e = 8.686\ dB \quad - (25)$$

Now,
from eq. (13) —
if $\sigma = 0$ — for lossless medium and free space
$\alpha = 0$, and the wave is not attenuated as it propagates

→ quantity $\beta$ is a measure of the phase shift per unit length in radians/meter and is called phase constant or wave number
the wave velocity

$$u = \frac{\omega}{\beta}$$

and wavelength $\lambda = \frac{2\pi}{\beta}$

→ we also notic from eq. (19) and (24) that $\bar{E}$ and $\bar{H}$ are out of phase by $\theta_\eta$ at any instant of time due to the complex intrinsic impedance of the medium. Thus at any time, $\bar{E}$ leads $\bar{H}$ by $\theta_\eta$.

$$\frac{|\bar{J}_{cs}|}{|\bar{J}_{ds}|} = \frac{|\sigma \bar{E}_s|}{|j\omega\epsilon \bar{E}_s|} = \frac{\sigma}{\omega\epsilon} = \tan\theta$$

or

$$\boxed{\tan\theta = \frac{\sigma}{\omega\epsilon}} \quad \text{— (27)}$$

$J_c$ = Conduction current density
$J_d$ = displacement current density.
$\tan\theta$ = loss tangent and
$\theta$ = loss angle of medium.

→ A medium is said to be a good (lossless or perfect) dielectric if $\tan\theta$ is very small ($\sigma << \omega\epsilon$) or a good conductor if $\tan\theta$ is very large ($\sigma >> \omega\epsilon$)

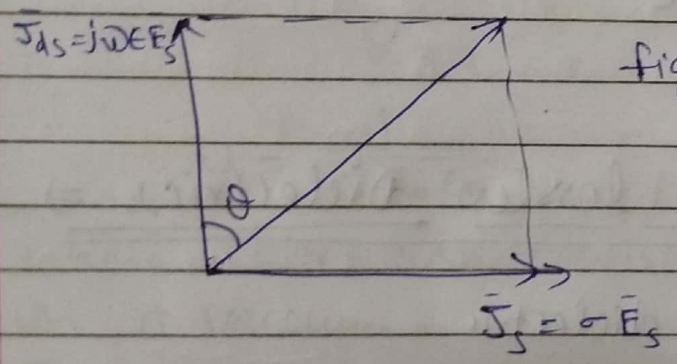



fig- loss angle of a lossy medium.

→ From the wave propagation point of view, the characteristics behaviour of a medium depends not only on it's constitutive parameters $\sigma$, $\epsilon$, and $\mu$ but also on frequencies of operation.

→ A medium that is regarded as good conductor at low frequencies may be a good dielectric at high frequencies.

from eq. (23) and (27)

$$\theta = 2\theta_n \quad \text{— (28)}$$

and from eq. (4)

$$\nabla \times \bar{H}_s = (\sigma + j\omega\epsilon)\bar{E}_s = j\omega\epsilon\left[1 - \frac{j\sigma}{\omega\epsilon}\right]\bar{E}_s \quad \text{— (29)}$$

$$= j\omega\epsilon_c\bar{E}_s$$

where

$$\epsilon_c = \epsilon\left[1 - j\frac{\sigma}{\omega\epsilon}\right] \quad \text{— (30) (a)}$$

or

$$\epsilon_c = \epsilon' - j\epsilon'' \quad \text{— (30) (b)}$$

with $\epsilon' = \epsilon$, $\epsilon'' = \sigma/\omega$;

$\epsilon_c$ is called Complex Permittivity of the medium.

and $$\tan\theta = \frac{\epsilon''}{\epsilon'} = \frac{\sigma}{\omega\epsilon} \quad \text{— (31)}$$

## Plane waves in lossless Dielectrics ⇒

in a lossless dielectric

$\sigma << \omega\epsilon$.

$$\boxed{\sigma \simeq 0, \quad \epsilon = \epsilon_0\epsilon_r \quad \text{and} \quad \mu = \mu_0\mu_r} \quad \text{— (32)}$$

Substituting these into eq. (13) and (14) gives

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\epsilon}\right]^2} - 1\right]} \quad \text{— (13)}$$

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\epsilon}\right]^2} + 1\right]} \quad \text{— (14)}$$

$$\alpha = 0, \quad \beta = \omega\sqrt{\mu\epsilon} \quad (33)(a)$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon}}, \quad \lambda = \frac{2\pi}{\beta} \quad (33)(b)$$

also, $$\eta = \sqrt{\frac{\mu}{\epsilon}} \angle 0^\circ \quad - (34)$$

and thus $\bar{E}$ and $\bar{H}$ are in time phase with each other.

## Plane waves in free space ⇒

$$\boxed{\sigma = 0, \quad \epsilon = \epsilon_0, \quad \mu = \mu_0} \quad - (35)$$

Put these value in eq. (13) and (14) we get

$$\alpha = 0, \quad \beta = \omega\sqrt{\mu_0\epsilon_0} = \frac{\omega}{c}$$

$$u = \frac{1}{\sqrt{\mu_0\epsilon_0}} = c, \quad \lambda = \frac{2\pi}{\beta}$$

where $c \cong 3\times10^8$ m/s, the speed of light in a vacuum.

→ Em waves travel in free space at the speed of light.

By substituting the constitutive parameters from (35) to (23), $\delta_n = 0$ and $\eta = \eta_0$.

where $\eta_0$ = intrinsic impedance of free space

$$\boxed{\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi \simeq 377\,\Omega} \quad (37)$$

$$\bar{E} = E_0 \cos(\omega t - \beta z)\,\bar{a}_x \quad - (38)(a)$$

then $$\bar{H} = H_0 \cos(\omega t - \beta z)\,\bar{a}_y = \frac{E_0}{\eta_0}\cos(\omega t - \beta z)\,\bar{a}_y \quad - (38)(b)$$

The plot of $\bar{E}$ and $\bar{H}$ are shown in fig. in general if $\bar{a}_E$, $\bar{a}_H$ and $\bar{a}_k$ are unit vectors ~~alog~~ along $\bar{E}$ field, $\bar{H}$ field and the direction of wave propagation, it can be shown that

$$\bar{a}_k \times \bar{a}_E = \bar{a}_H$$

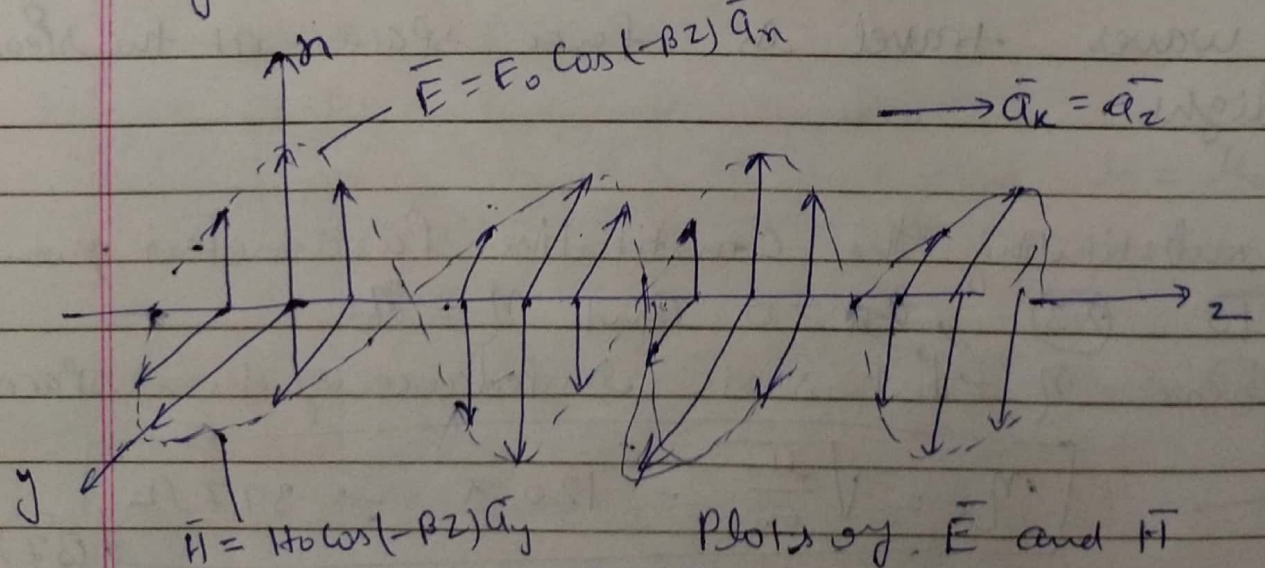
or

$$\bar{a}_k \times \bar{a}_H = -\bar{a}_E$$

or

$$\boxed{\bar{a}_E \times \bar{a}_H = \bar{a}_k} \quad \text{— (39)}$$

Both $\bar{E}$ and $\bar{H}$ fields of (EM waves) are everywhere normal to the direction of wave propagation $\bar{a}_k$. Such a wave is called a <u>transverse electromagnetic (TEM) wave</u>. A combination of $\bar{E}$ and $\bar{H}$ is called a <u>uniform plane wave</u> because $\bar{E}$ (or $\bar{H}$) has the same magnitude throughout any transverse plane, defined by $z$ = constant.




Plots of $\bar{E}$ and $\bar{H}$ as function of $z$ at $t=0$

fig (a)

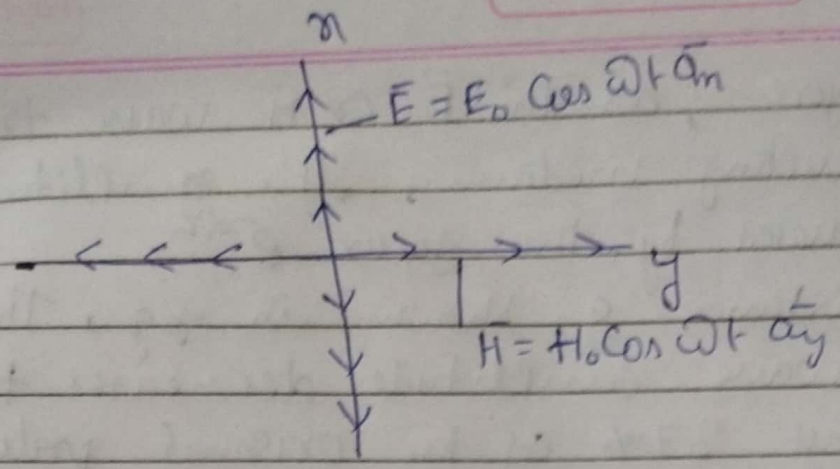



fig (b) - Plots of $\bar{E}$ and $\bar{H}$ - at $z=0$.

## Plane waves in Good conductors ⇒

A Perfect good conductor is one in which

$\sigma >> \omega\epsilon$ so that $\frac{\sigma}{\omega\epsilon} >> 1$; that is

$\sigma \simeq \infty$, $\epsilon = \epsilon_0$, $\mu = \mu_0 \mu_r$

Put these value in eq. (13) and (14)

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f \mu \sigma}$$

$$u = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}}, \quad \lambda = \frac{2\pi}{\beta}$$

also from eq. (22)

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ$$

and thus $\bar{E}$ leads $\bar{H}$ by $45^\circ$.

$$\text{or } \bar{E} = E_0 e^{-\alpha z} \cos(\omega t - \beta z) \bar{a}_x$$

$$\text{then } \bar{H} = \frac{E_0}{\sqrt{\frac{\omega\mu}{\sigma}}} e^{-\alpha z} \cos(\omega t - \beta z - 45^\circ) \bar{a}_y$$

therefore, as the $\bar{E}$ or $\bar{H}$ wave travels in a conducting medium, it's amplitude is attenuated by the factor $e^{-\alpha z}$.

→ The distance $\delta$, shown in fig, through which the wave amplitude decreases to a factor $e^{-1}$ (about 37% of the original value) is called <u>skin depth</u> or <u>penetration depth</u> of the medium.

that is,

$$E_0 e^{-\alpha \delta} = E_0 e^{-1}$$

or

$$\delta = \frac{1}{\alpha}$$

~~The~~ <u>skin depth</u> ⇒ The skin depth is a measure of the depth to which an EM wave can penetrate the medium.

for good conductors

$$\boxed{\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\alpha}}$$

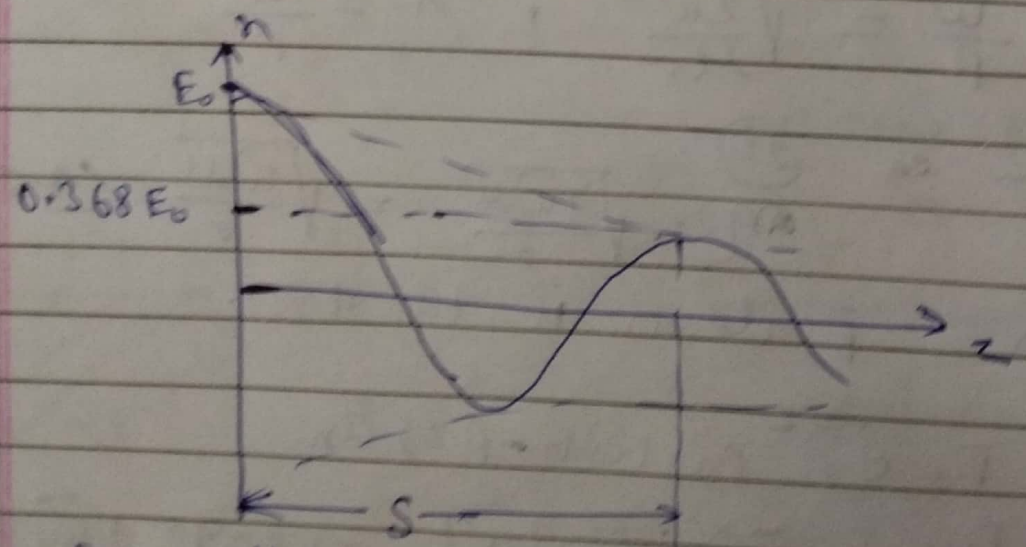



fig illustration of skin depth.

**Table 9.1** Skin Depth in Copper*

| Frequency (Hz) | 10 | 60 | 100 | 500 | $10^4$ | $10^8$ | $10^{10}$ |
|---|---|---|---|---|---|---|---|
| Skin depth (mm) | 20.8 | 8.6 | 6.6 | 2.99 | 0.66 | $6.6 \times 10^{-3}$ | $6.6 \times 10^{-4}$ |

*For copper, $\sigma = 5.8 \times 10^7$ S/m, $\mu = \mu_o$, $\delta = 66.1/\sqrt{f}$ (in mm).

The illustration in Figure 9.4 for a good conductor is exaggerated. However, for a partially conducting medium, the skin depth can be considerably large. Note from eqs. (9.41a), (9.42), and (9.44b) that for a good conductor.

$$\eta = \frac{1}{\sigma\delta}\sqrt{2}\, e^{j\pi/4} = \frac{1+j}{\sigma\delta} \tag{9.45}$$

Noting that for good conductors we have $\alpha = \beta = \dfrac{1}{\delta}$, eq. (9.43a) can be written as

$$\mathbf{E} = E_o e^{-z/\delta} \cos\left(\omega t - \frac{z}{\delta}\right) \mathbf{a}_x$$

showing that $\delta$ measures the exponential damping of the wave as it travels through the conductor. The skin depth in copper at various frequencies is shown in Table 9.1. From the table, we notice that the skin depth decreases with increasing frequency. Thus, **E** and **H** can hardly propagate through good conductors.

The phenomenon whereby field intensity in a conductor rapidly decreases is known as *skin effect.* It is a tendency of charges to migrate from the bulk of the conducting material to the surface, resulting in higher resistance. The fields and associated currents are confined to a very thin layer (the skin) of the conductor surface. For a wire of radius $a$, for example, it is a good approximation at high frequencies to assume that all of the current flows in the circular ring of thickness $\delta$ as shown in Figure 9.5. The skin effect appears in different guises in such problems as attenuation in waveguides, effective or ac resistance of transmission lines, and electromagnetic shielding. It is used to advantage in many applications. For example, because the skin depth in silver is very small, the difference in performance between a pure silver component and a silver-plated brass component is negligible, so silver plating is often used to reduce the material cost of waveguide components. For the

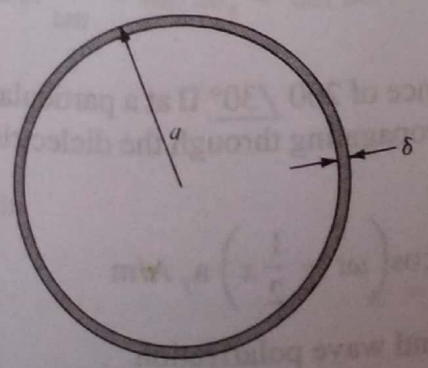



**Figure 9.5** Skin depth at high frequencies, $\delta \ll a$.

same reason, hollow tubular conductors are used instead of solid conductors in outdoor television antennas. Effective electromagnetic shielding of electrical devices can be provided by conductive enclosures a few skin depths in thickness.

The skin depth is useful in calculating the *ac resistance* due to skin effect. The resistance in eq. (4.16) is called the *dc resistance*, that is,

$$R_{dc} = \frac{\ell}{\sigma S} \tag{4.16}$$

We define the *surface or skin resistance* $R_s$ (in $\Omega$) as the real part of $\eta$ for a good conductor. Thus from eq. (9.45)

$$\boxed{R_s = \frac{1}{\sigma\delta} = \sqrt{\frac{\pi f \mu}{\sigma}}} \tag{9.46}$$

This is the resistance of a unit width and unit length of the conductor. It is equivalent to the dc resistance for a unit length of the conductor having cross-sectional area $1 \times \delta$. Thus for a given width $w$ and length $\ell$, the ac resistance is calculated by using the familiar dc resistance relation of eq. (4.16) and assuming a uniform current flow in the conductor of thickness $\delta$, that is,

$$\boxed{R_{ac} = \frac{\ell}{\sigma\delta w} = \frac{R_s \ell}{w}} \tag{9.47}$$

where $S \simeq \delta w$. For a conductor wire of radius $a$ (see Figure 9.5), $w = 2\pi a$, so

$$\frac{R_{ac}}{R_{dc}} = \frac{\dfrac{\ell}{\sigma 2\pi a \delta}}{\dfrac{\ell}{\sigma \pi a^2}} = \frac{a}{2\delta} = \frac{a}{2}\sqrt{\pi f \mu \delta}$$

Since $\delta \ll a$ at high frequencies, this shows that $R_{ac}$ is far greater than $R_{dc}$. In general, the ratio of the ac to the dc resistance starts at 1.0 for dc and very low frequencies and increases as the frequency increases. Also, although the bulk of the current is nonuniformly distributed over a thickness of $5\delta$ of the conductor, the power loss is the same as though it were uniformly distributed over a thickness of $\delta$ and zero elsewhere. This is one more reason why $\delta$ is referred to as the skin depth.

## .6 POWER AND THE POYNTING VECTOR

As mentioned before, energy can be transported from one point (where a transmitter is located) to another point (with a receiver) by means of EM waves. The rate of such energy transportation can be obtained from Maxwell's equations:

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \tag{9.48a}$$

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \tag{9.48b}$$

Dotting both sides of eq. (9.48b) with **E** gives

$$\mathbf{E} \cdot (\nabla \times \mathbf{H}) = \sigma E^2 + \mathbf{E} \cdot \varepsilon \frac{\partial \mathbf{E}}{\varepsilon t} \tag{9.49}$$

But for any vector fields **A** and **B** (see Appendix B.10)

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

Applying this vector identity to eq. (9.49) (letting $\mathbf{A} = \mathbf{H}$ and $\mathbf{B} = \mathbf{E}$) gives

$$\begin{aligned} \mathbf{H} \cdot (\nabla \times \mathbf{E}) + \nabla \cdot (\mathbf{H} \times \mathbf{E}) &= \sigma E^2 + \mathbf{E} \cdot \varepsilon \frac{\partial \mathbf{E}}{\partial t} \\ &= \sigma E^2 + \frac{1}{2} \varepsilon \frac{\partial}{\partial t} E^2 \end{aligned} \tag{9.50}$$

Dotting both sides of eq. (9.48a) with **H**, we write

$$\mathbf{H} \cdot (\nabla \times \mathbf{E}) = \mathbf{H} \cdot \left( -\mu \frac{\partial \mathbf{H}}{\partial t} \right) = -\frac{\mu}{2} \frac{\partial}{\partial t} (\mathbf{H} \cdot \mathbf{H}) \tag{9.51}$$

and thus eq. (9.50) becomes

$$-\frac{\mu}{2} \frac{\partial H^2}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{H}) = \sigma E^2 + \frac{1}{2} \varepsilon \frac{\partial E^2}{\partial t}$$

Rearranging terms and taking the volume integral of both sides,

$$\int_v \nabla \cdot (\mathbf{E} \times \mathbf{H})\, dv = -\frac{\partial}{\partial t} \int_v \left[ \frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 \right] dv - \int_v \sigma E^2\, dv \tag{9.52}$$

Applying the divergence theorem to the left-hand side gives

$$\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \int_v \left[ \frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 \right] dv - \int_v \sigma E^2\, dv \tag{9.53}$$

$$\underset{\substack{\text{total power}\\ \text{leaving the volume}}}{\downarrow} = \underset{\substack{\text{rate of decrease in}\\ \text{energy stored in electric}\\ \text{and magnetic fields}}}{\downarrow} - \underset{\substack{\text{ohmic power}\\ \text{dissipated}}}{\downarrow} \tag{9.54}$$

Equation (9.53) is referred to as *Poynting's theorem.*[2] The various terms in the equation are identified using energy-conservation arguments for EM fields. The first term on the right-hand side of eq. (9.53) is interpreted as the rate of decrease in energy stored in the electric and magnetic fields. The second term is the power dissipated because the medium is conducting ($\sigma \neq 0$). The quantity $\mathbf{E} \times \mathbf{H}$ on the left-hand side of eq. (9.53) is known as the *Poynting vector* $\mathcal{P}$, measured in watts per square meter (W/m$^2$); that is,

$$\boxed{\mathcal{P} = \mathbf{E} \times \mathbf{H}} \tag{9.55}$$

It represents the instantaneous power density vector associated with the EM field at a given point. The integration of the Poynting vector over any closed surface gives the net power flowing out of that surface.

**Poynting's theorem** states that the net power flowing out of a given volume $v$ is equal to the time rate of decrease in the energy stored within $v$ minus the ohmic losses.

The theorem is illustrated in Figure 9.6.

It should be noted that $\mathcal{P}$ is normal to both $\mathbf{E}$ and $\mathbf{H}$ and is therefore along the direction of wave propagation $\mathbf{a}_k$ for uniform plane waves. Thus

$$\mathbf{a}_k = \mathbf{a}_E \times \mathbf{a}_H \tag{9.39}$$

The fact that $\mathcal{P}$ points along $\mathbf{a}_k$ causes $\mathcal{P}$ to be regarded as a "pointing" vector.

Again, if we assume that

$$\mathbf{E}(z, t) = E_o e^{-\alpha z} \cos(\omega t - \beta z)\mathbf{a}_x$$

then

$$\mathbf{H}(z, t) = \frac{E_o}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta)\mathbf{a}_y$$

and

$$\begin{aligned} \mathcal{P}(z, t) &= \frac{E_o^2}{|\eta|} e^{-2\alpha z} \cos(\omega t - \beta z) \cos(\omega t - \beta z - \theta_\eta)\mathbf{a}_z \\ &= \frac{E_o^2}{2|\eta|} e^{-2\alpha z} [\cos \theta_\eta + \cos(2\omega t - 2\beta z - \theta_\eta)]\mathbf{a}_z \end{aligned} \tag{9.56}$$

[2] After J. H. Poynting, "On the Transfer of Energy in the Electromagnetic Field," *Philosophical Transactions*, vol. 174, 1883, p. 343.

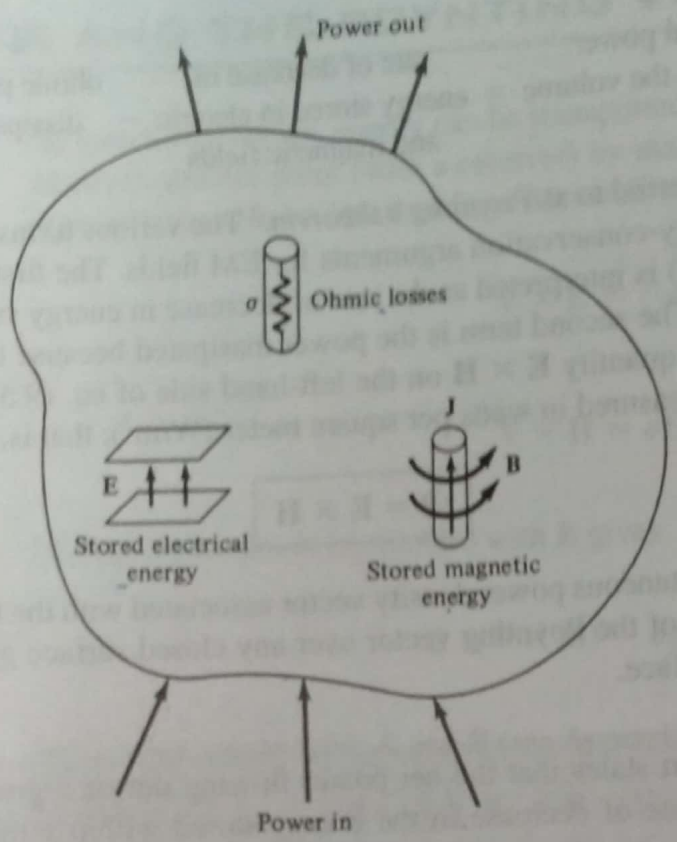



**Figure 9.6** Illustration of power balance for EM fields.

since $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$. To determine the time-average Poynting vector $\mathcal{P}_{ave}(z)$ (in W/m$^2$), which is of more practical value than the instantaneous Poynting vector $\mathcal{P}(z, t)$, we integrate eq. (9.56) over the period $T = 2\pi/\omega$; that is,

$$\mathcal{P}_{ave}(z) = \frac{1}{T}\int_0^T \mathcal{P}(z, t)\, dt \tag{9.57}$$

It can be shown (see Problem 9.30) that this is equivalent to

$$\boxed{\mathcal{P}_{ave}(z) = \frac{1}{2}\mathrm{Re}(\mathbf{E}_s \times \mathbf{H}_s^*)} \tag{9.58}$$

By substituting eq. (9.56) into eq. (9.57), we obtain

$$\boxed{\mathcal{P}_{ave}(z) = \frac{E_o^2}{2|\eta|} e^{-2\alpha z} \cos\theta_\eta\, \mathbf{a}_z} \tag{9.59}$$

The total time-average power crossing a given surface $S$ is given by

$$\boxed{P_{ave} = \int_S \mathcal{P}_{ave} \cdot d\mathbf{S}} \tag{9.60}$$

We should note the difference between $\mathcal{P}$, $\mathcal{P}_{\text{ave}}$, and $P_{\text{ave}}$: whereas $\mathcal{P}(x, y, z, t)$ is the Poynting vector in watts per square meter and is time varying, $\mathcal{P}_{\text{ave}}(x, y, z)$, also in watts per square meter, is the time average of the Poynting vector $\mathcal{P}$; it is a vector but is time invariant. Finally, $P_{\text{ave}}$ is a total time-average power through a surface in watts; it is a scalar.

**EXAMPLE 8.6**

The electric field and magnetic field in free space are given by

$$\mathbf{E} = \frac{50}{\rho}\cos(10^6 t + \beta z)\mathbf{a}_\phi \text{ V/m}$$

$$\mathbf{H} = \frac{H_o}{\rho}\cos(10^6 t + \beta z)\mathbf{a}_\rho \text{ A/m}$$

Express these in phasor form and determine the constants $H_o$ and $\beta$ such that the fields satisfy Maxwell's equations.

**Solution:**

The instantaneous forms of **E** and **H** are written as

$$\mathbf{E} = \text{Re}(\mathbf{E}_s e^{j\omega t}), \qquad \mathbf{H} = \text{Re}(\mathbf{H}_s e^{j\omega t}) \tag{8.7.1}$$

where $\omega = 10^6$ and phasors $\mathbf{E}_s$ and $\mathbf{H}_s$ are given by

$$\mathbf{E}_s = \frac{50}{\rho} e^{j\beta z}\mathbf{a}_\phi, \qquad \mathbf{H}_s = \frac{H_o}{\rho} e^{j\beta z}\mathbf{a}_\rho \tag{8.7.2}$$

For free space, $\rho_v = 0$, $\sigma = 0$, $\varepsilon = \varepsilon_o$, and $\mu = \mu_o$ so Maxwell's equations become

$$\nabla \cdot \mathbf{D} = \varepsilon_o \nabla \cdot \mathbf{E} = 0 \rightarrow \nabla \cdot \mathbf{E}_s = 0 \tag{8.7.3}$$

$$\nabla \cdot \mathbf{B} = \mu_o \nabla \cdot \mathbf{H} = 0 \rightarrow \nabla \cdot \mathbf{H}_s = 0 \tag{8.7.4}$$

$$\nabla \times \mathbf{H} = \sigma\mathbf{E} + \varepsilon_o \frac{\partial \mathbf{E}}{\partial t} \rightarrow \nabla \times \mathbf{H}_s = j\omega\varepsilon_o \mathbf{E}_s \tag{8.7.5}$$

$$\nabla \times \mathbf{E} = -\mu_o \frac{\partial \mathbf{H}}{\partial t} \rightarrow \nabla \times \mathbf{E}_s = -j\omega\mu_o \mathbf{H}_s \tag{8.7.6}$$

Substituting eq (8.7.2) into eqs. (8.7.3) and (8.7.4), it is readily verified that two Maxwell's equations are satisfied; that is,

$$\nabla \cdot \mathbf{E}_s = \frac{1}{\rho}\frac{\partial}{\partial \phi}(E_{\phi s}) = 0$$

$$\nabla \cdot \mathbf{H}_s = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_{\rho s}) = 0$$

Now

$$\nabla \times \mathbf{H}_s = \nabla \times \left(\frac{H_o}{\rho} e^{j\beta z}\mathbf{a}_\rho\right) = \frac{jH_o\beta}{\rho} e^{j\beta z}\mathbf{a}_\phi \tag{8.7.7}$$

Substituting eqs. (8.7.2) and (8.7.7) into eq. (8.7.5), we have

$$\frac{jH_o\beta}{\rho} e^{j\beta z}\mathbf{a}_\phi = j\omega\varepsilon_o \frac{50}{\rho} e^{j\beta z}\mathbf{a}_\phi$$

or

$$H_o\beta = 50\,\omega\varepsilon_o \tag{8.7.8}$$

Similarly, substituting eq. (8.7.2) into eq. (8.7.6) gives

$$-j\beta\frac{50}{\rho}e^{j\beta z}\mathbf{a}_\rho = -j\omega\mu_o\frac{H_o}{\rho}e^{j\beta z}\mathbf{a}_\rho$$

or

$$\frac{H_o}{\beta} = \frac{50}{\omega\mu_o} \tag{8.7.9}$$

Multiplying eq. (8.7.8) by eq. (8.7.9) yields

$$H_o^2 = (50)^2\frac{\varepsilon_o}{\mu_o}$$

or

$$H_o = \pm 50\sqrt{\varepsilon_o/\mu_o} = \pm\frac{50}{120\pi} = \pm 0.1326$$

Dividing eq. (8.7.8) by eq. (8.7.9), we get

$$\beta^2 = \omega^2\mu_o\varepsilon_o$$

or

$$\beta = \pm\omega\sqrt{\mu_o\varepsilon_o} = \pm\frac{\omega}{c} = \pm\frac{10^6}{3\times 10^8}$$
$$= \pm 3.33\times 10^{-3}$$

In view of eq. (8.7.8), $H_o = 0.1326$, $\beta = 3.33\times 10^{-3}$ or $H_o = -0.1326$, $\beta = -3.33\times 10^{-3}$; only these will satisfy Maxwell's four equations.

**PRACTICE EXERCISE 8.6**

In air, $\mathbf{E} = \dfrac{\sin\theta}{r}\cos(6\times 10^7 t - \beta r)\mathbf{a}_\phi$ V/m.

Find $\beta$ and $\mathbf{H}$.

**Answer:** 0.2 rad/m, $-\dfrac{1}{12\pi r^2}\cos\theta\sin(6\times 10^7 t - 0.2r)\mathbf{a}_r - \dfrac{1}{120\pi r}\sin\theta\times\cos(6\times 10^7 t - 0.2r)\mathbf{a}_\theta$ A/m.

**EXAMPLE 8.7**

In a medium characterized by $\sigma = 0$, $\mu = \mu_o$, $\varepsilon_o$, and

$$\mathbf{E} = 20\sin(10^8 t - \beta z)\mathbf{a}_y \text{ V/m}$$

calculate $\beta$ and $\mathbf{H}$.

**Solution:**

This problem can be solved directly in time domain or by using phasors. As in Example 8.7, we find $\beta$ and $\mathbf{H}$ by making $\mathbf{E}$ and $\mathbf{H}$ satisfy Maxwell's four equations.

**Method 1** (time domain):

Let us solve this problem the harder way—in time domain. It is evident that Gauss's law for electric fields is satisfied; that is,

$$\nabla \cdot \mathbf{E} = \frac{\partial E_y}{\partial y} = 0$$

From Faraday's law,

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad \rightarrow \quad \mathbf{H} = -\frac{1}{\mu} \int (\nabla \times \mathbf{E})\, dt$$

But

$$\nabla \times \mathbf{E} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} = -\frac{\partial E_y}{\partial z}\mathbf{a}_x + \frac{\partial E_y}{\partial x}\mathbf{a}_z$$
$$= 20\beta \cos(10^8 t - \beta z)\, \mathbf{a}_x + 0$$

Hence,

$$\mathbf{H} = -\frac{20\beta}{\mu} \int \cos(10^8 t - \beta z)\, dt\, \mathbf{a}_x$$
$$= -\frac{20\beta}{\mu 10^8} \sin(10^8 t - \beta z)\, \mathbf{a}_x \tag{8.8.1}$$

It is readily verified that

$$\nabla \cdot \mathbf{H} = \frac{\partial H_x}{\partial x} = 0$$

showing that Gauss's law for magnetic fields is satisfied. Lastly, from Ampère's law

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \quad \rightarrow \quad \mathbf{E} = \frac{1}{\varepsilon} \int (\nabla \times \mathbf{H})\, dt \tag{8.8.2}$$

because $\sigma = 0$.
But

$$\nabla \times \mathbf{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & 0 & 0 \end{vmatrix} = -\frac{\partial H_x}{\partial z}\mathbf{a}_y - \frac{\partial H_x}{\partial y}\mathbf{a}_z$$

$$= \frac{20\beta^2}{\mu 10^8} \cos(10^8 t - \beta z)\, \mathbf{a}_y + 0$$

where **H** in eq. (8.8.1) has been substituted. Thus eq. (8.8.2) becomes

$$\mathbf{E} = \frac{20\beta^2}{\mu\varepsilon 10^8} \int \cos(10^8 t - \beta z)\, dt\, \mathbf{a}_y$$

$$= \frac{20\beta^2}{\mu\varepsilon 10^{16}} \sin(10^8 t - \beta z)\, \mathbf{a}_y$$

Comparing this with the given **E**, we have

$$\frac{20\beta^2}{\mu\varepsilon 10^{16}} = 20$$

or

$$\beta = \pm 10^8 \sqrt{\mu\varepsilon} = \pm 10^8 \sqrt{\mu_o \cdot 4\varepsilon_o} = \pm \frac{10^8(2)}{c} = \pm \frac{10^8(2)}{3 \times 10^8}$$

$$= \pm \frac{2}{3}$$

From eq. (8.8.1),

$$\mathbf{H} = \pm \frac{20\,(2/3)}{4\pi \cdot 10^{-7}(10^8)} \sin\left(10^8 t \pm \frac{2z}{3}\right) \mathbf{a}_x$$

or

$$\mathbf{H} = \pm \frac{1}{3\pi} \sin\left(10^8 t \pm \frac{2z}{3}\right) \mathbf{a}_x \text{ A/m}$$

**Method 2** (using phasors):

$$\mathbf{E} = \text{Im}(E_s e^{j\omega t}) \quad \rightarrow \quad \mathbf{E}_s = 20 e^{-j\beta z}\, \mathbf{a}_y \tag{8.8.3}$$

where $\omega = 10^8$.
Again

$$\nabla \cdot \mathbf{E}_s = \frac{\partial E_{ys}}{\partial y} = 0$$

$$\nabla \times \mathbf{E}_s = -j\omega\mu \mathbf{H}_s \quad \rightarrow \quad \mathbf{H}_s = \frac{\nabla \times \mathbf{E}_s}{-j\omega\mu}$$

or

$$\mathbf{H}_s = \frac{1}{-j\omega\mu}\left[-\frac{\partial E_{ys}}{\partial z}\mathbf{a}_x\right] = -\frac{20\beta}{\omega\mu}e^{-j\beta z}\mathbf{a}_x \tag{8.8.4}$$

Notice that $\nabla \cdot \mathbf{H}_s = 0$ is satisfied.

$$\nabla \times \mathbf{H}_s = j\omega\varepsilon\mathbf{E}_s \quad \rightarrow \quad \mathbf{E}_s = \frac{\nabla \times \mathbf{H}_s}{j\omega\varepsilon} \tag{8.8.5}$$

Substituting $\mathbf{H}_s$ in eq. (8.8.4) into eq. (8.8.5) gives

$$\mathbf{E}_s = \frac{1}{j\omega\varepsilon}\frac{\partial H_{xs}}{\partial z}\mathbf{a}_y = \frac{20\beta^2 e^{-j\beta z}}{\omega^2\mu\varepsilon}\mathbf{a}_y$$

Comparing this with the given $\mathbf{E}_s$ in eq. (8.8.3), we have

$$20 = \frac{20\beta^2}{\omega^2\mu\varepsilon}$$

or

$$\beta = \pm\omega\sqrt{\mu\varepsilon} = \pm\frac{2}{3}$$

as obtained before. From eq. (8.8.4),

$$\mathbf{H}_s = \pm\frac{20(2/3)\,e^{\pm j\beta z}}{10^8(4\pi \times 10^{-7})}\mathbf{a}_x = \pm\frac{1}{3\pi}e^{\pm j\beta z}\mathbf{a}_x$$

$$\begin{aligned}\mathbf{H} &= \text{Im}(\mathbf{H}_s e^{j\omega t}) \\ &= \pm\frac{1}{3\pi}\sin(10^8 t \pm \beta z)\,\mathbf{a}_x \text{ A/m}\end{aligned}$$

as obtained before. It should be noticed that working with phasors provides a considerable simplification compared with working directly in time domain. Also, notice that we have used

$$\mathbf{A} = \text{Im}(\mathbf{A}_s e^{j\omega t})$$

because the given $\mathbf{E}$ is in sine form and not cosine. If we had used

$$\mathbf{A} = \text{Re}(\mathbf{A}_s e^{j\omega t})$$

sine would have to be expressed in terms of cosine, and eq. (8.8.3) would be

$$\mathbf{E} = 20\cos(10^8 t - \beta z - 90^\circ)\,\mathbf{a}_y = \mathrm{Re}(\mathbf{E}_s e^{j\omega t})$$

or

$$\mathbf{E}_s = 20e^{-j\beta z - j90^\circ}\mathbf{a}_y = -j20e^{-j\beta z}\mathbf{a}_y$$

and we follow the same procedure.

**PRACTICE EXERCISE 8.7**

A medium is characterized by $\sigma = 0$, $\mu = 2\mu_o$ and $\varepsilon = 5\varepsilon_o$. If $\mathbf{H} = 2\cos(\omega t - 3y)\,\mathbf{a}_z$ A/m, calculate $\omega$ and $\mathbf{E}$.

**Answer:** $2.846 \times 10^8$ rad/s, $-151.8\cos(2.846 \times 10^8 t - 3y)\,\mathbf{a}_x$ V/m.

**EXAMPLE 9.1**

A lossy dielectric has an intrinsic impedance of $200 \angle 30^\circ\ \Omega$ at a particular radian frequency $\omega$. If, at that frequency, the plane wave propagating through the dielectric has the magnetic field component

$$\mathbf{H} = 10\, e^{-\alpha x} \cos\left(\omega t - \frac{1}{2} x\right) \mathbf{a}_y \text{ A/m}$$

find $\mathbf{E}$ and $\alpha$. Determine the skin depth and wave polarization.

**Solution:**

The given wave travels along $\mathbf{a}_x$ so that $\mathbf{a}_k = \mathbf{a}_x$; $\mathbf{a}_H = \mathbf{a}_y$, so

$$-\mathbf{a}_E = \mathbf{a}_k \times \mathbf{a}_H = \mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$$

or

$$\mathbf{a}_E = -\mathbf{a}_z$$

Also $H_o = 10$, so

$$\frac{E_o}{H_o} = \eta = 200\,\underline{/30^\circ} = 200\,e^{j\pi/6} \rightarrow E_o = 2000e^{j\pi/6}$$

Except for the amplitude and phase difference, **E** and **H** always have the same form. Hence

$$\mathbf{E} = \mathrm{Re}(2000e^{j\pi/6}e^{-\gamma x}e^{j\omega t}\mathbf{a}_E)$$

or

$$\mathbf{E} = -2e^{-\alpha x}\cos\left(\omega t - \frac{x}{2} + \frac{\pi}{6}\right)\mathbf{a}_z\ \mathrm{kV/m}$$

Knowing that $\beta = 1/2$, we need to determine $\alpha$. Since

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1\right]}$$

and

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1\right]}$$

$$\frac{\alpha}{\beta} = \left[\frac{\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1}{\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1}\right]^{1/2}$$

But $\frac{\sigma}{\omega\varepsilon} = \tan 2\theta_\eta = \tan 60^\circ = \sqrt{3}$. Hence,

$$\frac{\alpha}{\beta} = \left[\frac{2-1}{2+1}\right]^{1/2} = \frac{1}{\sqrt{3}}$$

or

$$\alpha = \frac{\beta}{\sqrt{3}} = \frac{1}{2\sqrt{3}} = 0.2887\ \mathrm{Np/m}$$

and

$$\delta = \frac{1}{\alpha} = 2\sqrt{3} = 3.464 \text{ m}$$

The wave has only an $E_z$ component; hence it is polarized along the $z$-direction.

**PRACTICE EXERCISE 9.1**

A plane wave propagating through a medium with $\varepsilon_r = 8$, $\mu_r = 2$ has $\mathbf{E} = 0.5\, e^{-z/3} \sin(10^8 t - \beta z)\mathbf{a}_x$ V/m. Determine

(a) $\beta$
(b) The loss tangent
(c) Intrinsic impedance
(d) Wave velocity
(e) **H** field

**Answer:** (a) 1.374 rad/m, (b) 0.5154, (c) $177.72 \angle 13.63^\circ\ \Omega$, (d) $7.278 \times 10^7$ m/s, (e) $2.817 e^{-z/3} \sin(10^8 t - \beta z - 13.63^\circ)\mathbf{a}_y$ mA/m.

**EXAMPLE 9.2**

In a lossless dielectric for which $\eta = 60\pi$, $\mu_r = 1$, and $\mathbf{H} = -0.1 \cos(\omega t - z)\mathbf{a}_x + 0.5 \sin(\omega t - z)\mathbf{a}_y$ A/m, calculate $\varepsilon_r$, $\omega$, and **E**.

**Solution:**

In this case, $\sigma = 0$, $\alpha = 0$, and $\beta = 1$, so

$$\eta = \sqrt{\mu/\varepsilon} = \sqrt{\frac{\mu_o}{\varepsilon_o}}\sqrt{\frac{\mu_r}{\varepsilon_r}} = \frac{120\pi}{\sqrt{\varepsilon_r}}$$

or

$$\sqrt{\varepsilon_r} = \frac{120\pi}{\eta} = \frac{120\pi}{60\pi} = 2 \quad \rightarrow \quad \varepsilon_r = 4$$

$$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_o\varepsilon_o}\sqrt{\mu_r\varepsilon_r} = \frac{\omega}{c}\sqrt{4} = \frac{2\omega}{c}$$

or

$$\omega = \frac{\beta c}{2} = \frac{1\,(3 \times 10^8)}{2} = 1.5 \times 10^8 \text{ rad/s}$$

From the given **H** field, **E** can be calculated in two ways: by using the techniques (based on Maxwell's equations) developed in this chapter or directly, by using Maxwell's equations as in Chapter 8.

**Method 1:** To use the techniques developed in the present chapter, we let

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2$$

where $\mathbf{H}_1 = -0.1\cos(\omega t - z)\mathbf{a}_x$ and $\mathbf{H}_2 = 0.5\sin(\omega t - z)\mathbf{a}_y$ and the corresponding electric field

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$$

where $\mathbf{E}_1 = E_{1o}\cos(\omega t - z)\mathbf{a}_{E_1}$ and $\mathbf{E}_2 = E_{2o}\sin(\omega t - z)\mathbf{a}_{E_2}$. Notice that although $\mathbf{H}$ has components along $\mathbf{a}_x$ and $\mathbf{a}_y$, it has no component along the direction of propagation; it is therefore a TEM wave.

For $\mathbf{E}_1$,

$$\mathbf{a}_{E_1} = -(\mathbf{a}_k \times \mathbf{a}_{H_1}) = -(\mathbf{a}_z \times -\mathbf{a}_x) = \mathbf{a}_y$$

$$E_{1o} = \eta H_{1o} = 60\pi\,(0.1) = 6\pi$$

Hence

$$\mathbf{E}_1 = 6\pi\cos(\omega t - z)\mathbf{a}_y$$

For $\mathbf{E}_2$,

$$\mathbf{a}_{E_2} = -(\mathbf{a}_k \times \mathbf{a}_{H_2}) = -(\mathbf{a}_z \times \mathbf{a}_y) = \mathbf{a}_x$$

$$E_{2o} = \eta H_{2o} = 60\pi(0.5) = 30\pi$$

Hence

$$\mathbf{E}_2 = 30\pi\sin(\omega t - z)\mathbf{a}_x$$

Adding $\mathbf{E}_1$ and $\mathbf{E}_2$ gives $\mathbf{E}$; that is,

$$\mathbf{E} = 94.25\sin(1.5\times 10^8 t - z)\mathbf{a}_x + 18.85\cos(1.5\times 10^8 t - z)\mathbf{a}_y \text{ V/m}$$

**Method 2:** We may apply Maxwell's equations directly

$$\nabla\times\mathbf{H} = \underbrace{\cancel{\sigma\mathbf{E}}}_{0} + \varepsilon\frac{\partial\mathbf{E}}{\partial t} \quad\rightarrow\quad \mathbf{E} = \frac{1}{\varepsilon}\int\nabla\times\mathbf{H}\,dt$$

because $\sigma = 0$. But

$$\nabla\times\mathbf{H} = \begin{vmatrix} a_x & a_y & a_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x(z) & H_y(z) & 0 \end{vmatrix} = -\frac{\partial H_y}{\partial z}\mathbf{a}_x + \frac{\partial H_x}{\partial z}\mathbf{a}_y$$

$$= H_{2o}\cos(\omega t - z)\mathbf{a}_x + H_{1o}\sin(\omega t - z)\mathbf{a}_y$$

where $H_{1o} = -0.1$ and $H_{2o} = 0.5$. Hence

$$\mathbf{E} = \frac{1}{\varepsilon}\int \nabla \times \mathbf{H}\, dt = \frac{H_{2o}}{\varepsilon\omega}\sin(\omega t - z)\mathbf{a}_x - \frac{H_{1o}}{\varepsilon\omega}\cos(\omega t - z)\mathbf{a}_y$$
$$= 94.25\sin(\omega t - z)\mathbf{a}_x + 18.85\cos(\omega t - z)\mathbf{a}_y \text{ V/m}$$

as expected.

**PRACTICE EXERCISE 9.2**

A plane wave in a nonmagnetic medium has $\mathbf{E} = 50\sin(10^8 t + 2z)\mathbf{a}_y$ V/m. Find

(a) The direction of wave propagation

(b) $\lambda$, $f$, and $\varepsilon_r$

(c) $\mathbf{H}$

**Answer:** (a) in the $-z$-direction, (b) 3.142 m, 15.92 MHz, 36, (c) $0.7958\sin(10^8 t + 2z)\mathbf{a}_x$ A/m.

**EXAMPLE 9.3**

A uniform plane wave propagating in a medium has

$$\mathbf{E} = 2e^{-\alpha z}\sin(10^8 t - \beta z)\mathbf{a}_y \text{ V/m}$$

If the medium is characterized by $\varepsilon_r = 1$, $\mu_r = 20$, and $\sigma = 3$ S/m, find $\alpha$, $\beta$, and $\mathbf{H}$.

**Solution:**

We need to determine the loss tangent to be able to tell whether the medium is a lossy dielectric or a good conductor.

$$\frac{\sigma}{\omega\varepsilon} = \frac{3}{10^8 \times 1 \times \dfrac{10^{-9}}{36\pi}} = 3393 \gg 1$$

showing that the medium may be regarded as a good conductor at the frequency of operation. Hence,

$$\alpha = \beta = \sqrt{\frac{\mu\omega\sigma}{2}} = \left[\frac{4\pi \times 10^{-7} \times 20(10^8)(3)}{2}\right]^{1/2}$$
$$= 61.4$$
$$\alpha = 61.4 \text{ Np/m}, \qquad \beta = 61.4 \text{ rad/m}$$

Also

$$|\eta| = \sqrt{\frac{\mu\omega}{\sigma}} = \left[\frac{4\pi \times 10^{-7} \times 20(10^8)}{3}\right]^{1/2}$$
$$= \sqrt{\frac{800\pi}{3}}$$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon} = 3393 \quad \rightarrow \quad \theta_\eta = 45^\circ = \frac{\pi}{4}$$

Hence

$$\mathbf{H} = H_o e^{-\alpha z} \sin\left(\omega t - \beta z - \frac{\pi}{4}\right)\mathbf{a}_H$$

where

$$\mathbf{a}_H = \mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_z \times \mathbf{a}_y = -\mathbf{a}_x$$

and

$$H_o = \frac{E_o}{|\eta|} = 2\sqrt{\frac{3}{800\pi}} = 69.1 \times 10^{-3}$$

Thus

$$\mathbf{H} = -69.1\, e^{-61.4z} \sin\left(10^8 t - 61.42z - \frac{\pi}{4}\right)\mathbf{a}_x \text{ mA/m}$$



**PRACTICE EXERCISE 9.3**

A plane wave traveling in the +y-direction in a lossy medium ($\varepsilon_r = 4$, $\mu_r = 1$, $\sigma = 10^{-2}$ S/m) has $\mathbf{E} = 30\cos(10^9\pi t + \pi/4)\mathbf{a}_z$ V/m at $y = 0$. Find

(a) E at $y = 1$ m, $t = 2$ ns

(b) The distance traveled by the wave to have a phase shift of 10°

(c) The distance traveled by the wave to have its amplitude reduced by 40%

(d) **H** at $y = 2$ m, $t = 2$ ns

**Answer:** (a) $2.733\mathbf{a}_z$ V/m, (b) 8.349 mm, (c) 542 mm, (d) $-22.6\mathbf{a}_x$ mA/m.

**EXAMPLE 9.4**

A plane wave $\mathbf{E} = E_o\cos(\omega t - \beta z)\mathbf{a}_x$ is incident on a good conductor at $z \geq 0$. Find the current density in the conductor.

**Solution:**

Since the current density $\mathbf{J} = \sigma\mathbf{E}$, we expect **J** to satisfy the wave equation in eq. (9.7); that is, we expect to find

$$\nabla^2\mathbf{J}_s - \gamma^2\mathbf{J}_s = 0$$

Also the incident **E** has only an $x$-component and varies with $z$. Hence $\mathbf{J} = J_x(z, t)\mathbf{a}_x$ and

$$\frac{d^2}{dz^2}J_{sx} - \gamma^2 J_{sx} = 0$$

which is an ordinary differential equation with solution (see Case 2 of Example 5.5)

$$J_{sx} = Ae^{-\gamma z} + Be^{+\gamma z}$$

The constant $B$ must be zero because $J_{sx}$ is finite as $z \to \infty$. But in a good conductor, $\sigma \gg \omega\varepsilon$ so that $\alpha = \beta = 1/\delta$. Hence

$$\gamma = \alpha + j\beta = \alpha(1 + j) = \frac{(1 + j)}{\delta}$$

and

$$J_{sx} = Ae^{-z(1+j)/\delta}$$

or

$$J_{sx} = J_{sx}(0)\, e^{-z(1+j)/\delta}$$

where $J_{sx}(0)$ is the current density on the conductor surface.

**PRACTICE EXERCISE 9.4**

Given the current density of Example 9.4, find the magnitude of the total current through a strip of the conductor of infinite depth along $z$ and width $w$ along $y$.

**Answer:** $\dfrac{J_{sx}(0)w\delta}{\sqrt{2}}$.

**EXAMPLE 9.5**

For the copper coaxial cable of Figure 6.12, let $a = 2$ mm, $b = 6$ mm, and $t = 1$ mm. Calculate the resistance of a 2 m length of the cable at dc and at 100 MHz.

**Solution:**

Let

$$R = R_o + R_i$$

where $R_o$ and $R_i$ are the resistances of the inner and outer conductors. At dc,

$$R_i = \frac{\ell}{\sigma S} = \frac{\ell}{\sigma \pi a^2} = \frac{2}{5.8 \times 10^7 \pi [2 \times 10^{-3}]^2} = 2.744 \text{ m}\Omega$$

$$R_o = \frac{\ell}{\sigma S} = \frac{\ell}{\sigma\pi[[b+t]^2 - b^2]} = \frac{\ell}{\sigma\pi[t^2 + 2bt]}$$

$$= \frac{2}{5.8 \times 10^7 \pi\,[1 + 12] \times 10^{-6}}$$

$$= 0.8429 \text{ m}\Omega$$

Hence $R_{dc} = 2.744 + 0.8429 = 3.587$ mΩ.

At $f = 100$ MHz,

$$R_i = \frac{R_s \ell}{w} = \frac{\ell}{\sigma\delta 2\pi a} = \frac{\ell}{2\pi a}\sqrt{\frac{\pi f \mu}{\sigma}}$$

$$= \frac{2}{2\pi \times 2 \times 10^{-3}}\sqrt{\frac{\pi \times 10^8 \times 4\pi \times 10^{-7}}{5.8 \times 10^7}}$$

$$= 0.41\ \Omega$$

Since $\delta = 6.6\ \mu\text{m} \ll t = 1$ mm, $w = 2\pi b$ for the outer conductor. Hence,

$$R_o = \frac{R_s \ell}{w} = \frac{\ell}{2\pi b}\sqrt{\frac{\pi f \mu}{\sigma}}$$

$$= \frac{2}{2\pi \times 6 \times 10^{-3}}\sqrt{\frac{\pi \times 10^8 \times 4\pi \times 10^{-7}}{5.8 \times 10^7}}$$

$$= 0.1384\ \Omega$$

Hence,

$$R_{ac} = 0.41 + 0.1384 = 0.5484\ \Omega$$

which is about 150 times greater than $R_{dc}$. Thus, for the same effective current $i$, the ohmic loss ($i^2R$) of the cable at 100 MHz is greater than the dc power loss by a factor of 150.

**PRACTICE EXERCISE 9.5**

For an aluminum wire having a diameter 2.6 mm, calculate the ratio of ac to dc resistance at

(a) 10 MHz

(b) 2 GHz

**Answer:** (a) 24.16, (b) 341.7.

AMPLE 9.6

In a nonmagnetic medium

$$\mathbf{E} = 4\sin(2\pi \times 10^7 t - 0.8x)\mathbf{a}_z \text{ V/m}$$

Find

(a) $\varepsilon_r, \eta$

(b) The time-average power carried by the wave

(c) The total power crossing 100 cm$^2$ of plane $2x + y = 5$

**Solution:**

(a) Since $\alpha = 0$ and $\beta \neq \omega/c$, the medium is not free space but a lossless medium:

$$\beta = 0.8, \quad \omega = 2\pi \times 10^7, \quad \mu = \mu_o \text{ (nonmagnetic)}, \quad \varepsilon = \varepsilon_o\varepsilon_r$$

Hence

$$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_o\varepsilon_o\varepsilon_r} = \frac{\omega}{c}\sqrt{\varepsilon_r}$$

or

$$\sqrt{\varepsilon_r} = \frac{\beta c}{\omega} = \frac{0.8(3 \times 10^8)}{2\pi \times 10^7} = \frac{12}{\pi}$$

$$\varepsilon_r = 14.59$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_o}{\varepsilon_o\varepsilon_r}} = \frac{120\pi}{\sqrt{\varepsilon_r}} = 120\pi \cdot \frac{\pi}{12} = 10\pi^2$$

$$= 98.7\ \Omega$$

(b) $\mathcal{P} = \mathbf{E} \times \mathbf{H} = \dfrac{E_o^2}{\eta}\sin^2(\omega t - \beta x)\mathbf{a}_x$

$$\mathcal{P}_{ave} = \frac{1}{T}\int_0^T \mathcal{P}\, dt = \frac{E_o^2}{2\eta}\mathbf{a}_x = \frac{16}{2 \times 10\pi^2}\mathbf{a}_x$$

$$= 81\mathbf{a}_x \text{ mW/m}^2$$

$F(x,y,z) = 2x + y - 5$

(c) On plane $2x + y = 5$ (see Example 2.5 or 7.5),

$\nabla F = 2\bar{a}_x + \bar{a}_y$

unit vector normal to the Plane $\bar{a}_n = \dfrac{\nabla F}{|\nabla F|} = \dfrac{2\bar{a}_x + \bar{a}_y}{\sqrt{4+1}}$

$$\mathbf{a}_n = \frac{2\mathbf{a}_x + \mathbf{a}_y}{\sqrt{5}}$$

Hence the total power is

$$P_{ave} = \int \mathcal{P}_{ave} \cdot d\mathbf{S} = \mathcal{P}_{ave} \cdot S\,\mathbf{a}_n$$

$$= (81 \times 10^{-3}\mathbf{a}_x) \cdot (100 \times 10^{-4}) \left[\frac{2\mathbf{a}_x + \mathbf{a}_y}{\sqrt{5}}\right]$$

$$= \frac{162 \times 10^{-5}}{\sqrt{5}} = 724.5\ \mu\text{W}$$

**PRACTICE EXERCISE 9.6**

In free space, $\mathbf{H} = 0.2\cos(\omega t - \beta x)\mathbf{a}_z$ A/m. Find the total power passing through:

(a) A square plate of side 10 cm on plane $x + y = 1$

(b) A circular disk of radius 5 cm on plane $x = 1$.

**Answer:** (a) 53.31 mW, (b) 59.22 mW.

# REFERENCES


- Engineering Electromagnetic by william H. Hayt, Jr. John A. Buck
- Principals of Electromagnetics by Matthew N. O. Sadiku.
- CBS problems and solution series ( Problems and solution of Engineering Electromagnetics).
- https://nptel.ac.in/courses/108/106/108106073/
- https://nptel.ac.in/courses/117/103/117103065/

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# Thank you!