Jaipur Engineering College & Research Centre, Jaipur

JECRC

Session 2020-21

Notes – Unit IV

Electromagnetic Fields (3EE4-08)

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Vision and Mission of Institute

Vision of institute

To become a renowned centre of outcome based learning, and work towards, professional, cultural and social enrichment of the lives of individuals and communities.

Mission of institute

M1.Focus on evaluation of learning outcomes and motivate students to inculcate research aptitude by project based learning.

M2.Identify, based on informed perception of Indian, regional and global needs, the areas of focus and provide platform to gain knowledge and solutions.

M3.Offer opportunities for interaction between academia and industry.

M4.Develop human potential to its fullest extent so that intellectually capable and imaginatively gifted leaders may emerge in a range of professions

Vision and Mission of Electrical Engineering Department

Vision of department

The Electrical Engineering department strives to recognized globally for outcome based technical knowledge and produce quality human being who can manage the advance technologies and contribute to society.

Mission Of department

M1. To impart quality technical knowledge to the learners to make them globally competitive Electrical Engineers.

M2. To provide the learners ethical guidelines along with excellent academic environment for a long productive career.

M3. To promote industry- institute relationship.

Syllabus of Electromagnetic fields

unit 1- Review of Vector Calculus
Vector algebra- addition, subtraction, components of vectors, scalar and vector multiplications, triple products, three orthogonal coordinate systems (rectangular, cylindrical and spherical). Vector calculus differentiation, partial differentiation, integration, vector operator del,gradient, divergence and curl; integral theorems of vectors. Conversion of a vector from one coordinate system to another.

Unit 2- Static Electric Field
Coulomb's law, Electric field intensity, Electrical field due to point charges. Line, Surface and Volume charge distributions. Gauss law and its applications. Absolute Electric potential, Potential difference, Calculation of potential differences for different configurations. Electric dipole, Electrostatic Energy and Energy density.

Unit 3- Conductors, Dielectrics and Capacitance
Current and current density, Ohms Law in Point form, Continuity of current, Boundary conditions of perfect dielectric materials. Permittivity of dielectric materials, Capacitance, Capacitance of a two wire line, Poisson's equation, Laplace's equation, Solution of Laplace and Poisson's equation, Application of Laplace's and Poisson's equations.

unit 4- Static Magnetic Fields
Biot-Savart Law, Ampere Law, Magnetic flux and magnetic flux density, Scalar and Vector Magnetic potentials. Steady magnetic fields produced by current carrying conductors.

Unit5- Magnetic Forces, Materials and Inductance
Force on a moving charge, Force on a differential current element, Force between differential current elements, Nature of magnetic materials, Magnetization and permeability, Magnetic boundary conditions, Magnetic circuits, inductances and mutual inductances.

Unit 6- Time Varying Fields and Maxwell's Equations
Faraday's law for Electromagnetic induction, Displacement current, Point form of Maxwell's equation, Integral form of Maxwell's equations, Motional Electromotive forces. Boundary Conditions

Unit 7- Electromagnetic Waves
Derivation of Wave Equation, Uniform Plane Waves, Maxwell's equation in Phasor form, Wave equation in Phasor form, Plane waves in free space and in a homogenous material. Wave equation for a conducting medium, Plane waves in lossy dielectrics, Propagation in good conductors, Skin effect. Poynting theorem.

Course outcomes for Electromagnetic fields

CO1- Acquire basic understanding of vectors , their representation and conversion in different coordinate systems.

CO2- Able to compute the force, fields & energy of the electrostatic & magneto static fields. Able to analyze the materials, conductors, dielectrics, inductances and capacitances.

CO3- Understand the concept of time varying field and able to solve electromagnetic relation using Maxwell equations. Also able to analyze the electromagnetic waves.

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Unit - 4

Magnetostatic fields ⇒

There are similarities and dissimilarities b/w electric and magnetic fields.

As $\bar{E}$ and $\bar{D}$ are related to according to $\bar{D} = \epsilon \bar{E}$ for linear, isotropic material space.

$\bar{H}$ and $\bar{B}$ are related according to $\bar{B} = \mu \bar{H}$

Biot - Savart's Law ⇒

Biot - Savart's law states that the differential magnetic field intensity dH Produced at a Point P, by differential current element Idl is Proportional to the Product Idl and the sine of the angle α b/w the element and the line joining P to the element and is inversely Proportional to the square of the distance R between P and the element.

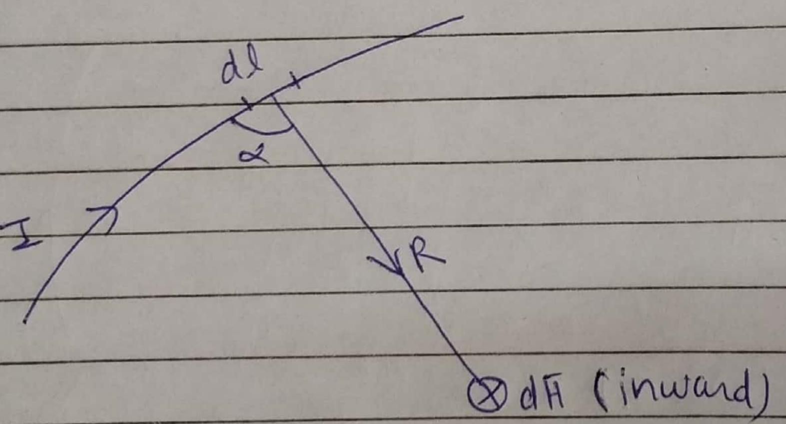

Fig:-1 Magnetic field $d\bar{H}$ at P due to current element Idl.

That is,

$$dH \propto \frac{I\,dl \sin\alpha}{R^2} \quad \text{—(I)}$$

or

$$d\bar{H} = \frac{K I\,dl \sin\alpha}{R^2} \quad \text{—(II)}$$

where k is the constant of Proportionality.

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In SI units, $k = \frac{1}{4\pi}$ so

$$dH = \frac{I\,dl \sin\alpha}{4\pi R^2} \quad \text{— (iii)}$$

from the definition of cross product

$$\boxed{d\bar{H} = \frac{I\,d\bar{l} \times \bar{a}_R}{4\pi R^2} = \frac{I\,d\bar{l} \times \bar{R}}{4\pi R^3}} \quad \text{—(iv)}$$

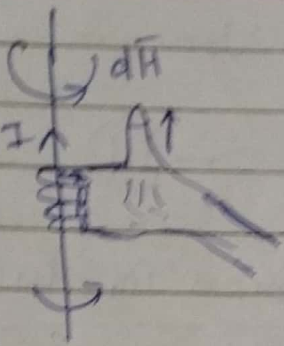
where $R = |\bar{R}|$ and $\bar{a}_R = \bar{R}/R$

The direction $d\bar{H}$ can be determined by the right hand rule with the right-hand thumb pointing in the direction of the current and the right-hand fingers encircling the wire in the direction of $d\bar{H}$ as shown in fig.2

Alternatively, we can use the right-handed ~~screw~~ screw rule to determine the direction of $d\bar{H}$: with the screw placed along the wire and pointed in the direction of current flow, the direction of advance of the screw is the direction of $d\bar{H}$ as in fig 3.

it is customary to represent the direction of the magnetic field intensity $\bar{H}$ or (current I) by a small circle with a dot or cross sign depending on whether $\bar{H}$ (or I) is out of, or into the page as shown in fig 4.

→ Just as we can have different charge configurations, we can have different current distributions: line current, surface current and volume current as shown in fig (5) if we define $\bar{K}$ as the surface current density in (A/m) and $\bar{J}$ as the volume current density in (A/m²)



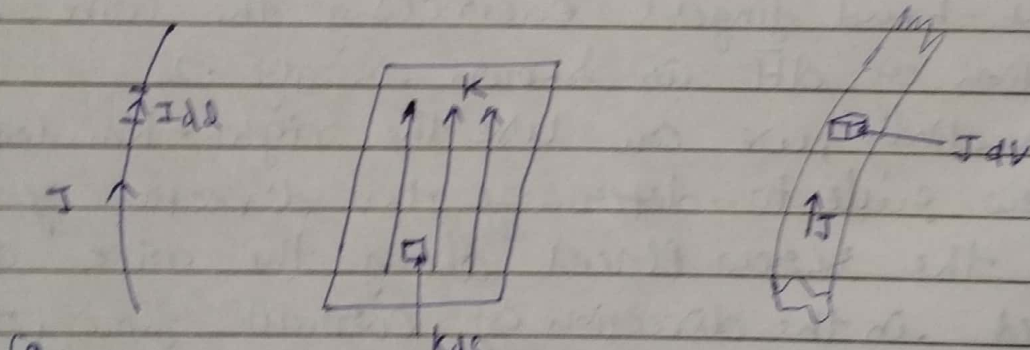
fig(2) Determination of direction using Right hand rule

fig(3) Right hand screw rule

$\bar{H}$ or I is out — $\odot$ (a)

$\bar{H}$ or I is in — $\otimes$ (b)

fig 4 - Conventional representation of $\bar{H}$ or (I)



(a) (b) (c)

fig 5 - Current distributions (a) line current (b) surface current (c) volume current

→ Thus in terms of distributed current sources, the Biot - savart law =

$$\bar{H} = \int_L \frac{I\, d\bar{l} \times \bar{a}_R}{4\pi R^2} \quad \text{— (V) (line current)}$$

$$\bar{H} = \int_S \frac{\bar{K}\, dS \times \bar{a}_R}{4\pi R^2} \quad \text{— (VI) (Surface current)}$$

$$\bar{H} = \int_v \frac{\bar{J}\, dv \times \bar{a}_R}{4\pi R^2} \quad \text{— (VII) (Volume current)}$$

where $\bar{a}_R$ is a unit vector pointing from the differential element of current to the point of interest.

⇒ Field due to a straight current carrying filamentary conductor of finite length ⇒

Consider a straight current carrying filamentary conductor of finite length AB.

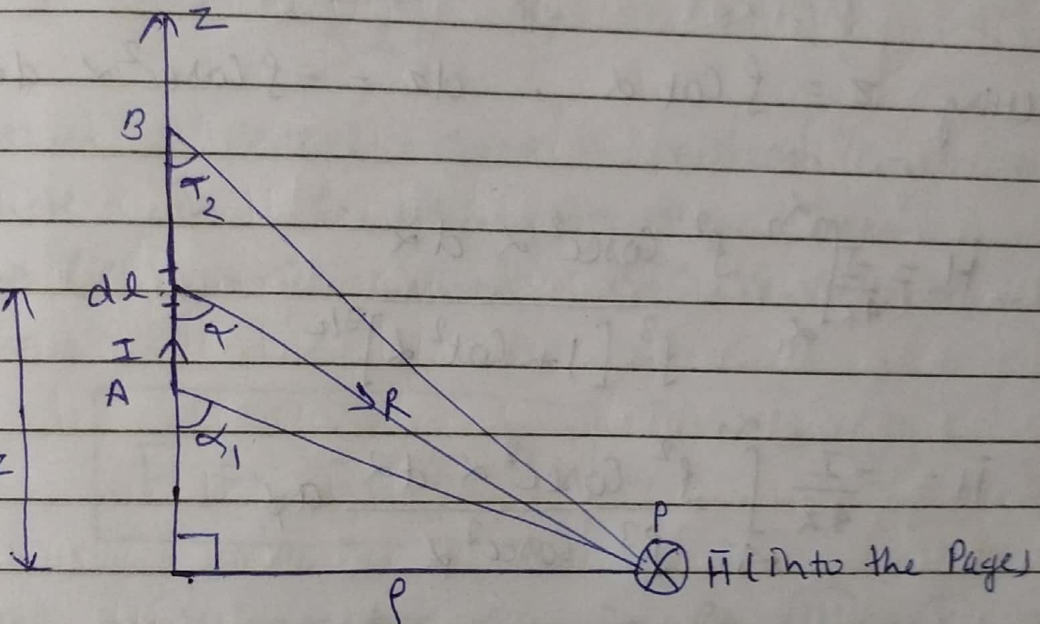


fig - Field at point P due to a straight filamentary conductor.

We assume that the conductor is along the z-axis with it's upper and lower ends, respectively, subtending angles α_2 and α_1 at P, the point at which $\bar{H}$ is to be determined.
note that current flows from point A, where $\alpha = \alpha_1$, to point B, where $\alpha = \alpha_2$. if

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We Consider the Contribution $d\bar{H}$ at P due to an element dl at $(0,0,z)$,

$$d\bar{H} = \frac{I\, d\bar{l} \times \bar{R}}{4\pi R^3} \quad \text{— (I)}$$

But $d\bar{l} = dz\, \bar{a}_z$ and $\bar{R} = \rho\, \bar{a}_\rho - z\, \bar{a}_z$, so

$$d\bar{l} \times \bar{R} = \rho\, dz\, \bar{a}_\phi$$

hence, $$\bar{H} = \int \frac{I \rho\, dz}{4\pi [\rho^2 + z^2]^{3/2}}\, \bar{a}_\phi \quad \text{— (II)}$$

letting $z = \rho \cot\alpha$, $dz = -\rho\, \mathrm{cosec}^2\alpha\, d\alpha$

$$\bar{H} = \frac{-I}{4\pi} \int_{\alpha_1}^{\alpha_2} \frac{\rho^2\, \mathrm{cosec}^2\alpha\, d\alpha}{\rho^3 [1 + \cot^2\alpha]^{3/2}}$$

$$\bar{H} = \frac{-I}{4\pi} \int_{\alpha_1}^{\alpha_2} \frac{\rho^2\, \mathrm{cosec}^2\alpha\, d\alpha}{\rho^3\, \mathrm{cosec}^3\alpha}\, a_\phi$$

$$\bar{H} = \frac{-I}{4\pi\rho}\, \bar{a}_\phi \int_{\alpha_1}^{\alpha_2} \sin\alpha\, d\alpha$$

or

$$\boxed{\bar{H} = \frac{I}{4\pi\rho} (\cos\alpha_2 - \cos\alpha_1)\, \bar{a}_\phi} \quad \text{— (III)}$$

This expression is generally applicable for any straight filamentary Conductor of finite length.

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The conductor need not lie on the z axis, but it must be straight. From eq. (III) it is clear that $\bar{H}$ is always along the unit vector $\bar{a}_\phi$ (i.e along concentric circular path) irrespective of the length of the wire or the point of interest P.

→ As a special case, when the conductor is semi-infinite (with respect to P) so that point A is now at O (0,0,0) while B is at (0,0,∞), $\alpha_1 = 90°$, $\alpha_2 = 0$ eq. becomes

$$\bar{H} = \frac{I}{4\pi\rho}\,\bar{a}_\phi \qquad \text{—(IV)}$$

→ Another special case — when conductor is infinite in lenght for this case —
A → (0,0,−∞) and B(0,0,∞), $\alpha_1 = 180°$, $\alpha_2 = 0°$

$$\boxed{\bar{H} = \frac{I}{2\pi\rho}\,\bar{a}_\phi} \qquad \text{—(V)}$$

To find unit vector a_ϕ in eq. (III) - (V) ~~is not always easy~~

$$\boxed{\bar{a}_\phi = \bar{a}_l \times \bar{a}_\rho} \qquad \text{—(VI)}$$

when $\bar{a}_l$ is the unit vector along the line current and $\bar{a}_\rho$ is a unit vector along the perpendicular line from the line current to the field point.

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Ampere's circuit law – Maxwell's equation ⇒

"Ampere's circuit law states that the line integral of $\bar{H}$ around a closed path is the same as the net current I_{enc} enclosed by the path".

In other words, the circulation of $\bar{H}$ equals I_{enc}; that is,

$$\boxed{\oint \bar{H} \cdot d\bar{l} = I_{enc}} \quad \text{—(I)}$$

Ampere's law is similar to Gauss's law. Since Ampere's law is easily applied to determine $\bar{H}$ when the current distribution is symmetrical.

⇒ Ampere's law is a special case of Biot-Savart's law; the former may be derived from the latter.
By applying Stoke's theorem to the left-hand side of eq. (I) we obtain

$$I_{enc} = \oint_L \bar{H} \cdot d\bar{l} = \int_S (\nabla \times \bar{H}) \cdot d\bar{S} \quad \text{—(II)}$$

But $\quad I_{enc} = \int_S \bar{J} \cdot d\bar{S} \quad$ —(III)

Comparing eq. (II) and (III)

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$$\boxed{\nabla \times \bar{H} = \bar{J}} \quad \text{--- (IV)}$$

This is the third maxwell's equation to be derived; it is essentially ampere's law in differential form, where as eq (I) is the integral form.

→ from eq (IV) it is observed that $\nabla \times \bar{H} = \bar{J} \neq 0$ that is a magnetic field is not conservative

Applications of Ampere's law ⇒

Now apply Ampere's circuital law to determine $\bar{H}$ for some symmetrical current distribution.
We will consider infinite line current, an infinite current sheet eg, and an infinitely long co-axial transmission line.
In each case, we apply $\oint_L \bar{H} \cdot d\bar{l} = I_{enc}$.
for symmetrical current distribution, $\bar{H}$ is either parallel or perpendicular to $d\bar{l}$. when $\bar{H}$ is parallel to $d\bar{l}$, $|\bar{H}|$ = constant.

(1) infinite line current ⇒

Consider an infinitely long filamentary current I along the z-axis as shown in fig. To determine $\bar{H}$ at an observation point P, we allow a closed path to pass

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through P. This Path, on which Ampere's law is to be applied, is known as an "Amperian Path" (Analogous to the term Gaussian surface). We ~~Choose~~ Choose a concentric circle as the Amperian Path in view of eq. which shows that -

$$\bar{H} = \frac{1}{2\pi\rho}\,\bar{a}_\phi ,$$

$\bar{H}$ is constant provided ρ is constant. since this Path encloses the whole current I, according to Ampere's law -

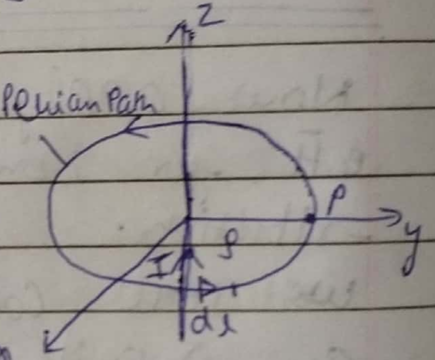
$$I = \int H_\phi\,\bar{a}_\phi \cdot \rho\, d\phi\, \bar{a}_\phi = H_\phi \int \rho\, d\phi$$

$$I = H_\phi \cdot 2\pi\rho$$

or

$$\boxed{H = \frac{I}{2\pi\rho}\, a_\phi}$$

as expected.



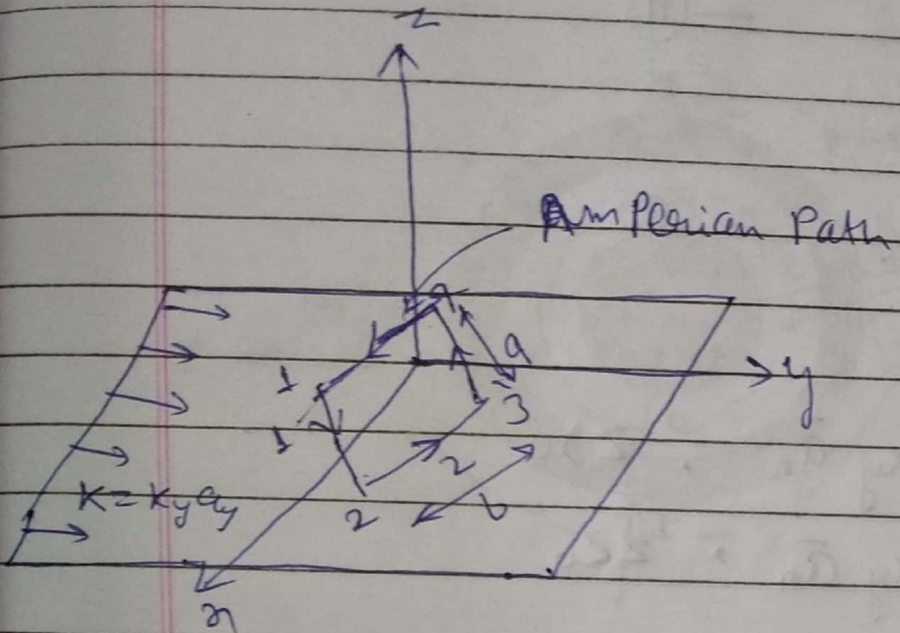
Ampere's law applied to filamentary line current.

② Infinite Sheet of Current =)

Consider an infinite sheet of current in the z=0 Plane. The surface current density is $\bar{K}$. The current is flowing in Positive y direction hence $\bar{K} = K_y\,\bar{a}_y$

Consider a closed Path 1-2-3-4-1 as shown in fig.

The current flowing across the distance b is given by $K_y\, b$.



(a)

$$I_{enc} = K_y b.$$

Applying Ampere's law to rectangular closed path

$$\oint \bar{H} \cdot d\bar{l} = I_{enc} = K_y b \quad \text{— (1)}$$

if the sheet is subdevided into a no. of filaments, it is evident that no filament can produce an H_y component. Moreover, the Biot - Savart law shows that the contributions to H_z produced by a symmetrically located pair of filaments cancel. Thus H_z is zero also. Only an H_x component is present

$$\bar{H} = H_x \bar{a}_x \quad \text{--- for } z > 0$$

$$\bar{H} = -H_x \bar{a}_x \quad \text{for } z < 0$$

$$\oint \bar{H} \cdot d\bar{l} = \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1 \right) \bar{H} \cdot d\bar{l}$$

$$= 0(-a) + (-H_x)(-b) + 0(a) + H_x b$$

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$$\oint \bar{H}\cdot d\bar{l} = 2H_x b \quad \text{—(11)}$$

but $\oint H\cdot dl = k_y b$

So $H_x = \frac{1}{2} k_y$

or

$$\bar{H} = \begin{cases} \frac{1}{2} k_y \bar{a}_x , & z>0 \\ -\frac{1}{2} k_y \bar{a}_x , & z<0 \end{cases}$$

In general, for an infinite sheet of current density K A/m,

$$\boxed{\bar{H} = \frac{1}{2} \bar{K} \times \bar{a}_n}$$

where $\bar{a}_n$ is a unit normal vector directed from the current sheet to the point of interest.

(3) Infinitely long coaxial transmission line ⇒

Consider an infinitely long transmission line consisting of two concentric cylinders having their axes along z-axis. The cross section of the line is shown in fig. where the z-axis is out of page.
The inner conductor has radius a and carries current I, while the outer conductor has

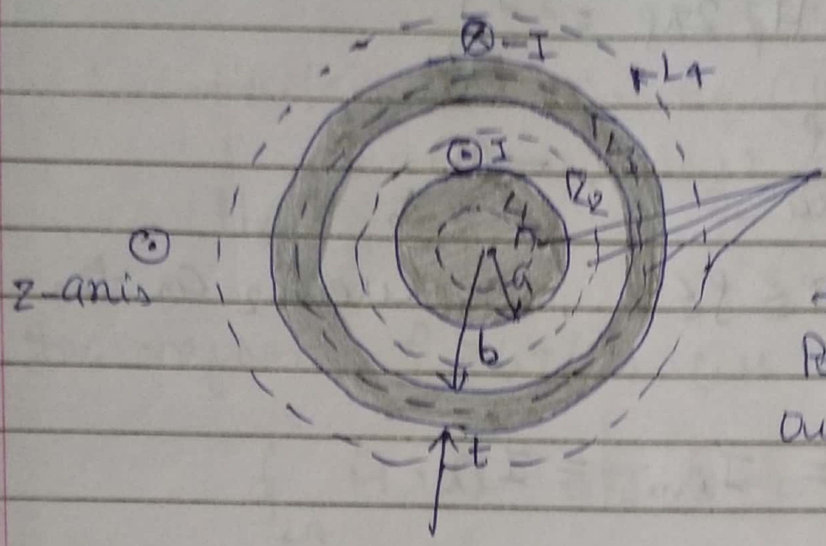

fig- Cross section of the transmission line; the Positive z-direction is out of the page.

inner radius b and thickness t and carries return current $-I$. We want to determine $\bar{H}$ everywhere, assuming that current is uniformly distributed in both conductors.
Since the current distribution is symmetrical we apply ampere's law along the Amperian Path for each of the four Possible regions:

$0 \le \rho \le a$, $a \le \rho \le b$; $b \le \rho \le b+t$ and $\rho \ge b+t$

for region $0 \le \rho \le a$, we apply ampere's law to Path L_1, giving

$$\oint_{L_1} \bar{H} \cdot d\bar{l} = I_{enc} = \int \bar{J} \cdot d\bar{s} \quad -(1)$$

Since the current is uniformly distributed over the cross section,

$$\bar{J} = \frac{I}{\pi a^2} \bar{a}_z , \quad d\bar{s} = \rho \, d\phi \, d\rho \, \bar{a}_z$$

$$I_{enc} = \int \bar{J} \cdot d\bar{s} = \frac{I}{\pi a^2} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{a} \rho \, d\phi \, d\rho = \frac{I}{\pi a^2} \pi \rho^2$$
$$= I\rho^2 / a^2$$

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Hence eq. (1) becomes

$$H_\phi \int_{L_1} dl = H_\phi 2\pi\rho = \frac{I\rho^2}{a^2}$$

or $$H_\phi = \frac{I\rho}{2\pi a^2} \quad \text{--- (II)}$$

for region $a \le \rho \le b$, we use L_2 as the Amperian Path

$$\oint_{L_2} \bar{H}\cdot d\bar{l} = I_{enc} = I$$

$$H_\phi 2\pi\rho = I$$

or
$$H_\phi = \frac{I}{2\pi\rho} \quad \text{--- (III)}$$

Since the whole current I is enclosed by L_2. eq. (III) is independent of a.

for region $b \le \rho \le b+t$ we use L_3, getting

$$\oint_{L_3} \bar{H}\cdot d\bar{l} = H_\phi \cdot 2\pi\rho = I_{enc} \quad \text{--- (IV)}$$

where $I_{enc} = I + \int \bar{J}\cdot d\bar{s}$

and J in this case is the current density of the outer conductor and is along $-\bar{a}_z$, that is,

$$\bar{J} = -\frac{I}{\pi[(b+t)^2 - b^2]}\bar{a}_z$$

thus

$$I_{enc} = I - \frac{I}{\pi[(b+t)^2-b^2]}\int_{\phi=0}^{2\pi}\int_{\rho=b}^{\rho} \rho \, d\rho \, d\phi$$

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$$I_{enc} = I\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right]$$

Substitute this in eq. (IV)

$$H_\phi = \frac{I}{2\pi\rho}\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right] \quad \text{— (V)}$$

for region $\rho \geq b+t$, we use path L_4, getting

$$\oint_{L4} \bar{H} \cdot d\bar{l} = I - I = 0$$

$$\text{or} \quad H_\phi = 0 \quad \text{— (VI)}$$

Putting eq. (II), (III), (V) & (VI) together gives

$$\bar{H} = \begin{cases} \dfrac{I\rho}{2\pi a^2}\,\bar{a}_\phi, & 0 < \rho \leq a \\ \dfrac{I}{2\pi\rho}\,\bar{a}_\phi, & a \leq \rho \leq b. \\ \dfrac{I}{2\pi\rho}\left[1 - \dfrac{\rho^2 - b^2}{t^2 + 2bt}\right]\bar{a}_\phi, & b \leq \rho \leq b+t \\ 0 & ; \rho \geq b+t \end{cases}$$

the magnitude of $\bar{H}$ is sketched in fig-

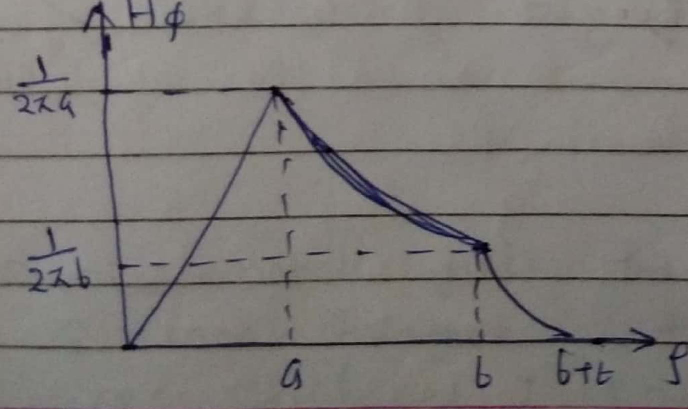

fig- Plot of H_ϕ against ρ.

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Magnetic flux density — Maxwell's equation ⇒

The magnetic flux density $\bar{B}$ is similar to the electric flux density $\bar{D}$. As $\bar{D} = \epsilon_0 \bar{E}$ in free space. The magnetic flux density $\bar{B}$ is related to magnetic field intensity $\bar{H}$ according to

$$\boxed{\bar{B} = \mu_0 \bar{H}} \quad \text{— (I)}$$

Where μ_0 is a constant known as permeability of free space. The constant is in H/m. and has the value of

$$\boxed{\mu_0 = 4\pi \times 10^{-7} \text{ H/m}} \quad \text{— (II)}$$

The magnetic flux through the surface S is given by

$$\boxed{\Psi = \int_S \bar{B} \cdot d\bar{s}} \quad \text{— (III)}$$

where magnetic flux Ψ is in webers (wb) and the magnetic flux density in in wb/m² or teslas (T).

A magnetic flux lines is a Path to which $\bar{B}$ is tangential at every point on the line. it is a line along which the needle of a magnetic compass will orient itself if placed in the presence of magnetic field.

for ex- the magnetic flux lines due to a straight long wire are shown in fig. The flux lines are determined by using the same principle followed for the electric flux lines. The direction $\bar{B}$ is taken as that indicated as north by the needle of the magnetic compass. notic that each flux line is closed and has no beginning or end.

This fig (a) is for a straight, current carrying conductor, it is generally true that magnetic flux lines are closed and do not cross each other regardless of the current distribution.

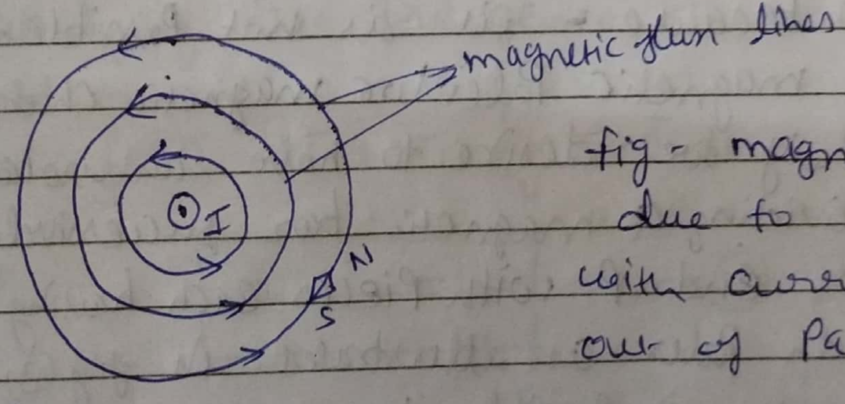

fig - magnetic flux lines due to a straight wire with current comming out of Page.

fig (a).

→ In an electrostatic field, the flux passing through a closed surface is the same as the charge enclosed; i.e - $\Psi = \oint \bar{D} \cdot d\bar{S} = Q$.

Thus it is Possible to have an isolated electric charge as shown in fig 2 (a), which also reveals that the electric flux lines are not necessarily closed.
Unlike electric flux lines, magnetic flux lines always close upon themselves as in fig 2(b)

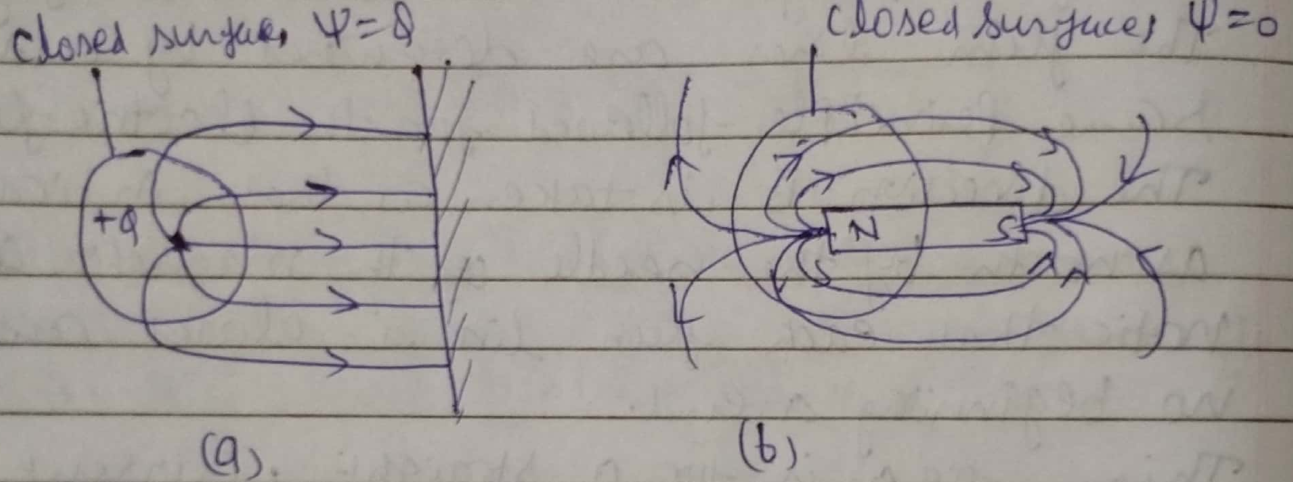

fig 2 - flux leaving a closed surface due to (a) isolated electric charge $\Psi = \oint \bar{D} \cdot d\bar{s} = Q$. (b) magnetic Charge $\Psi = \oint \bar{B} \cdot d\bar{s} = 0$

This is because - it is not Possible to have isolated magnetic Poles (or magnetic charges). for ex- if we desire to have an isolated magnetic Pole by dividing a magnetic bar successively into two, we end up with Pieces each having both end south Poles as illustrated in fig (3)
we find it impossible to seperate the north Pole from the south Pole.

"An isolated magnetic charge does not exist".

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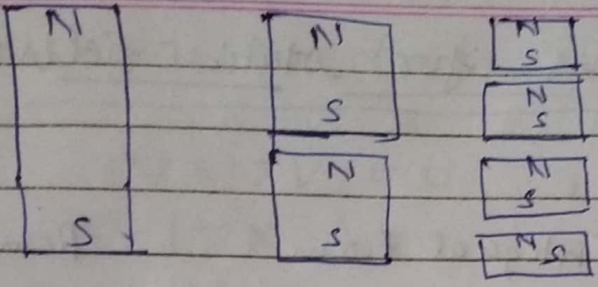

fig.3. Successive division of a bar magnet results in a pieces with north and south pole, showing ~~that magnetic~~ poles can't be isolated.

Thus the total flux through a closed surface in magnetic field must be zero, that is

$$\boxed{\oint \bar{B} \cdot d\bar{S} = 0} \quad \text{— (I)}$$

This equation is reffered to as the law of conservation of magnetic flux or Gauss's law for magnetostatic field. Just as $\oint \bar{D} \cdot d\bar{S} = Q$ is Gauss's law for electro-static fields. Although the magnetostatic field is not conservative, magnetic flux is conserved.

By applying the divergence theorem to eq. (I) we obtain

$$\oint \bar{B} \cdot d\bar{S} = \int_v \nabla \cdot \bar{B} \, dv = 0$$

$$\boxed{\bar{\nabla} \cdot \bar{B} = 0} \quad \text{— (II)}$$

This equation is the fourth maxwell's equation to be derived.

→ eq. (I) & (II) shows that magnetostatic field have no source or sinks.

→ eq. (II) suggested that magnetic field lines are always continuous.

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Maxwell's equations for static fields ⇒

Differential (or Point) form	Integral form	Remarks
$\nabla \cdot \bar{D} = \rho_v$	$\oint_S \bar{D} \cdot d\bar{S} = \int_V \rho_v \, dv$	Gauss's law
$\nabla \cdot \bar{B} = 0$	$\oint \bar{B} \cdot d\bar{S} = 0$	Nonexistance of magnetic monopole
$\nabla \times \bar{E} = 0$	$\oint_L \bar{E} \cdot d\bar{l} = 0$	Conservative nature of electrostatic field
$\nabla \times \bar{H} = \bar{J}$	$\oint_L \bar{H} \cdot d\bar{l} = \int_S \bar{J} \cdot d\bar{S}$	Ampere's law.

Magnetic scalar and vector Potentials ⇒

In electrostatics, it is seen that there exists a scalar electric Potential V which is related to the electric field intensity $\bar{E}$ as $\bar{E} = -\nabla V$.

In case of magnetic fields there are two types of Potentials which can be defined:

1- The scalar magnetic Potential denoted as V_m
2. The vector magnetic Potential denoted as $\bar{A}$.

To define scalar and vector magnetic Potentials, let us use two vector identities which are listed as the properties of curl, earlier

$$\nabla \times \nabla V = 0 \quad , \quad V = \text{scalar} \quad \text{— (I)}$$
$$\nabla \cdot (\nabla \times \bar{A}) = 0 \quad , \quad \bar{A} = \text{vector} \quad \text{— (II)}$$

(A) Scalar magnetic Potential ⇒ If V_m is the scalar magnetic potential then it must satisfy the equation (I),

$$\therefore \quad \nabla \times \nabla V_m = 0 \quad \text{— (III)}$$

But the scalar magnetic potential is related to the magnetic field intensity $\bar{H}$ as,

$$\bar{H} = -\nabla V_m \quad \text{— (IV)}$$

Using in eq. (III),

$$\nabla \times (-\bar{H}) = 0 \quad \text{i.e} \quad \nabla \times \bar{H} = 0 \quad \text{— (V)}$$

But $\nabla \times \bar{H} = \bar{J}$ i.e $\bar{J} = 0$ — (VI)

Thus scalar magnetic potential V_m can be defined for source free region where $\bar{J}$ i.e current density is zero.

$\bar{H} = -\nabla V_m$ only for $\bar{J} = 0$ — (VII)

Similar to the relation b/w $\bar{E}$ and electric scalar potential, magnetic scalar potential can be expressed interms of $\bar{H}$ as,

$$V_m = -\int_b^a \bar{H} \cdot dl \quad \text{-- -- specified path}$$

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(B) Laplace's equation for scalar magnetic Potential =>

it is known that as monoPole of magnetic field is non existing,

$$\oint \bar{B} \cdot d\bar{s} = 0 \qquad \text{--(VIII)}$$

using divergente theorem

$$\oint \bar{B} \cdot d\bar{s} = \int_{vol} (\nabla \cdot \bar{B})\, dv = 0 \qquad \text{--(IX)}$$

$$\therefore \quad \nabla \cdot \bar{B} = 0 \qquad \text{--(X)}$$

$$\nabla \cdot (\mu_0 \bar{H}) = 0 \qquad \text{but } \mu_0 \neq 0$$

$$\nabla \cdot \bar{H} = 0$$

$$\nabla \cdot (-\nabla V_m) = 0 \qquad \text{--- using } \bar{H} = -\nabla V_m$$

$$\boxed{\nabla^2 V_m = 0 \qquad \text{for } \bar{J} = 0} \qquad \text{--(XI)}$$

This is laplace's equation for scalar magnetic Potential. This is similar to the laplace's equation for scalar electric Potential $\nabla^2 V = 0$

(C) Vector magnetic Potential =>

The vector magnetic Potential is denoted as $\bar{A}$ and measured in Wb/m it has to satisfy eq. (II)

$$\nabla \cdot (\nabla \times \bar{A}) = 0$$

But $\quad \nabla \cdot \bar{B} = 0 \qquad$ --- from eq. (X)

$$\therefore \quad \boxed{\bar{B} = \nabla \times \bar{A}} \qquad \text{--- (XII)}$$

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Thus Curl of vector magnetic Potential is the flux density.

Now $\nabla \times \bar{H} = \bar{J}$

$\therefore \nabla \times \frac{\bar{B}}{\mu_0} = \bar{J}$

$\nabla \times \bar{B} = \mu_0 \bar{J}$

$\nabla \times \nabla \times \bar{A} = \mu_0 \bar{J}$ — (XIII)

using vector identity to express left hand side we can write,

$\nabla(\nabla \cdot \bar{A}) - \nabla^2 A = \mu_0 \bar{J}$

$$\bar{J} = \frac{1}{\mu_0}\left[\nabla \times \nabla \times \bar{A}\right] = \frac{1}{\mu_0}\left[\nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A}\right] \quad \text{— (XIV)}$$

Thus if vector magnetic Potential is known then the current density $\bar{J}$ can be obtained. For defining $\bar{A}$ the current density need not be zero.

(D) Poisson's equation for magnetic field ⇒

in a vector algebra, a vector can be fully defined if it's curl and divergence are defined.

For a vector magnetic Potential $\bar{A}$; it's curl is defined as $\nabla \times \bar{A} = \bar{B}$ which is known. But to completely define $\bar{A}$ is divergence must be known. Assume that $\nabla \cdot \bar{A} = 0$. This consistent with some other conditions to be studied later in time varying magnetic fields using in eq. (XIV)

$$\bar{J} = \frac{1}{\mu_0}\left[-\nabla^2 \bar{A}\right]$$

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$$\boxed{\nabla^2 \bar{A} = -\mu_0 J}$$

This is the Poisson's equation for magnetostatic fields.

(E) $\bar{A}$ due to differential current element ⇒

Consider the differential element $d\bar{l}$ carrying current I. Then according to Biot-savart law the vector magnetic potential $\bar{A}$ at a distance R from the differential current element is given by.

$$\bar{A} = \oint \frac{\mu_0 I \, dl}{4\pi R} \quad Wb/m$$

For distributed current sources, $I d\bar{l}$ can be replaced by $K ds$ where $\bar{K}$ is surface current density.

$$\boxed{\bar{A} = \oint_s \frac{\mu_0 \bar{K} \, ds}{4\pi R} \quad Wb/m}$$

The line integral becomes a surface integral if the volume current density J is given in A/m^2 then $I d\bar{l}$ can be replaced by $\bar{J} dv$ where dv is differential volume element.

$$\boxed{\bar{A} = \oint_{vol} \frac{\mu_0 \bar{J} \, dV}{4\pi R} \quad Wb/m}$$

It can be noted that

1. The zero reference for $\bar{A}$ is at infinity
2. No finite current can produce the contribution as $R \rightarrow \infty$.

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Table 6.1 Analogy between Electric and Magnetic Fields*

Term	Electric	Magnetic
Basic laws	$\mathbf{F} = \frac{Q_1 Q_2}{4\pi\varepsilon_r^2}\mathbf{a}_r$	$d\mathbf{B} = \frac{\mu_o I\, d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2}$
	$\oint \mathbf{D} \cdot d\mathbf{S} = Q_{enc}$	$\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc}$
Force law	$\mathbf{F} = Q\mathbf{E}$	$\mathbf{F} = Q\mathbf{u} \times \mathbf{B}$
Source element	dQ	$Q\mathbf{u} = I\, d\mathbf{l}$
Field intensity	$E = \frac{V}{\ell}$ (V/m)	$H = \frac{I}{\ell}$ (A/m)
Flux density	$\mathbf{D} = \frac{\Psi}{S}$ (C/m^2)	$\mathbf{B} = \frac{\Psi}{S}$ (Wb/m^2)
Relationship between fields	$\mathbf{D} = \varepsilon\mathbf{E}$	$\mathbf{B} = \mu\mathbf{H}$
Potentials	$\mathbf{E} = -\nabla V$	$\mathbf{H} = -\nabla V_m$ ($\mathbf{J} = 0$)
	$V = \int \frac{\rho_L dl}{4\pi\varepsilon r}$	$\mathbf{A} = \int \frac{\mu I\, d\mathbf{l}}{4\pi R}$
Flux	$\Psi = \int \mathbf{D} \cdot d\mathbf{S}$	$\Psi = \int \mathbf{B} \cdot d\mathbf{S}$
	$\Psi = Q = CV$	$\Psi = LI$
	$I = C\frac{dV}{dt}$	$V = L\frac{dI}{dt}$
Energy density	$w_E = \frac{1}{2}\mathbf{D} \cdot \mathbf{E}$	$w_m = \frac{1}{2}\mathbf{B} \cdot \mathbf{H}$
Poisson's equation	$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$	$\nabla^2 \mathbf{A} = -\mu\mathbf{J}$

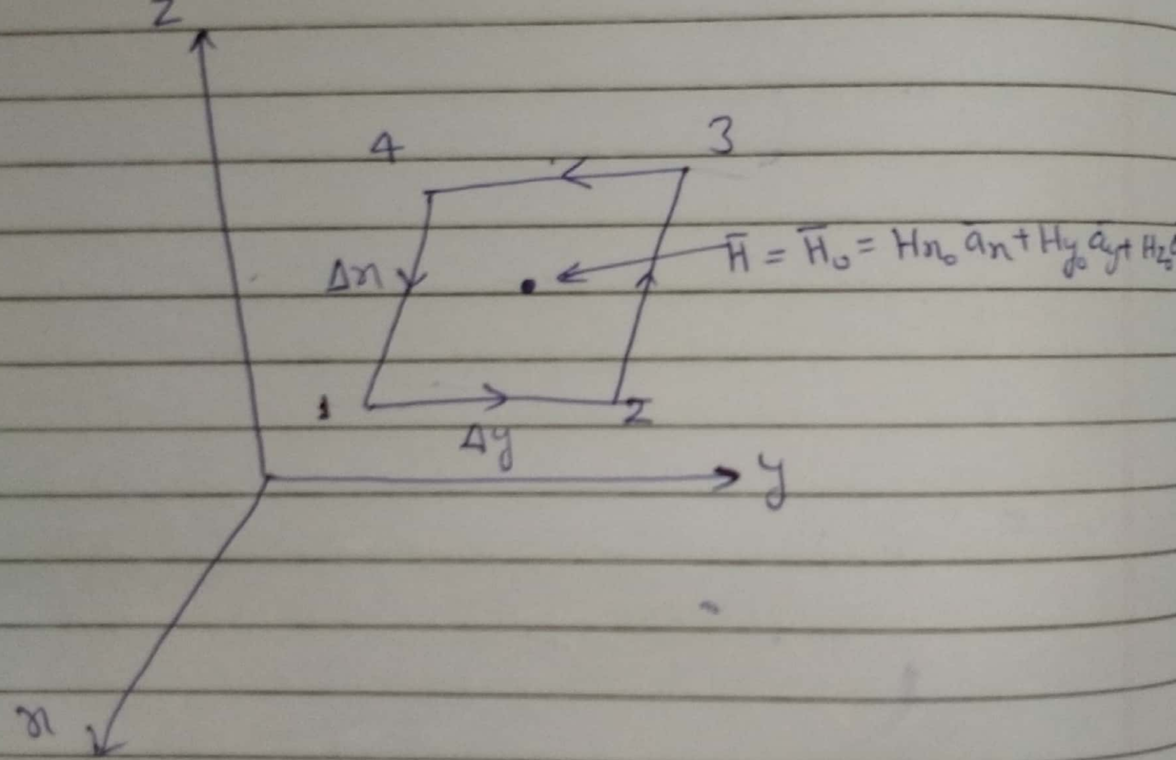
*A similar analogy can be found in R. S. Elliot, "Electromagnetic theory: a simplified representation,"

Curl ⟹

Let us consider an incremental closed path of sides Δx and Δy. and choose cartesian coordinate system. Assume that some current produces a reference value of $\bar{H}$ at the center of this small rectangle,

$$H_0 = H_{x_0}\bar{a}_x + H_{y_0}\bar{a}_y + H_{z_0}\bar{a}_z$$

The closed line integral of $\bar{H}$ about this path is then approximately the sum of the four values of $\bar{H}\cdot\Delta\bar{L}$ on each side. We choose the



direction of traverse as 1-2-3-4-1, which is corresponds to a current in the $\bar{a}_z$ direction, and the first contribution is therefore

$$(\bar{H}\cdot\Delta\bar{L})_{1-2} = H_{y,1-2}\,\Delta y$$

$$H_{y\,1-2} = H_{y_0} + \frac{\partial H_y}{\partial x}\left(\frac{1}{2}\Delta x\right)$$

Thus $(\bar{H}\cdot\Delta\bar{L})_{1-2} = \left(H_{y_0} + \frac{1}{2}\frac{\partial H_y}{\partial x}\Delta x\right)\Delta y$

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Along the next section of the path we have

$$(\bar{H}\cdot\Delta\bar{L})_{2-3} = H_{x,2-3}(-\Delta x) = -\left(H_{x_0} + \frac{1}{2}\frac{\partial H_x}{\partial y}\Delta y\right)\Delta x$$

$$(\bar{H}\cdot\Delta\bar{L})_{2-3} = -\left(H_{x_0} + \frac{1}{2}\frac{\partial H_x}{\partial y}\Delta y\right)\Delta x$$

Continuing for the remaining two segments and adding the results,

$$\oint \bar{H}\cdot d\bar{L} = \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\Delta x\,\Delta y$$

By Ampere's circuital law, this result must be equal to the current enclosed by the path. If we assume a general current density J, the enclosed current is then $\Delta I = J_z\,\Delta x\,\Delta y$, and

$$\oint \bar{H}\cdot d\bar{L} = \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\Delta x\Delta y = J_z\,\Delta x\,\Delta y$$

or $$\frac{\oint \bar{H}\cdot d\bar{L}}{\Delta x\,\Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z$$

As we cause the closed path to shrink, the above expression becomes more nearly exact, and in the limit we have the equality.

$$\lim_{\Delta x,\Delta y\to 0}\frac{\oint \bar{H}\cdot d\bar{L}}{\Delta x\,\Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z \quad -①$$

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If we choose closed path which are oriented perpendicularly to each of the remaining two coordinates axes, analogous processes lead to expression for y and z components of the current density.

$$\lim_{\Delta y, \Delta z \to 0} \frac{\oint \bar{H} \cdot d\bar{L}}{\Delta y \Delta z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x \quad \text{—(II)}$$

and

$$\lim_{\Delta z, \Delta x \to 0} \frac{\oint \bar{H} \cdot d\bar{L}}{\Delta z \Delta x} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y \quad \text{—(III)}$$

Comparing eq. (I), (II) & (III) we see that a component of current density is given by the limit of the quotient of the closed line integral of $\bar{H}$ about a small path in a plane normal to that component and of the area enclosed as the path shrinks to zero.
This defined as curl.
The mathematical form of definition is

$$\boxed{(\text{curl } H)_N = \lim_{\Delta S_N \to 0} \frac{\oint \bar{H} \cdot d\bar{L}}{\Delta S_N}} \quad \text{—(IV)}$$

Where S_N is the planner area enclosed by the closed line integral. The N subscript indicates that the component of the curl is that component which is normal to the surface enclosed by the closed path.
In cartesian coordinates definition (IV) shows that the x, y, and z components of the curl $\bar{H}$ are

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given by (I), (II) and (III) and therefore

$$\boxed{\text{curl } \bar{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\bar{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\bar{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\bar{a}_z}$$

— (V)

this result may be written in the form of a determinant.

$$\boxed{\text{Curl } \bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ H_x & H_y & H_z \end{vmatrix}}$$ — (VI)

and may also be written in terms of vector operator,

$$\boxed{\text{Curl } \bar{H} = \nabla \times \bar{H}} = J \quad \text{— (VII)}$$

$$\nabla \times \bar{H} = \left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\bar{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)a_\phi + \left(\frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial H_\rho}{\partial \phi}\right)\bar{a}_z$$ — Cylindrical

— (VIII)

$$\nabla \times \bar{H} = \frac{1}{r\sin\theta}\left(\frac{\partial(H_\phi \sin\theta)}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi}\right)\bar{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial H_r}{\partial \phi} - \frac{\partial(rH_\phi)}{\partial r}\right)\bar{a}_\theta + \frac{1}{r}\left(\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right)\bar{a}_\phi$$ (Spherical)

— (IX)

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Physical significance of curl ⇒

The curl is a closed line integral per unit area as the area shrinks to a point. it gives the circulation per unit area i.e. circulation density of a vector about a point at which area is going to shrink. Thus curl of a vector at a point quantifies the circulation of a vector around that point. in general if there is no rotation, there is no curl while large angular velocities means greater values of curl. The curl also gives the direction, which is along the axis through a point at which curl is defined.

The magnetic field lines produced by current carrying conductor are rotating in the form of concentric circles around the conductor. Thus their exist a curl of magnetic field intensity which we have defined as $\nabla \times \bar{H}$. The direction of curl is along the axis about which rotation of vector field exists and the proper direction is to be obtained by right handed screw rule. If the direction of rotation of vector field about a point reverses, the sign of the curl also reverses.

The water velocity in a river which increases linearly towards the surface, the magnetic field lines due to current carrying conductor, the body rotating about a fixed axis are few examples of a curl.

Stoke's theorem ⇒ Analogous to divergence theorem in electrostatics, there exists Stoke's theorem in magnetostatics. The Stoke's theorem relates the line integral to a surface integral. Basically it is a mathematical theorem which is to be applied in magnetostatics. Stoke's theorem states that -
"The line integral of a vector $\bar{A}$ around a closed path L is equal to the integral of curl of $\bar{A}$ over the open surface S enclosed by the closed path L".
• The theorem is applicable only when $\bar{A}$ and $\nabla \times \bar{A}$ are continuous on surface S.

Proof of Stoke's theorem ⇒ Consider a surface S which is splitted into no. of incremental surfaces. Each incremental surface is having area ΔS as shown in fig (a)

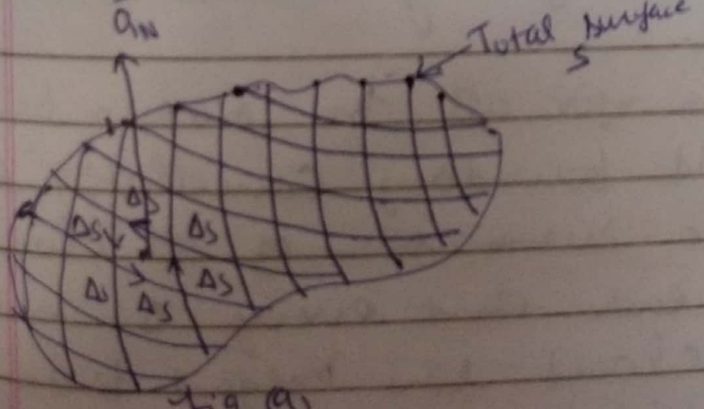

fig (a)


fig (b)

Applying definition of the curl to any of these incremental surfaces we can write,

$$(\nabla \times \bar{H})_N = \frac{\oint \bar{H} \cdot d\bar{L}_{\Delta S}}{\Delta S} \quad \text{—①}$$

where N = Normal to ΔS according to right hand rule
$dL_{\Delta S}$ = Perimeter of the incremental surface ΔS

Now the curl of $\bar{H}$ in the normal direction is the dot product of curl of $\bar{H}$ with $\bar{a}_N$, where $\bar{a}_N$ is the unit vector, normal to surface ΔS

$$(\nabla \times \bar{H})_N = (\nabla \times \bar{H}) \cdot \bar{a}_N \qquad \text{using eq. (1)}$$

$$\oint \bar{H} \cdot d\bar{L}_{\Delta S} = (\nabla \times \bar{H}) \cdot \bar{a}_N \Delta S$$

$$\therefore \oint \bar{H} \cdot d\bar{L}_{\Delta S} = (\nabla \times \bar{H}) \cdot \Delta \bar{S} \qquad \text{— (II)}$$

To obtain the total curl for every incremental surface, add the closed line integrals for each ΔS. From fig. (6) it can be seen that at a common boundary b/w the two incremental surfaces the line integral is getting cancelled as the boundary is getting traced in two opposite directions.

∴ This happens for all the interior boundaries only at the outside boundary cancellation does not exist. Hence summation of all closed line integrals for each and every ΔS ends up in a single closed line integral to be obtained for the outer boundary of the total surface S.

Hence the eq. (III) becomes:

$$\boxed{\oint_L \bar{H} \cdot d\bar{L} = \int_S (\nabla \times H) \cdot ds}$$

where dL = Perimeter of the total surface S

Thus line integral can be expressed as surface integrals.

EXAMPLE 6.1

The conducting triangular loop in Figure 6.6(a) carries a current of 10 A. Find **H** at (0, 0, 5) due to side 1 of the loop.

Solution:

This example illustrates how eq. (6.12) is applied to any straight, thin, current-carrying conductor. The key point to keep in mind in applying eq. (6.12) is figuring out α_1, α_2, ρ, and $\mathbf{a}_\phi$. To find **H** at (0, 0, 5) due to side 1 of the loop in Figure 6.6(a), consider Figure 6.6(b), where side 1 is treated as a straight conductor. Notice that we join the point of interest (0, 0, 5) to the beginning and end of the line current. Observe that α_1, α_2, and ρ are assigned in the same manner as in Figure 6.5 on which eq. (6.12) is based:

$$\cos \alpha_1 = \cos 90° = 0, \qquad \cos \alpha_2 = \frac{2}{\sqrt{29}}, \qquad \rho = 5$$

To determine $\mathbf{a}_\phi$ is often the hardest part of applying eq. (6.12). According to eq. (6.15), $\mathbf{a}_\ell = \mathbf{a}_x$ and $\mathbf{a}_\rho = \mathbf{a}_z$, so

$$\mathbf{a}_\phi = \mathbf{a}_x \times \mathbf{a}_z = -\mathbf{a}_y$$

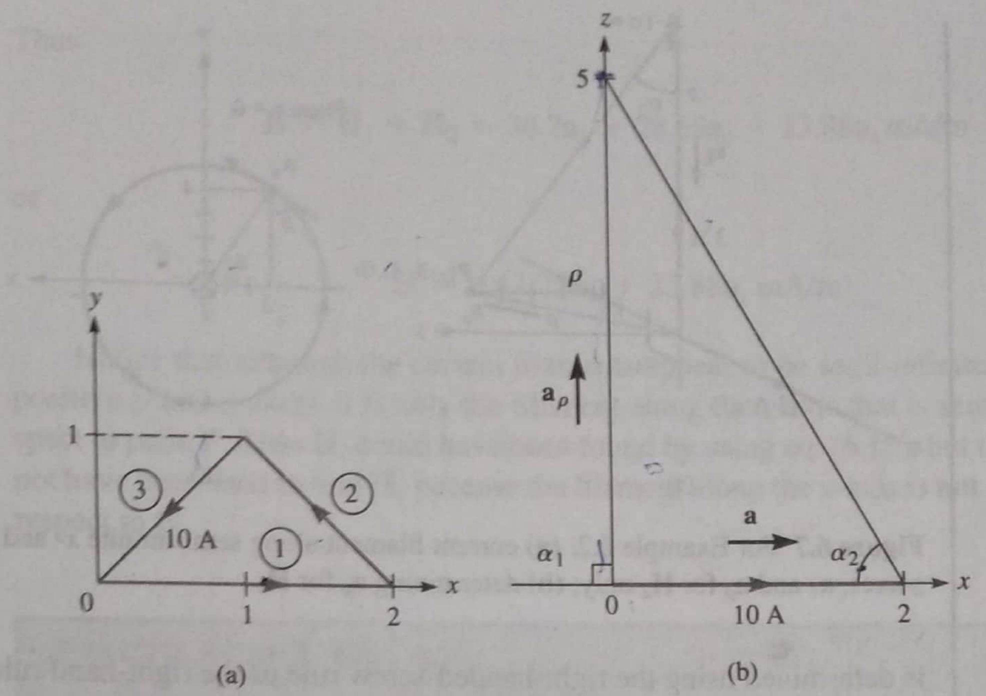

Figure 6.6 For Example 6.1: (**a**) conducting triangular loop, (**b**) side 1 of the loop.

Hence,

$$\mathbf{H}_1 = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\mathbf{a}_\phi = \frac{10}{4\pi(5)}\left(\frac{2}{\sqrt{29}} - 0\right)(-\mathbf{a}_y)$$
$$= -59.1\mathbf{a}_y \text{ mA/m}$$

PRACTICE EXERCISE 6.1

Find **H** at (0, 0, 5) due to side 3 of the triangular loop in Figure 6.6(a).

Answer: $-30.63\mathbf{a}_x + 30.63\mathbf{a}_y$ mA/m.

EXAMPLE 6.3

A circular loop located on $x^2 + y^2 = 9, z = 0$ carries a direct current of 10 A along $\mathbf{a}_\phi$. Determine $\mathbf{H}$ at (0, 0, 4) and (0, 0, −4).

Solution:

Consider the circular loop shown in Figure 6.8(a). The magnetic field intensity $d\mathbf{H}$ at point $P(0, 0, h)$ contributed by current element $I\, d\mathbf{l}$ is given by Biot–Savart's law:

$$d\mathbf{H} = \frac{I\, d\mathbf{l} \times \mathbf{R}}{4\pi R^3}$$

where $d\mathbf{l} = \rho\, d\phi\, \mathbf{a}_\phi$, $\mathbf{R} = (0, 0, h) - (x, y, 0) = -\rho \mathbf{a}_\rho + h\mathbf{a}_z$, and

$$d\mathbf{l} \times \mathbf{R} = \begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ 0 & \rho\, d\phi & 0 \\ -\rho & 0 & h \end{vmatrix} = \rho h\, d\phi\, \mathbf{a}_\rho + \rho^2\, d\phi\, \mathbf{a}_z$$

Hence,

$$d\mathbf{H} = \frac{I}{4\pi[\rho^2 + h^2]^{3/2}} (\rho h\, d\phi\, \mathbf{a}_\rho + \rho^2\, d\phi\, \mathbf{a}_z) = dH_\rho\, \mathbf{a}_\rho + dH_z\, \mathbf{a}_z$$

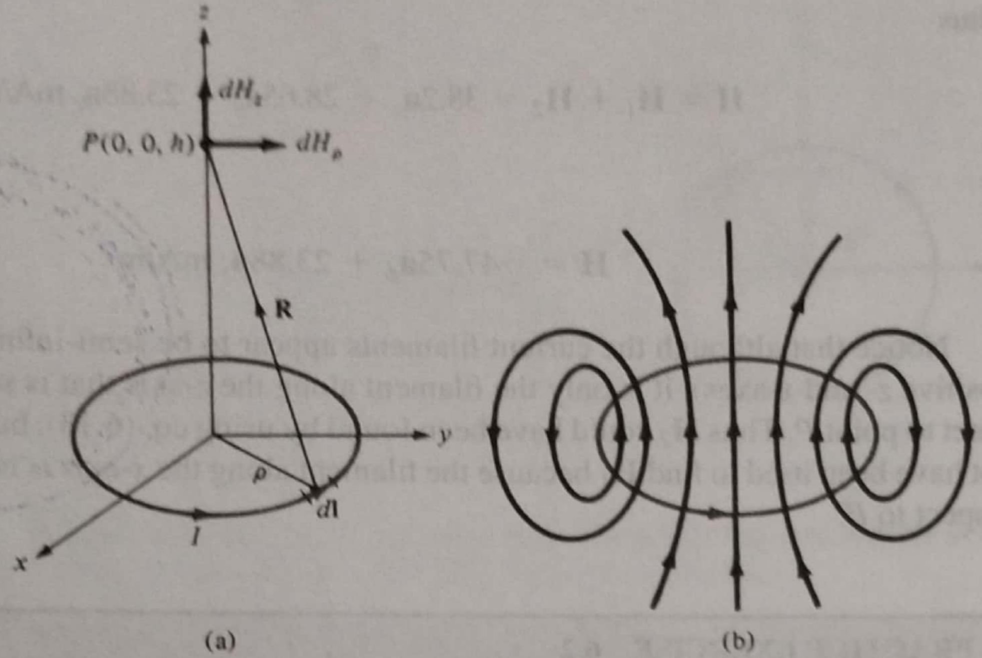

Figure 6.8 (**a**) circular current loop, (**b**) flux lines due to the current loop.

By symmetry, the contributions along $\mathbf{a}_\rho$ add up to zero because the radial components produced by current element pairs 180° apart cancel. This may also be shown mathematically by writing $\mathbf{a}_\rho$ in rectangular coordinate systems (i.e., $\mathbf{a}_\rho = \cos\phi\, \mathbf{a}_x + \sin\phi\, \mathbf{a}_y$). Integrating $\cos\phi$ or $\sin\phi$ over $0 \le \phi \le 2\pi$ gives zero, thereby showing that $\mathbf{H}_\rho = 0$. Thus

$$\mathbf{H} = \int dH_z\, \mathbf{a}_z = \int_0^{2\pi} \frac{I\rho^2\, d\phi\, \mathbf{a}_z}{4\pi[\rho^2 + h^2]^{3/2}} = \frac{I\rho^2 2\pi \mathbf{a}_z}{4\pi[\rho^2 + h^2]^{3/2}}$$

or

$$\mathbf{H} = \frac{I\rho^2 \mathbf{a}_z}{2[\rho^2 + h^2]^{3/2}}$$

(a) Substituting $I = 10\,A$, $\rho = 3$, $h = 4$ gives

$$\mathbf{H}(0, 0, 4) = \frac{10\,(3)^2 \mathbf{a}_z}{2[9 + 16]^{3/2}} = 0.36\mathbf{a}_z \text{ A/m}$$

(b) Notice from $d\mathbf{l} \times \mathbf{R}$ in the Biot–Savart law that if h is replaced by $-h$, the z-component of $d\mathbf{H}$ remains the same while the ρ-component still adds up to zero due to the axial symmetry of the loop. Hence

$$\mathbf{H}(0, 0, -4) = \mathbf{H}(0, 0, 4) = 0.36\mathbf{a}_z \text{ A/m}$$

The flux lines due to the circular current loop are sketched in Figure 6.8(b).

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EXAMPLE 6.4

A solenoid of length ℓ and radius a consists of N turns of wire carrying current I. Show that at point P along its axis,

$$\mathbf{H} = \frac{nI}{2}(\cos \theta_2 - \cos \theta_1)\mathbf{a}_z$$

where $n = N/\ell$, θ_1 and θ_2 are the angles subtended at P by the end turns as illustrated in Figure 6.9. Also show that if $\ell \gg a$, at the center of the solenoid,

$$\mathbf{H} = nI\mathbf{a}_z$$

Solution:

Consider the cross section of the solenoid as shown in Figure 6.9. Since the solenoid consists of circular loops, we apply the result of Example 6.3. The contribution to the magnetic field H at P by an element of the solenoid of length dz is

$$dH_z = \frac{I\, dl\, a^2}{2[a^2 + z^2]^{3/2}} = \frac{Ia^2 n\, dz}{2[a^2 + z^2]^{3/2}}$$

where $dl = n\, dz = (N/\ell)\, dz$. From Figure 6.9, $\tan \theta = a/z$; that is,

$$dz = -a \operatorname{cosec}^2 \theta\, d\theta = -\frac{[z^2 + a^2]^{3/2}}{a^2} \sin \theta\, d\theta$$

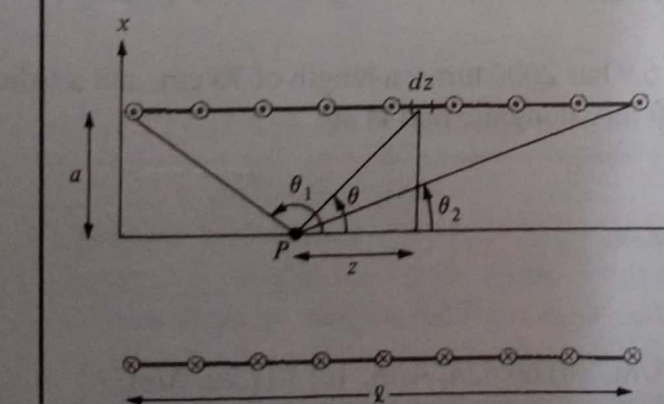

Figure 6.9 For Example 6.4; cross section of a solenoid.

Hence,

$$dH_z = -\frac{nI}{2} \sin \theta \, d\theta$$

or

$$H_z = -\frac{nI}{2} \int_{\theta_1}^{\theta_2} \sin \theta \, d\theta$$

Thus

$$\mathbf{H} = \frac{nI}{2} (\cos \theta_2 - \cos \theta_1)\mathbf{a}_z$$

as required. Substituting $n = N/\ell$ gives

$$\mathbf{H} = \frac{NI}{2\ell} (\cos \theta_2 - \cos \theta_1)\mathbf{a}_z$$

At the center of the solenoid,

$$\cos \theta_2 = \frac{\ell/2}{[a^2 + \ell^2/4]^{1/2}} = -\cos \theta_1$$

and

$$\mathbf{H} = \frac{In\ell}{2[a^2 + \ell^2/4]^{1/2}} \mathbf{a}_z$$

If $\ell \gg a$ or $\theta_2 \simeq 0°$, $\theta_1 \simeq 180°$,

$$\mathbf{H} = nI\,\mathbf{a}_z = \frac{NI}{\ell} \mathbf{a}_z$$

PRACTICE EXERCISE 6.4

The solenoid of Figure 6.9 has 2000 turns, a length of 75 cm, and a radius of 5 cm. If it carries a current of 50 mA along $\mathbf{a}_\phi$, find $\mathbf{H}$ at

(a) (0, 0, 0)

(b) (0, 0, 75 cm)

(c) (0, 0, 50 cm)

Answer: (a) $66.52\mathbf{a}_z$ A/m, (b) $66.52\mathbf{a}_z$ A/m, (c) $131.7\mathbf{a}_z$ A/m.

EXAMPLE 6.5

Planes $z = 0$ and $z = 4$ carry current $\mathbf{K} = -10\mathbf{a}_x$ A/m and $\mathbf{K} = 10\mathbf{a}_x$ A/m, respectively. Determine $\mathbf{H}$ at

(a) (1, 1, 1)

(b) (0, −3, 10)

Solution:

The parallel current sheets are shown in Figure 6.14. Let

$$\mathbf{H} = \mathbf{H}_o + \mathbf{H}_4$$

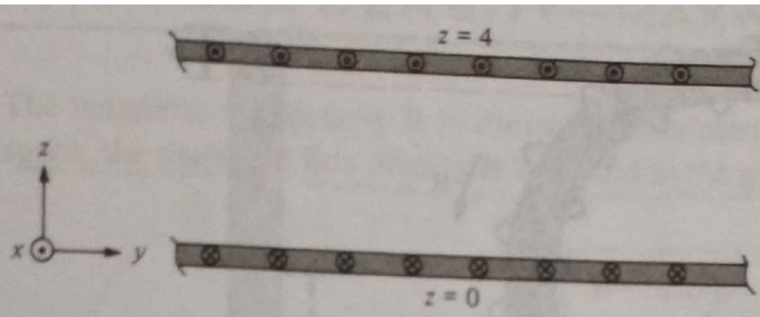

Figure 6.14 For Example 6.5: parallel infinite current sheets.

where $\mathbf{H}_o$ and $\mathbf{H}_4$ are the contributions due to the current sheets $z = 0$ and $z = 4$, respectively. We make use of eq. (6.23).

(a) At (1, 1, 1), which is between the plates ($0 < z = 1 < 4$),

$$\mathbf{H}_o = 1/2\ \mathbf{K} \times \mathbf{a}_n = 1/2\,(-10\mathbf{a}_x) \times \mathbf{a}_z = 5\mathbf{a}_y \text{ A/m}$$

$$\mathbf{H}_4 = 1/2\ \mathbf{K} \times \mathbf{a}_n = 1/2\,(10\mathbf{a}_x) \times (-\mathbf{a}_z) = 5\mathbf{a}_y \text{ A/m}$$

Hence,

$$\mathbf{H} = 10\mathbf{a}_y \text{ A/m}$$

(b) At (0, −3, 10), which is above the two sheets ($z = 10 > 4 > 0$),

$$\mathbf{H}_o = 1/2\,(-10\mathbf{a}_x) \times \mathbf{a}_z = 5\mathbf{a}_y \text{ A/m}$$

$$\mathbf{H}_4 = 1/2\,(10\mathbf{a}_x) \times \mathbf{a}_z = -5\mathbf{a}_y \text{ A/m}$$

Hence,

$$\mathbf{H} = 0 \text{ A/m}$$

PRACTICE EXERCISE 6.5

Plane $y = 1$ carries current $\mathbf{K} = 50\mathbf{a}_z$ mA/m. Find $\mathbf{H}$ at

(a) (0, 0, 0)

(b) (1, 5, −3)

Answer: (a) $25\mathbf{a}_x$ mA/m, (b) $-25\mathbf{a}_x$ mA/m.

EXAMPLE 6.6

A toroid whose dimensions are shown in Figure 6.15 has N turns and carries current I. Determine H inside and outside the toroid.

Solution:

We apply Ampère's circuit law to the Amperian path, which is a circle of radius ρ shown dashed in Figure 6.15. Since N wires cut through this path each carrying current I, the net

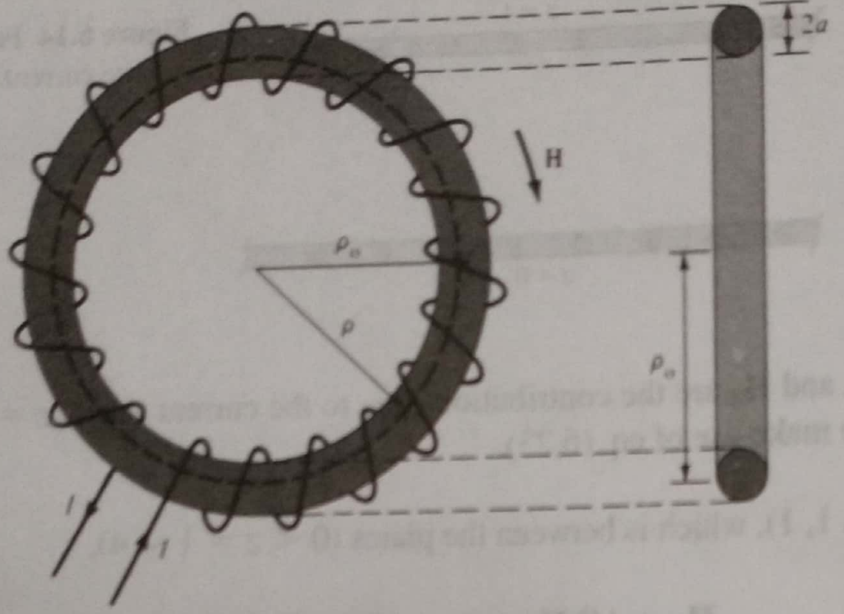

Figure 6.15 For Example 6.6: a toroid with a circular cross section.

current enclosed by the Amperian path is NI. Hence,

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc} \rightarrow H \cdot 2\pi\rho = NI$$

or

$$H = \frac{NI}{2\pi\rho}, \qquad \text{for } \rho_o - a < \rho < \rho_o + a$$

where ρ_o is the mean radius of the toroid as shown in Figure 6.15. An approximate value of H is

$$H_{approx} = \frac{NI}{2\pi\rho_o} = \frac{NI}{\ell}$$

Notice that this is the same as the formula obtained for H for points well inside a very long solenoid ($\ell \gg a$). Thus a straight solenoid may be regarded as a special toroidal coil for which $\rho_o \rightarrow \infty$. Outside the toroid, the current enclosed by an Amperian path is $NI - NI = 0$ and hence $H = 0$.

PRACTICE EXERCISE 6.6

A toroid of circular cross section whose center is at the origin and axis the same as the z-axis has 1000 turns with $\rho_o = 10$ cm, $a = 1$ cm. If the toroid carries a 100 mA current, find $|H|$ at

(a) (3 cm, −4 cm, 0)

(b) (6 cm, 9 cm, 0)

Answer: (a) 0, (b) 147.1 A/m.

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EXAMPLE 6.7

Given the magnetic vector potential $\mathbf{A} = -\rho^2/4\ \mathbf{a}_z$ Wb/m, calculate the total magnetic flux crossing the surface $\phi = \pi/2$, $1 \leq \rho \leq 2$ m, $0 \leq z \leq 5$ m.

Solution:

We can solve this problem in two different ways: using eq. (6.32) or eq. (6.51).

Method 1:

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial \rho}\mathbf{a}_\phi = \frac{\rho}{2}\mathbf{a}_\phi, \qquad d\mathbf{S} = d\rho\, dz\, \mathbf{a}_\phi$$

Hence,

$$\Psi = \int \mathbf{B} \cdot d\mathbf{S} = \frac{1}{2}\int_{z=0}^{5}\int_{\rho=1}^{2} \rho\, d\rho\, dz = \frac{1}{4}\rho^2 \Big|_2^1 (5) = \frac{15}{4}$$

$$\Psi = 3.75 \text{ Wb}$$

Method 2:
We use

$$\Psi = \oint_L \mathbf{A} \cdot d\mathbf{l} = \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4$$

where L is the path bounding surface S; Ψ_1, Ψ_2, Ψ_3, and Ψ_4 are, respectively, the evaluations of $\int \mathbf{A} \cdot d\mathbf{l}$ along the segments of L labeled 1 to 4 in Figure 6.20. Since $\mathbf{A}$ has only a z-component,

$$\Psi_1 = 0 = \Psi_3$$

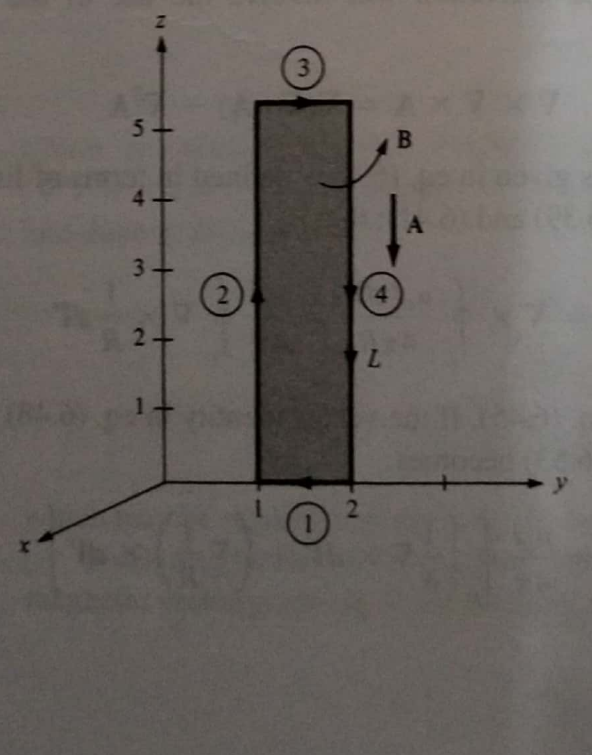

Figure 6.20 For Example 6.7.

That is,

$$\Psi = \Psi_2 + \Psi_4 = -\frac{1}{4}\left[(1)^2 \int_0^5 dz + (2)^2 \int_5^0 dz\right]$$

$$= -\frac{1}{4}(1 - 4)(5) = \frac{15}{4}$$

$$= 3.75 \text{ Wb}$$

as obtained by Method 1. Note that the direction of the path L must agree with that of $d\mathbf{S}$.

PRACTICE EXERCISE 6.7

A current distribution gives rise to the vector magnetic potential $\mathbf{A} = x^2y\mathbf{a}_x + y^2x\mathbf{a}_y - 4xyz\mathbf{a}_z$ Wb/m. Calculate the following:

(a) $\mathbf{B}$ at $(-1, 2, 5)$

(b) The flux through the surface defined by $z = 1, 0 \leq x \leq 1, -1 \leq y \leq 4$

Answer: (a) $20\mathbf{a}_x + 40\mathbf{a}_y + 3\mathbf{a}_z$ Wb/m^2, (b) 20 Wb.

EXAMPLE 7.3

A charged particle moves with a uniform velocity $4\mathbf{a}_x$ m/s in a region where $\mathbf{E} = 20\,\mathbf{a}_y$ V/m and $\mathbf{B} = B_o\mathbf{a}_z$ Wb/m^2. Determine B_o such that the velocity of the particle remains constant.

Solution:

If the particle moves with a constant velocity, it is implied that its acceleration is zero. In other words, the particle experiences no net force. Hence,

$$0 = \mathbf{F} = m\mathbf{a} = Q\,(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

$$0 = Q\,(20\mathbf{a}_y + 4\mathbf{a}_x \times B_o\mathbf{a}_z)$$

or

$$-20\mathbf{a}_y = -4B_o\mathbf{a}_y$$

Thus $B_o = 5$.

This example illustrates an important principle employed in a velocity filter shown in Figure 7.3. In this application, **E**, **B**, and **u** are mutually perpendicular so that $Q\mathbf{u} \times \mathbf{B}$ is directed opposite to $Q\mathbf{E}$, regardless of the sign of the charge. When the magnitudes of the two vectors are equal,

$$QuB = QE$$

or

$$u = \frac{E}{B}$$

This is the required (critical) speed to balance out the two parts of the Lorentz force. Particles with this speed are undeflected by the fields; they are "filtered" through the aperture. Particles with other speeds are deflected down or up, depending on whether their speeds are greater or less than this critical speed.

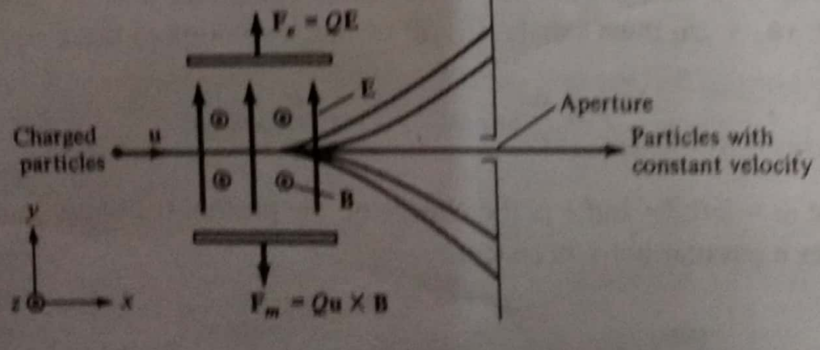

Figure 7.3 A velocity filter for charged particles.

EXAMPLE 7.5

Determine the magnetic moment of an electric circuit formed by the triangular loop of Figure 7.9.

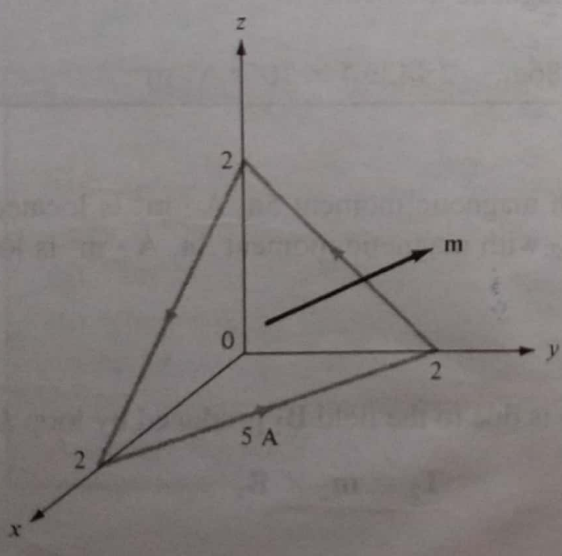

Figure 7.9 Triangular loop of Example 7.5.

Solution:

The equation of a plane is given by $Ax + By + Cz + D = 0$, where $D = -(A^2 + B^2 + C^2)$. Since points $(2, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$ lie on the plane, these points must satisfy the equation of the plane, and the constants A, B, C, and D can be determined. Doing this gives $x + y + z = 2$ as the plane on which the loop lies. Thus we can use

$$\mathbf{m} = IS\mathbf{a}_n$$

where

$$S = \text{loop area} = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2}(2\sqrt{2})(2\sqrt{2})\sin 60^\circ$$
$$= 4 \sin 60^\circ$$

If we define the plane surface by a function

$$f(x, y, z) = x + y + z - 2 = 0$$

$$\mathbf{a}_n = \pm\frac{\nabla f}{|\nabla f|} = \pm\frac{(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)}{\sqrt{3}}$$

We choose the plus sign in view of the direction of the current in the loop (using the right-hand rule, $\mathbf{m}$ is directed as in Figure 7.9). Hence

$$\mathbf{m} = 5(4 \sin 60^\circ)\frac{(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)}{\sqrt{3}}$$
$$= 10(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)\ \text{A} \cdot \text{m}^2$$

PRACTICE EXERCISE 7.5

A rectangular coil of area 10 cm^2 carrying current of 50 A lies on plane $2x + 6y - 3z = 7$ such that the magnetic moment of the coil is directed away from the origin. Calculate its magnetic moment.

Answer: $(1.429\mathbf{a}_x + 4.286\mathbf{a}_y - 2.143\mathbf{a}_z) \times 10^{-2}\ \text{A} \cdot \text{m}^2$.

AMPLE 7.6

A small current loop L_1 with magnetic moment $5\mathbf{a}_z\ \text{A} \cdot \text{m}^2$ is located at the origin while another small loop current L_2 with magnetic moment $3\mathbf{a}_y\ \text{A} \cdot \text{m}^2$ is located at $(4, -3, 10)$. Determine the torque on L_2.

Solution:

The torque $\mathbf{T}_2$ on the loop L_2 is due to the field $\mathbf{B}_1$ produced by loop L_1. Hence,

$$\mathbf{T}_2 = \mathbf{m}_2 \times \mathbf{B}_1$$

Since $\mathbf{m}_1$ for loop L_1 is along $\mathbf{a}_z$, we find $\mathbf{B}_1$ using eq. (7.22):

$$\mathbf{B}_1 = \frac{\mu_o m_1}{4\pi r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)$$

Using eq. (1.23), we transform $\mathbf{m}_2$ from Cartesian to spherical coordinates:

$$\mathbf{m}_2 = 3\mathbf{a}_y = 3(\sin\theta\sin\phi\,\mathbf{a}_r + \cos\theta\sin\phi\,\mathbf{a}_\theta + \cos\phi\,\mathbf{a}_\phi)$$

At (4, −3, 10),

$$r = \sqrt{4^2 + (-3)^2 + 10^2} = 5\sqrt{5}$$

$$\tan\theta = \frac{\rho}{z} = \frac{5}{10} = \frac{1}{2} \rightarrow \sin\theta = \frac{1}{\sqrt{5}}, \qquad \cos\theta = \frac{2}{\sqrt{5}}$$

$$\tan\phi = \frac{y}{x} = \frac{-3}{4} \rightarrow \sin\phi = \frac{-3}{5}, \qquad \cos\phi = \frac{4}{5}$$

Hence,

$$\mathbf{B}_1 = \frac{4\pi \times 10^{-7} \times 5}{4\pi\, 625\sqrt{5}}\left(\frac{4}{\sqrt{5}}\mathbf{a}_r + \frac{1}{\sqrt{5}}\mathbf{a}_\theta\right)$$

$$= \frac{10^{-7}}{625}(4\mathbf{a}_r + \mathbf{a}_\theta)$$

$$\mathbf{m}_2 = 3\left[-\frac{3\mathbf{a}_r}{5\sqrt{5}} - \frac{6\mathbf{a}_\theta}{5\sqrt{5}} + \frac{4\mathbf{a}_\phi}{5}\right]$$

and

$$\mathbf{T} = \frac{10^{-7}(3)}{625\,(5\sqrt{5})}(-3\mathbf{a}_r - 6\mathbf{a}_\theta + 4\sqrt{5}\mathbf{a}_\phi) \times (4\mathbf{a}_r + \mathbf{a}_\phi)$$

$$= 4.293 \times 10^{-11}(-8.944\mathbf{a}_r + 35.777\mathbf{a}_\theta + 21\mathbf{a}_\phi)$$

$$= -0.384\mathbf{a}_r + 1.536\mathbf{a}_\theta + 0.9015\mathbf{a}_\phi \text{ nN}\cdot\text{m}$$

PRACTICE EXERCISE 7.6

The coil of Practice Exercise 7.5 is surrounded by a uniform field $0.6\mathbf{a}_x + 0.4\mathbf{a}_y + 0.5\mathbf{a}_z$ Wb/m^2.

(a) Find the torque on the coil.

(b) Show that the torque on the coil is maximum if placed on plane $2x - 8y + 4z = \sqrt{84}$. Calculate the magnitude of the maximum torque.

Answer: (a) $0.03\mathbf{a}_x - 0.02\mathbf{a}_y - 0.02\mathbf{a}_z$ N · m, (b) 0.0439 N · m.

EXAMPLE 7.7

Given that $\mathbf{H}_1 = -2\mathbf{a}_x + 6\mathbf{a}_y + 4\mathbf{a}_z$ A/m in region $y - x - 2 \leq 0$, where $\mu_1 = 5\mu_o$, calculate

(a) $\mathbf{M}_1$ and $\mathbf{B}_1$

(b) $\mathbf{H}_2$ and $\mathbf{B}_2$ in region $y - x - 2 \geq 0$, where $\mu_2 = 2\mu_o$

Solution:

Since $y - x - 2 = 0$ is a plane, $y - x \leq 2$ or $y \leq x + 2$ is region 1 in Figure 7.14. A point in this region may be used to confirm this. For example, the origin (0, 0) is in this region because $0 - 0 - 2 < 0$. If we let the surface of the plane be described by $f(x, y) = y - x - 2$, a unit vector normal to the plane is given by

$$\mathbf{a}_n = \frac{\nabla f}{|\nabla f|} = \frac{\mathbf{a}_y - \mathbf{a}_x}{\sqrt{2}}$$

(a)

$$\mathbf{M}_1 = \chi_{m1}\mathbf{H}_1 = (\mu_{r1} - 1)\,\mathbf{H}_1 = (5 - 1)(-2, 6, 4)$$
$$= -8\mathbf{a}_x + 24\mathbf{a}_y + 16\mathbf{a}_z \text{ A/m}$$

$$\mathbf{B}_1 = \mu_1\mathbf{H}_1 = \mu_o\mu_{r1}\mathbf{H}_1 = 4\pi \times 10^{-7}(5)(-2, 6, 4)$$
$$= -12.57\mathbf{a}_x + 37.7\mathbf{a}_y + 25.13\mathbf{a}_z \ \mu \text{ Wb/m}^2$$

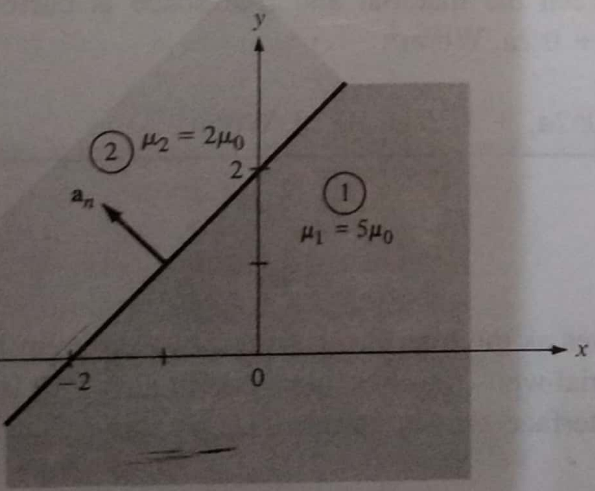

Figure 7.14 For Example 7.8.

(b) $\mathbf{H}_{1n} = (\mathbf{H}_1 \cdot \mathbf{a}_n)\mathbf{a}_n = \left[(-2, 6, 4) \cdot \frac{(-1, 1, 0)}{\sqrt{2}}\right] \frac{(-1, 1, 0)}{\sqrt{2}}$

$= -4\mathbf{a}_x + 4\mathbf{a}_y$

But

$$\mathbf{H}_1 = \mathbf{H}_{1n} + \mathbf{H}_{1t}$$

Hence,

$$\begin{aligned}\mathbf{H}_{1t} &= \mathbf{H}_1 - \mathbf{H}_{1n} = (-2, 6, 4) - (-4, 4, 0) \\ &= 2\mathbf{a}_x + 2\mathbf{a}_y + 4\mathbf{a}_z\end{aligned}$$

Using the boundary conditions, we have

$$\mathbf{H}_{2t} = \mathbf{H}_{1t} = 2\mathbf{a}_x + 2\mathbf{a}_y + 4\mathbf{a}_z$$
$$\mathbf{B}_{2n} = \mathbf{B}_{1n} \rightarrow \mu_2\mathbf{H}_{2n} = \mu_1\mathbf{H}_{1n}$$

or

$$\mathbf{H}_{2n} = \frac{\mu_1}{\mu_2}\mathbf{H}_{1n} = \frac{5}{2}(-4\mathbf{a}_x + 4\mathbf{a}_y) = -10\mathbf{a}_x + 10\mathbf{a}_y$$

Thus

$$\mathbf{H}_2 = \mathbf{H}_{2n} + \mathbf{H}_{2t} = -8\mathbf{a}_x + 12\mathbf{a}_y + 4\mathbf{a}_z \text{ A/m}$$

and

$$\begin{aligned}\mathbf{B}_2 &= \mu_2\mathbf{H}_2 = \mu_o\mu_{r2}\mathbf{H}_2 = (4\pi \times 10^{-7})(2)(-8, 12, 4) \\ &= -20.11\mathbf{a}_x + 30.16\mathbf{a}_y + 10.05\mathbf{a}_z\ \mu \text{ Wb/m}^2\end{aligned}$$

PRACTICE EXERCISE 7.7

Region 1, described by $3x + 4y \geq 10$, is free space, whereas region 2, described by $3x + 4y \leq 10$, is a magnetic material for which $\mu \simeq 10\mu_0$. Assuming that the boundary between the material and free space is current free, find $\mathbf{B}_2$ if $\mathbf{B}_1 = 0.1\mathbf{a}_x + 0.4\mathbf{a}_y + 0.2\mathbf{a}_z$ Wb/m^2.

Answer: $-1.052\mathbf{a}_x + 1.264\mathbf{a}_y + 2\mathbf{a}_z$ Wb/m^2.

EXAMPLE 7.8

The *xy*-plane serves as the interface between two different media. Medium 1 ($z < 0$) is filled with a material whose $\mu_r = 6$, and medium 2 ($z > 0$) is filled with a material whose $\mu_r = 4$. If the interface carries current $(1/\mu_o)\,\mathbf{a}_y$ mA/m, and $\mathbf{B}_2 = 5\mathbf{a}_x + 8\mathbf{a}_z$ mWb/m^2, find $\mathbf{H}_1$ and $\mathbf{B}_1$.

Solution:

In Example 7.7, $\mathbf{K} = 0$, so eq. (7.46) was appropriate. In this example, however, $\mathbf{K} \neq 0$, and we must resort to eq. (7.45) in addition to eq. (7.41). Consider the problem as illustrated in Figure 7.15. Let $\mathbf{B}_1 = (B_x, B_y, B_z)$ in mWb/m^2.

$$\mathbf{B}_{1n} = \mathbf{B}_{2n} = 8\mathbf{a}_z \rightarrow B_z = 8 \tag{7.9.1}$$

But

$$\mathbf{H}_2 = \frac{\mathbf{B}_2}{\mu_2} = \frac{1}{4\mu_o}(5\mathbf{a}_x + 8\mathbf{a}_z) \text{ mA/m} \tag{7.9.2}$$

and

$$\mathbf{H}_1 = \frac{\mathbf{B}_1}{\mu_1} = \frac{1}{6\mu_o}(B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z) \text{ mA/m} \tag{7.9.3}$$

Having found the normal components, we can find the tangential components by using

$$(\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{n12} = \mathbf{K}$$

or

$$\mathbf{H}_1 \times \mathbf{a}_{n12} = \mathbf{H}_2 \times \mathbf{a}_{n12} + \mathbf{K} \tag{7.9.4}$$

Substituting eqs. (7.9.2) and (7.9.3) into eq. (7.9.4) gives

$$\frac{1}{6\mu_o}(B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z) \times \mathbf{a}_z = \frac{1}{4\mu_o}(5\mathbf{a}_x + 8\mathbf{a}_z) \times \mathbf{a}_z + \frac{1}{\mu_o}\mathbf{a}_y$$

Equating components yields

$$B_y = 0, \qquad \frac{-B_x}{6} = \frac{-5}{4} + 1, \quad \text{or} \quad B_x = \frac{6}{4} = 1.5 \tag{7.9.5}$$

From eqs. (7.9.1) and (7.9.5), we have

$$\mathbf{B}_1 = 1.5\mathbf{a}_x + 8\mathbf{a}_z \text{ mWb/m}^2$$

$$\mathbf{H}_1 = \frac{\mathbf{B}_1}{\mu_1} = \frac{1}{\mu_o}(0.25\mathbf{a}_x + 1.33\mathbf{a}_z) \text{ mA/m}$$

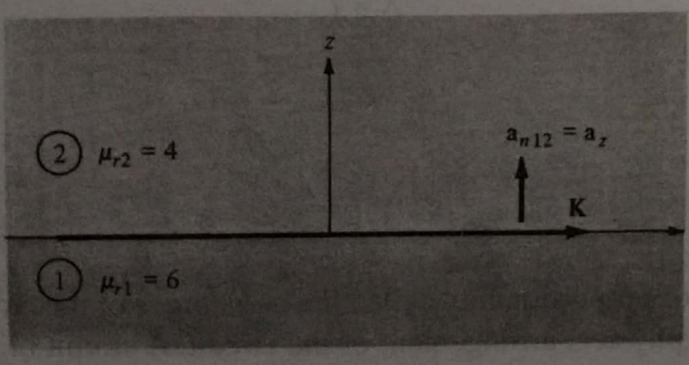

Figure 7.15 For Example 7.9.

and

$$\mathbf{H}_2 = \frac{1}{\mu_o}(1.25\mathbf{a}_x + 2\mathbf{a}_z) \text{ mA/m}$$

Note that H_{1x} is $1/\mu_o$ mA/m less than H_{2x} because of the current sheet and also that $B_{1n} = B_{2n}$.

PRACTICE EXERCISE 7.8

A unit normal vector from region 2 ($\mu = 2\mu_o$) to region 1 ($\mu = \mu_o$) is $\mathbf{a}_{n21} = (6\mathbf{a}_x + 2\mathbf{a}_y - 3\mathbf{a}_z)/7$. If $\mathbf{H}_1 = 10\mathbf{a}_x + \mathbf{a}_y + 12\mathbf{a}_z$ A/m and $\mathbf{H}_2 = H_{2x}\mathbf{a}_x - 5\mathbf{a}_y + 4\mathbf{a}_z$ A/m, determine

(a) $\mathbf{H}_{2x}$

(b) The surface current density **K** on the interface

(c) The angles $\mathbf{B}_1$ and $\mathbf{B}_2$ make with the normal to the interface

Answer: (a) 5.833, (b) $4.86\mathbf{a}_x - 8.64\mathbf{a}_y + 3.95\mathbf{a}_z$ A/m, (c) 76.27°, 77.62°.

EXAMPLE 7.9

Calculate the self-inductance per unit length of an infinitely long solenoid.

Solution:

We recall from Example 6.4 that for an infinitely long solenoid, the magnetic flux inside the solenoid per unit length is

$$B = \mu H = \mu I n$$

where $n = N/\ell$ = number of turns per unit length. If S is the cross-sectional area of the solenoid, the total flux through the cross section is

$$\Psi = BS = \mu I n S$$

Since this flux is only for a unit length of the solenoid, the linkage per unit length is

$$\lambda' = \frac{\lambda}{\ell} = n\Psi = \mu n^2 I S$$

and thus the inductance per unit length is

$$L' = \frac{L}{\ell} = \frac{\lambda'}{I} = \mu n^2 S$$

$$\boxed{L' = \mu n^2 S} \qquad \text{H/m}$$

PRACTICE EXERCISE 7.9

A very long solenoid with 2 × 2 cm cross section has an iron core ($\mu_r = 1000$) and 4000 turns per meter. It carries a current of 500 mA. Find the following:

(a) Its self-inductance per meter

(b) The energy per meter stored in its field

Answer: (a) 8.042 H/m, (b) 1.005 J/m.

EXAMPLE 7.10

Determine the self-inductance of a coaxial cable of inner radius a and outer radius b.

Solution:

The self-inductance of the inductor can be found in two different ways: by taking the four steps given in Section 7.8 or by using eqs. (7.54) and (7.66).

Method 1: Consider the cross section of the cable as shown in Figure 7.19. We recall from eq. (6.29) that by applying Ampère's circuit law, we obtained for region 1 ($0 \leq \rho \leq a$),

$$\mathbf{B}_1 = \frac{\mu I \rho}{2\pi a^2}\,\mathbf{a}_\phi$$

and for region 2 ($a \leq \rho \leq b$),

$$\mathbf{B}_2 = \frac{\mu I}{2\pi \rho}\,\mathbf{a}_\phi$$

We first find the internal inductance L_{in} by considering the flux linkages due to the inner conductor. From Figure 7.22(a), the flux leaving a differential shell of thickness $d\rho$ is

$$d\Psi_1 = B_1\, d\rho\, dz = \frac{\mu I \rho}{2\pi a^2}\, d\rho\, dz$$

The flux linkage is $d\Psi_1$ multiplied by the ratio of the area within the path enclosing the flux to the total area, that is,

$$d\lambda_1 = d\Psi_1 \cdot \frac{I_{\text{enc}}}{I} = d\Psi_1 \cdot \frac{\pi \rho^2}{\pi a^2}$$

because I is uniformly distributed over the cross section for dc excitation. Thus, the total flux linkages within the differential flux element are

$$d\lambda_1 = \frac{\mu I \rho\, d\rho\, dz}{2\pi a^2} \cdot \frac{\rho^2}{a^2}$$

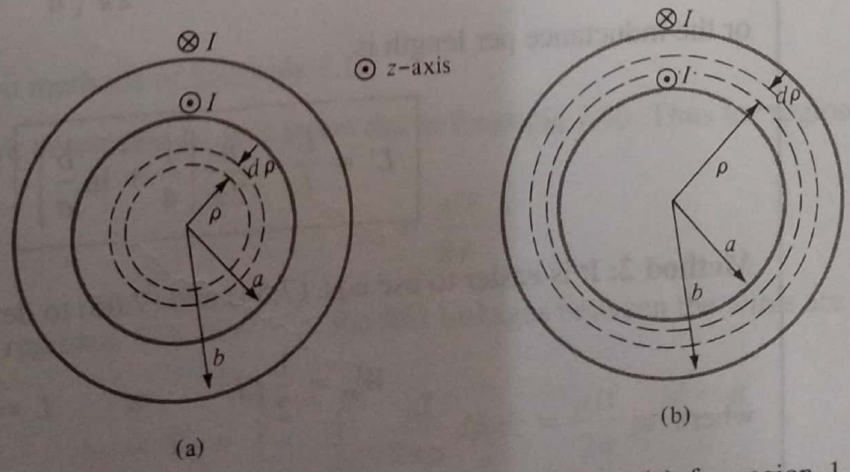

Figure 7.19 Cross section of the coaxial cable: **(a)** for region 1, $0 < \rho < a$, **(b)** for region 2, $a < \rho < b$; for Example 7.10.

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For length ℓ of the cable,

$$\lambda_1 = \int_{\rho=0}^{a} \int_{z=0}^{\ell} \frac{\mu I \rho^3 \, d\rho \, dz}{2\pi a^4} = \frac{\mu I \ell}{8\pi}$$

$$L_{\text{in}} = \frac{\lambda_1}{I} = \frac{\mu \ell}{8\pi} \tag{7.11.1}$$

The internal inductance per unit length, given by

$$\boxed{L'_{\text{in}} = \frac{L_{\text{in}}}{\ell} = \frac{\mu}{8\pi}} \qquad \text{H/m} \tag{7.11.2}$$

is independent of the radius of the conductor or wire. Thus eqs. (7.11.1) and (7.11.2) are also applicable to finding the inductance of any infinitely long straight conductor of finite radius.

We now determine the external inductance L_{ext} by considering the flux linkages between the inner and the outer conductor as in Figure 7.19(b). For a differential shell of thickness $d\rho$,

$$d\Psi_2 = B_2 \, d\rho \, dz = \frac{\mu I}{2\pi\rho} \, d\rho \, dz$$

In this case, the total current I is enclosed within the path enclosing the flux. Hence,

$$\lambda_2 = \Psi_2 = \int_{\rho=a}^{b} \int_{z=0}^{\ell} \frac{\mu I \, d\rho \, dz}{2\pi\rho} = \frac{\mu I \ell}{2\pi} \ln \frac{b}{a}$$

$$L_{\text{ext}} = \frac{\lambda_2}{I} = \frac{\mu \ell}{2\pi} \ln \frac{b}{a}$$

Thus

$$L = L_{\text{in}} + L_{\text{ext}} = \frac{\mu \ell}{2\pi} \left[\frac{1}{4} + \ln \frac{b}{a} \right]$$

or the inductance per length is

$$\boxed{L' = \frac{L}{\ell} = \frac{\mu}{2\pi} \left[\frac{1}{4} + \ln \frac{b}{a} \right]} \qquad \text{H/m}$$

Method 2: It is easier to use eqs. (7.54) and (7.66) to determine L, that is,

$$W_m = \frac{1}{2} L I^2 \qquad \text{or} \qquad L = \frac{2W_m}{I^2}$$

where

$$W_m = \frac{1}{2} \int \mathbf{B} \cdot \mathbf{H} \, dv = \int \frac{B^2}{2\mu} \, dv$$

Hence

$$L_{in} = \frac{2}{I^2}\int \frac{B_1^2}{2\mu}\,dv = \frac{1}{I^2\mu}\iiint \frac{\mu^2 I^2 \rho^2}{4\pi^2 a^4}\,\rho\,d\rho\,d\phi\,dz$$

$$= \frac{\mu}{4\pi^2 a^4}\int_0^{\ell} dz \int_0^{2\pi} d\phi \int_0^{a} \rho^3\,d\rho = \frac{\mu \ell}{8\pi}$$

$$L_{ext} = \frac{2}{I^2}\int \frac{B_2^2}{2\mu}\,dv = \frac{1}{I^2\mu}\iiint \frac{\mu^2 I^2}{4\pi^2 \rho^2}\,\rho\,d\rho\,d\phi\,dz$$

$$= \frac{\mu}{4\pi^2}\int_0^{\ell} dz \int_0^{2\pi} d\phi \int_a^{b} \frac{d\rho}{\rho} = \frac{\mu \ell}{2\pi}\ln\frac{b}{a}$$

and

$$L = L_{in} + L_{ext} = \frac{\mu\ell}{2\pi}\left[\frac{1}{4} + \ln\frac{b}{a}\right]$$

as obtained previously.

PRACTICE EXERCISE 7.10

Calculate the self-inductance of the coaxial cable of Example 7.10 if the space between the line conductor and the outer conductor is made of an inhomogeneous material having $\mu = 2\mu_o/(1 + \rho)$.

Answer: $\frac{\mu_o\ell}{8\pi} + \frac{\mu_o\ell}{\pi}\left[\ln\frac{b}{a} - \ln\frac{(1 + b)}{(1 + a)}\right]$.

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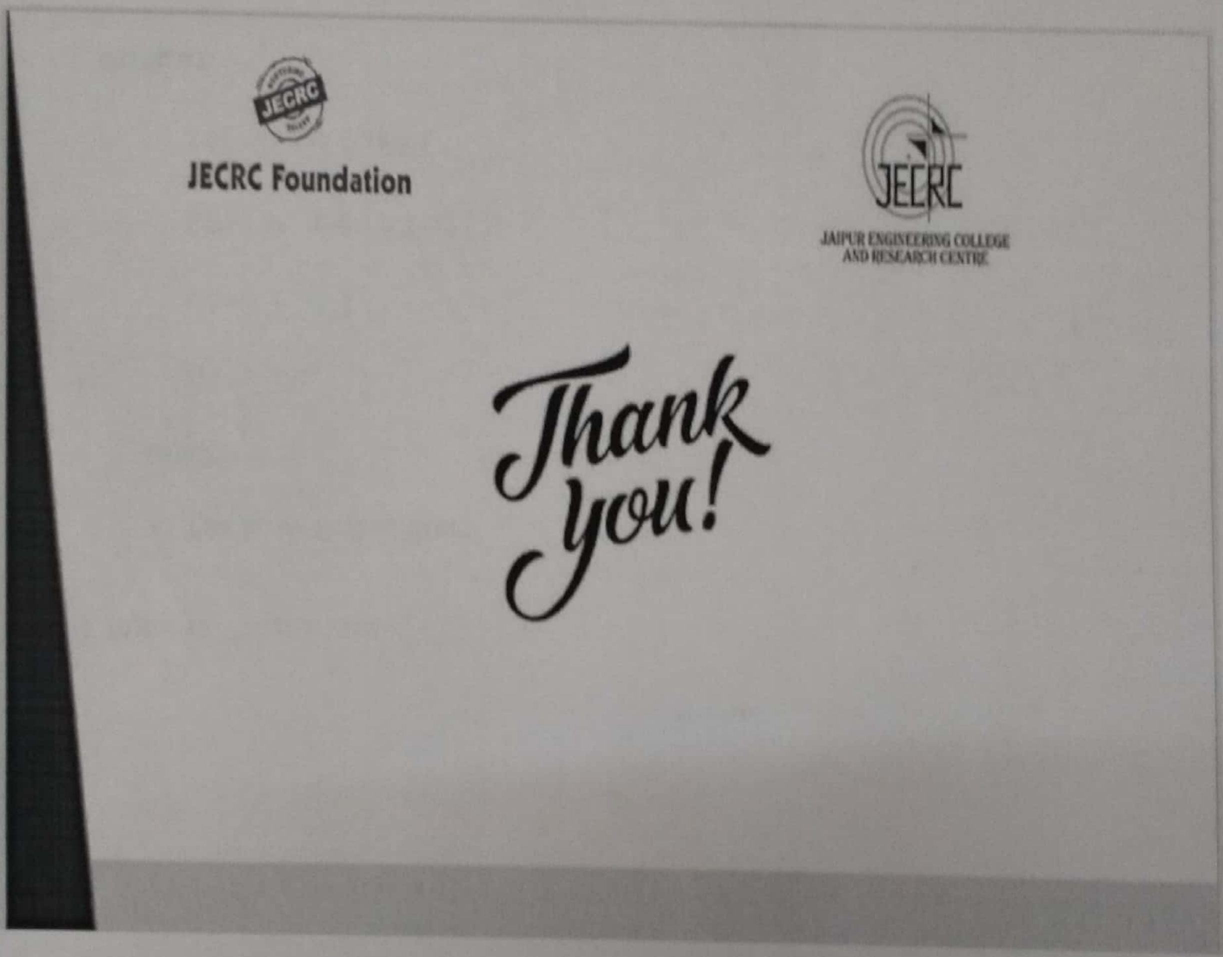
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