Jaipur Engineering College & Research Centre, Jaipur

Session 2020-21

Notes – Unit II

Electromagnetic Fields (3EE4-08)

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Vision and Mission of Institute

Vision of institute

To become a renowned centre of outcome based learning, and work towards, professional, cultural and social enrichment of the lives of individuals and communities.

Mission of institute

M1.Focus on evaluation of learning outcomes and motivate students to inculcate research aptitude by project based learning.

M2.Identify, based on informed perception of Indian, regional and global needs, the areas of focus and provide platform to gain knowledge and solutions.

M3.Offer opportunities for interaction between academia and industry.

M4.Develop human potential to its fullest extent so that intellectually capable and imaginatively gifted leaders may emerge in a range of professions

Vision and Mission of Electrical Engineering Department

Vision of department

The Electrical Engineering department strives to recognized globally for outcome based technical knowledge and produce quality human being who can manage the advance technologies and contribute to society.

Mission Of department

M1. To impart quality technical knowledge to the learners to make them globally competitive Electrical Engineers.

M2. To provide the learners ethical guidelines along with excellent academic environment for a long productive career.

M3. To promote industry- institute relationship.

Syllabus of Electromagnetic fields

unit 1- Review of Vector Calculus

Vector algebra- addition, subtraction, components of vectors, scalar and vector multiplications, triple products, three orthogonal coordinate systems (rectangular, cylindrical and spherical). Vector calculus differentiation, partial differentiation, integration, vector operator del,gradient, divergence and curl; integral theorems of vectors. Conversion of a vector from one coordinate system to another.

Unit 2- Static Electric Field

Coulomb's law, Electric field intensity, Electrical field due to point charges. Line, Surface and Volume charge distributions. Gauss law and its applications. Absolute Electric potential, Potential difference, Calculation of potential differences for different configurations. Electric dipole, Electrostatic Energy and Energy density.

Unit 3- Conductors, Dielectrics and Capacitance

Current and current density, Ohms Law in Point form, Continuity of current, Boundary conditions of perfect dielectric materials. Permittivity of dielectric materials, Capacitance, Capacitance of a two wire line, Poisson's equation, Laplace's equation, Solution of Laplace and Poisson's equation, Application of Laplace's and Poisson's equations.

unit 4- Static Magnetic Fields

Biot-Savart Law, Ampere Law, Magnetic flux and magnetic flux density, Scalar and Vector Magnetic potentials. Steady magnetic fields produced by current carrying conductors.

Unit5- Magnetic Forces, Materials and Inductance

Force on a moving charge, Force on a differential current element, Force between differential current elements, Nature of magnetic materials, Magnetization and permeability, Magnetic boundary conditions, Magnetic circuits, inductances and mutual inductances.

Unit 6- Time Varying Fields and Maxwell's Equations

Faraday's law for Electromagnetic induction, Displacement current, Point form of Maxwell's equation, Integral form of Maxwell's equations, Motional Electromotive forces. Boundary Conditions

Unit 7- Electromagnetic Waves

Derivation of Wave Equation, Uniform Plane Waves, Maxwell's equation in Phasor form, Wave equation in Phasor form, Plane waves in free space and in a homogenous material. Wave equation for a conducting medium, Plane waves in lossy dielectrics, Propagation in good conductors, Skin effect. Poynting theorem.

Course outcomes for Electromagnetic fields

CO1- Acquire basic understanding of vectors , their representation and conversion in different coordinate systems.

CO2- Able to compute the force, fields & energy of the electrostatic & magneto static fields. Able to analyze the materials, conductors, dielectrics, inductances and capacitances.

CO3- Understand the concept of time varying field and able to solve electromagnetic relation using Maxwell equations. Also able to analyze the electromagnetic waves.

Electric flux density and Gauss's Law

→ **Stream lines or flux lines =>**

If a unit test charge is placed near a point charge it experiences a force. The direction of this force can be represented by the lines, radially outward from the positive charge. These lines are called stream-lines or flux lines.

Thus the electric field due to a charge can be imagined to be present around it in terms of quantity called electric flux.

The flux lines gives the pictorial representation of distribution of electric flux around a charge.

Electric flux =>

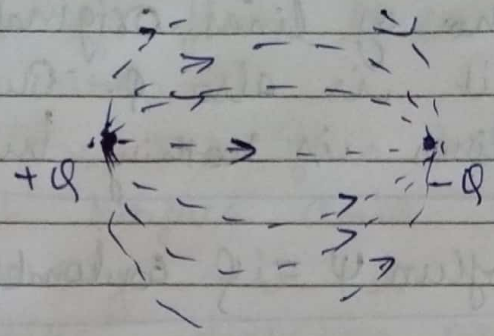
Total no. of lines of force in any particular electric field is called the electric flux.

it is represented by the symbol ψ. Similar to the charge, unit of electric flux is also Coulomb C.

Properties of Flux lines =>

1. The flux lines starts from positive charge and terminate on the negative charge as..

Shown.



Flux lines

2. if negative charge is absent, then the flux lines terminates at infinity as shown. while in absence of Positive charge, the electric flux terminates on the negative charge from infinity.

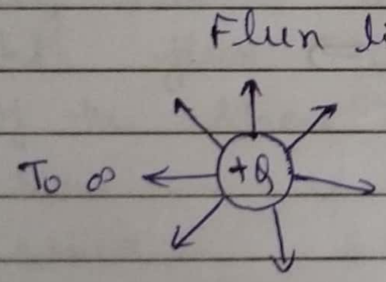


fig (a)

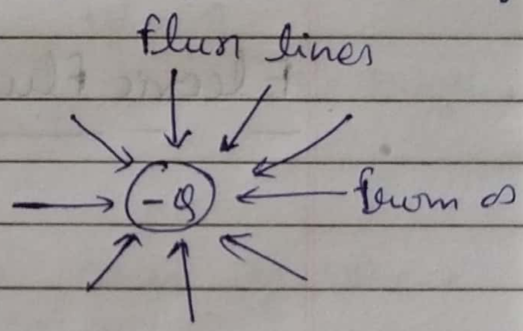


fig (b)

(3) There are more no. of lines i.e crowding of lines if electric field is stronger.

(4) These lines are Parallel and never cross each other

(5) The lines are independent of medium in which charges are placed.

(6). The lines always enter or leave the charged surface, normally.

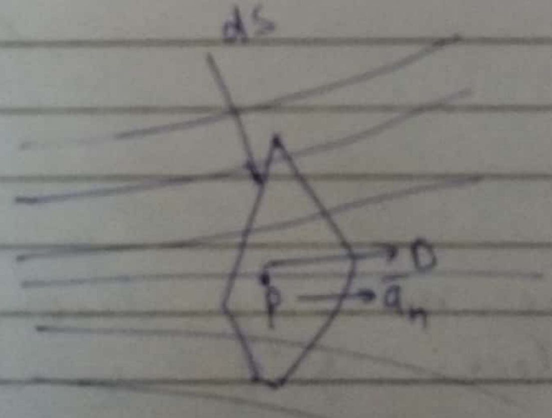
(7) if the charge on a body is $\pm Q$ Coulombs, then the total no of lines originating or terminating on it is also Q. But the total no of lines is nothing but a flux

$$\boxed{\therefore \quad \text{Electric flux } \Psi = Q \text{ coulombs (numerically)}}$$

The electric flux is also called displacement flux.

The Flux is a scalar field.

Electric Flux Density $\bar{D}$ ⟹



flux through ds

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Consider a unit surface area as show in fig. The no. of flux lines are Passing through this surface area.

The net flux Passing normal through the unit surface area is called the electric flux density. it is denoted as $\bar{D}$. it has a specific direction which is normal to the surface area under consideration hence it is a vector field.

Consider a sphere with a charge Q Placed at it's centre. There are no other charges Present around. The total flux distribution radially around the charge is $\psi = Q$.
This flux distributes uniformly over the surface of the shere.

Now ψ = total flux
while, S = total surface area of sphere

then electric flux density is defined as

$$\boxed{D = \frac{\psi}{S}}$$ in magnitude

as ψ is measured in Coulombs and S in square meters, the units of D are C/m^2.
This is also called displacement flux density or displacement density.

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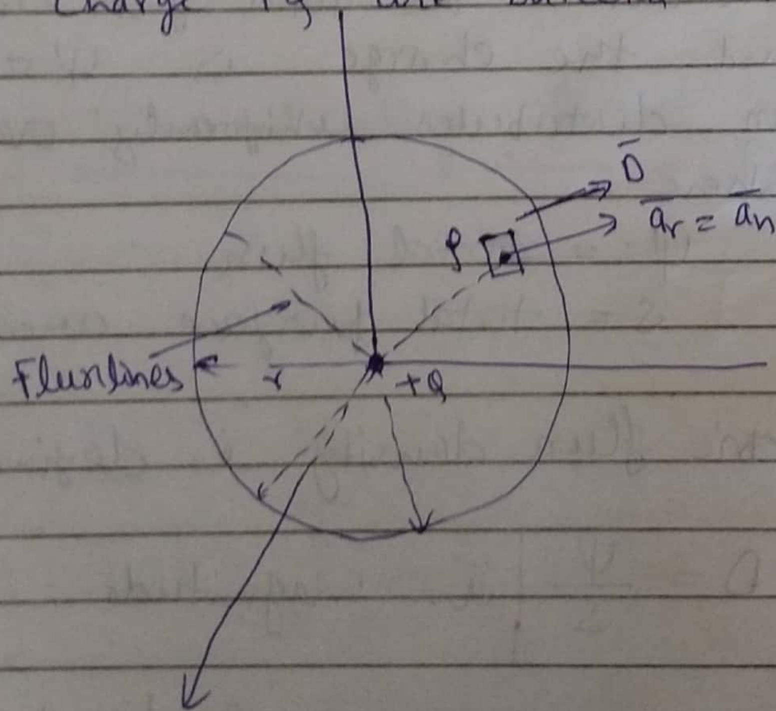
Vector form of electric flux density ⇒

$$\boxed{\bar{D} = \frac{d\psi}{ds}\,\bar{a}_n \quad C/m^2}$$

$\bar{D}$ due to a Point Charge Q ⇒

Consider a point charge $+Q$ placed at the centre of the imaginary sphere of radius r.
The flux lines originating from the point charge $+Q$ are directed radially outwards.



The magnitude of flux density at any point on the surface is,

$$|\bar{D}| = \frac{\text{Total flux } \psi}{\text{Total surface area } S}$$

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but $\Psi = Q =$ total flux
and $S = 4\pi r^2$ = total surface area

$$|\bar{D}| = \frac{Q}{4\pi r^2}$$

The unit vector directed radially outwards and normal to the surface at any point on the sphere is $\bar{a}_n = \bar{a}_r$

Thus in the vector form, electric flux density at a point which is at a distance of r, from the point charge +Q is given by,

$$\boxed{\bar{D} = \frac{Q}{4\pi r^2}\bar{a}_r \ C/m^2}$$

Relationship between $\bar{D}$ and $\bar{E}$

as we know that

$$\bar{E} = \frac{Q}{4\pi\epsilon_0 r^2}\bar{a}_r$$

so

$$\frac{\bar{D}}{\bar{E}} = \frac{\frac{Q}{4\pi r^2}\bar{a}_r}{\frac{Q}{4\pi\epsilon_0 r^2}\bar{a}_r} = \epsilon_0$$

$$\boxed{\bar{D} = \epsilon_0 \bar{E}}$$ ---- for free space -①

Thus $\bar{D}$ and $\bar{E}$ are related through the permittivity if the medium in which charge is located is other than free space having relative permittivity ϵ_r. then.

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$$\bar{D} = \epsilon_0 \epsilon_r \bar{E}$$

i.e $\cancel{\bar{D} = \epsilon_0}$

i.e $\bar{D} = \epsilon \bar{E}$

Electric flux density for various charge distributions →

① Line Charge –

$$Q = \int_L \rho_l \, dl$$

$$D = \frac{Q}{4\pi r^2} \bar{a}_r = \frac{\int_L \rho_L \, dl}{4\pi r^2} \bar{a}_r$$

if line charge is infinite then

$$\bar{E} = \frac{\rho_L}{2\pi \epsilon_0 r} \bar{a}_r$$

and $\bar{D} = \epsilon_0 \bar{E}$

$$\boxed{\bar{D} = \frac{\rho_L}{2\pi r} \bar{a}_r}$$

② Surface Charge –

$$Q = \int_S \rho_S \, ds$$

$$\bar{D} = \frac{Q}{4\pi r^2} \bar{a}_r = \frac{\int_S \rho_S \, ds}{4\pi r^2} \bar{a}_r$$

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if sheet charge is infinite then $\bar{E}$,

$$\bar{E} = \frac{\rho_s}{2\epsilon_0}\bar{a}_n$$

$$\bar{D} = \epsilon_0 E$$

$$\boxed{\bar{D} = \frac{\rho_s}{2}\bar{a}_n}$$

③ Volume charge =>

$$Q = \int_{vol} \rho_v \, dv$$

$$\bar{E} = \frac{\int_{vol} \rho_v \, dv}{4\pi\epsilon_0 r^2}\bar{a}_r$$

$$\bar{D} = \epsilon_0 \bar{E}$$

$$\boxed{\bar{D} = \frac{\int_{vol} \rho_v \, dv}{4\pi r^2}\bar{a}_r}$$

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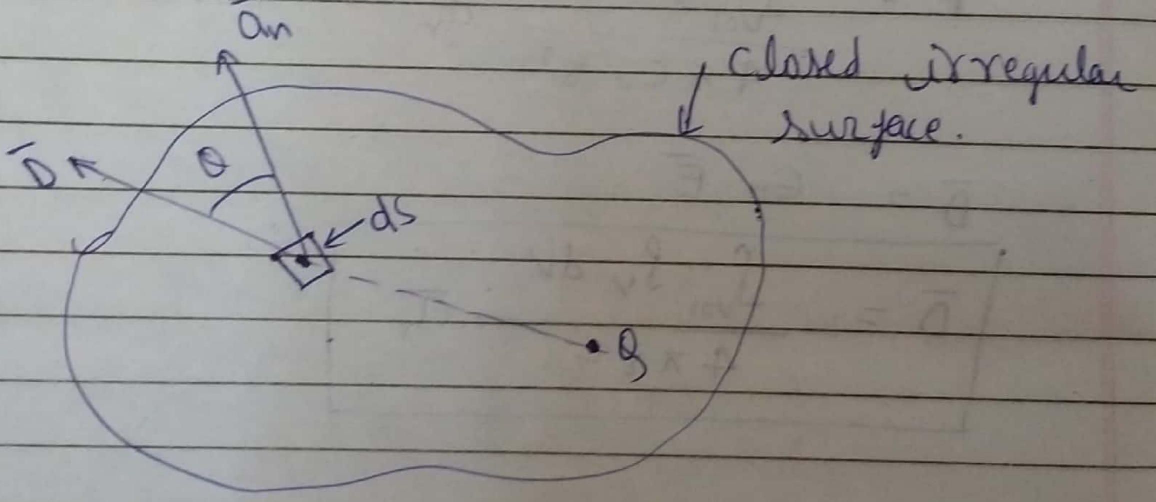
Gauss's law ⇒

Gauss's law states that -
" The total electric flux Ψ through any closed surface is equal to the total charge enclosed by that surface."

Mathematical representation of Gauss's law ⇒

Consider any irregular surface as shown in fig.



The total charge enclosed by the irregular closed surface is Q coulombs. Hence the total flux passing through the closed surface is Q. — $\Psi = Q$
Consider a small differential surface ds at Point P. As the surface is irregular the direction of $\bar{D}$ as well as it's magnitude

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is going to change from Point to Point on the surface.

ds in vector form $\overline{ds} = ds\,\bar{a}_n$

where $\bar{a}_n$ = normal to the surface ds at Point P

The flux density at Point P is $\bar{D}$ and it's direction is such that it makes an angle θ with the normal direction at Point P.

flux $d\psi$ is represented as

$$d\psi = \bar{D}_n\,\overline{ds}$$

D_n = Component of $\bar{D}$ in the direction of normal to the surface ds

but $D_n = |\bar{D}|\cos\theta$ so $d\psi = |\bar{D}|\cos\theta\, ds$

$$d\psi = \bar{D}\cdot\overline{ds}$$

hence total flux Passing through the entire closed surface is

$$\boxed{\psi = \int d\psi = \oint_s \bar{D}\cdot\overline{ds}}$$

such a closed surface over which the integration is carried out is called Gaussian surface.

Now

$$\boxed{\psi = \oint_s \bar{D}\cdot\overline{ds} = Q = \text{charge enclosed}}$$

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• This is the mathematical form of Gauss's law.

• The charge enclosed may take any of the following forms. —

① if there are no. of point charges $Q_1, Q_2, \ldots Q_n$ enclosed by the surface then

$$Q = Q_1 + Q_2 + \cdots Q_n = \Sigma Q_n$$

$$\psi = Q = \Sigma Q_n$$

② if there is a line charge with line charge density ρ_L then

$$\psi = Q = \int_L \rho_L \, dl$$

③ if there is a surface charge with surface charge density ρ_s

$$\psi = Q = \int_s \rho_s \, ds$$

④ if there is a volume charge with volume charge ~~den~~ density ρ_v

$$\psi = Q = \int_v \rho_v \, dV$$

⇒ The common form used to represent Gauss's law mathematically is,

$$\boxed{\psi = Q = \oint_s \bar{D} \cdot \bar{ds} = \int_v \rho_v \, dV}$$

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Applications of Gauss's law ⇒

The Gauss's law can be used to find $\bar{E}$ or $\bar{D}$ for symmeterical charge distributions, such as - Point charge, infinite line charge, sheet charge and a spherical distribution of charge.
it is also used to find the charge enclosed or the flux passing through the closed surface.

~~Symmetoric~~ once it has been found that symmetoric charge distribution exists, we construct a mathematical closed surface (known as Gaussian surface).

The surface is choosen such that $\bar{D}$ is normal or tangential to the Gaussian surface.
when -
→ $\bar{D}$ is normal to the surface, $\bar{D} \cdot d\bar{S} = D\,dS$
→ $\bar{D}$ is tangential to the surface, $\bar{D} \cdot d\bar{S} = 0$

we shall now apply these basic ideas to the following cases-

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(1) Point charge ⇒ Let us consider a point charge Q at origin of a spherical coordinate system and decide on a suitable closed surface which will meet the two requirements listed above. The surface is a ~~the~~ spherical surface centered at origin and of radius r. D_r is everywhere normal to the surface. So D_r has same value at all the points on the surface.

$$Q = \oint_S \bar{D} \cdot d\bar{S} = \oint_{Sph} D_r \, dS$$

$$= D_r \oint_{Sph} dS = D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$Q = 4\pi r^2 D_r$$

hence $D_r = \dfrac{Q}{4\pi r^2}$

since D_r is directed radially outward

$$\bar{D} = \frac{Q}{4\pi r^2} \bar{a}_r \quad \text{and} \quad \bar{E} = \frac{Q}{4\pi \epsilon_0 r^2} \bar{a}_r$$

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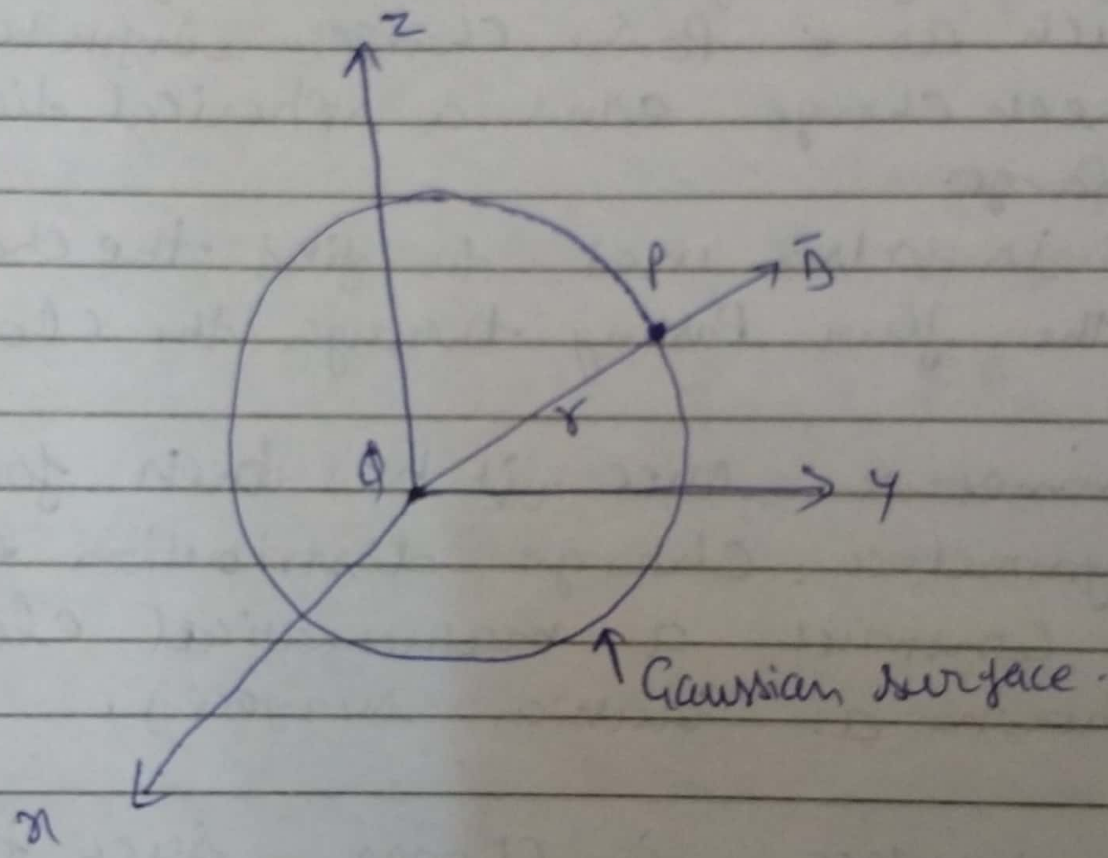
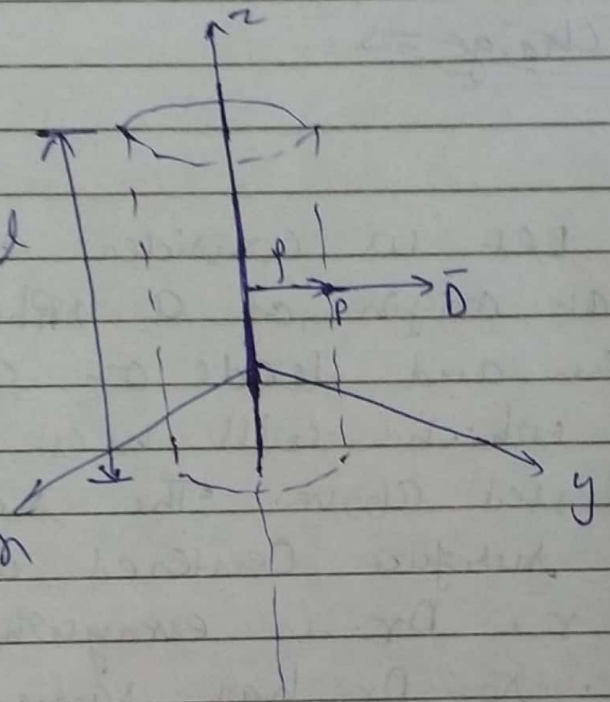

z
P
D̄
r
Q
y
x
Gaussian surface.

(2) Infinite line charge ⇒

Let us consider a uniform line charge distribution ρ_L lying along the z axis and extending from $-\infty$ to $+\infty$.



we choose a cylindrical surface containing P to satisfy the symmetry condition as shown in fig.

apply Gauss's law.

$$Q = \oint_{cyl} \bar{D}_S \cdot d\bar{S} = D_\rho \int_{sided} dS + 0 \int_{top} dS + 0 \int_{bottom} dS$$

$$= D_S \int_{z=0}^{l} \int_{\phi=0}^{2\pi} \rho \, d\phi \, dz = D_S \rho \, [\phi]_0^{2\pi} \, [z]_0^{l}$$

$$Q = D_S \rho \cdot 2\pi l.$$

$$D_S = D_\rho = \frac{Q}{2\pi \rho L}$$

in terms of Charge density ρ_L, the total charge enclosed is

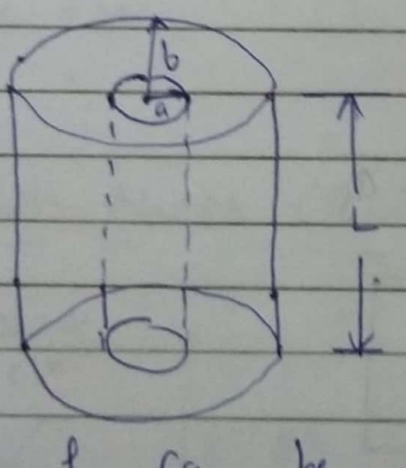
$$Q = \rho_L l$$

giving $D_\rho = \frac{\rho_L}{2\pi\rho}$ and $E_\rho = \frac{\rho_L}{2\pi\epsilon_0 \rho}$

this is the required result.

③ Co-axial Cables

Consider the two co-axial cylindrical conductors forming a co-axial cable. The radius of inner conductor is 'a' while the radius of outer conductor is 'b'. the length of the cable is L



The charge distribution on the outer surface area of the inner conductor is having density ρ_s C/m^2 The total outer surface area of the inner conductor is $2\pi a L$.

hence ρ_s can be expressed in terms of ρ_L

$$\rho_L = \frac{\rho_s \times \text{surface area}}{\text{Total length}} = \frac{\rho_s \times 2\pi a L}{L}$$

$$\rho_L = 2\pi a \rho_s \text{ C/m}$$

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since $Q = \bar{D}_s \cdot dS$

$$Q = D_s \, 2\pi \rho L \quad -(I) \qquad \text{where } a < \rho < b$$

the total charge on ~~the~~ a length L of the inner conductor is

$$Q = \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho_s \, ds = \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho_s \, a \, d\phi \, dz$$

$$\because ds = \rho \, d\phi \, dz \quad \text{and } \rho = a$$

$$Q = \rho_s \, a \, [\phi]_0^{2\pi} \, [z]_0^{L} = 2\pi \rho_s \, a L = 2\pi a L \rho_s \quad -(II)$$

from (I) & (II)

$$D_s \, 2\pi \rho . L = 2\pi a L \rho_s$$

$$D_s = \frac{a \rho_s}{\rho} \quad \text{or} \quad \bar{D} = \frac{a \rho_s}{\rho} \, a_\rho \quad (a < \rho < b)$$

This result may be expressed in terms of ρ_L.

$$\bar{D}_s = \frac{\cancel{a} \cdot \rho_L}{2\pi \cancel{a} \, \rho} \, a_\rho$$

$$\boxed{\bar{D} = \frac{\rho_L}{2\pi \rho} \, a_\rho}$$

This solution has a form identical with that of the infinite line charge.
Every flux line starting from the positive charge on the inner cylinder must terminate on the negative charge on the inner surface of

the outer cylinder. Hence the total charge on the inner surface of the outer cylinder is,

$$Q_{\text{outer cylinder}} = -2\pi a L \rho_{s(\text{inner})}$$

But $Q_{\text{outer cyl.}} = 2\pi b L \rho_s (\text{outer})$

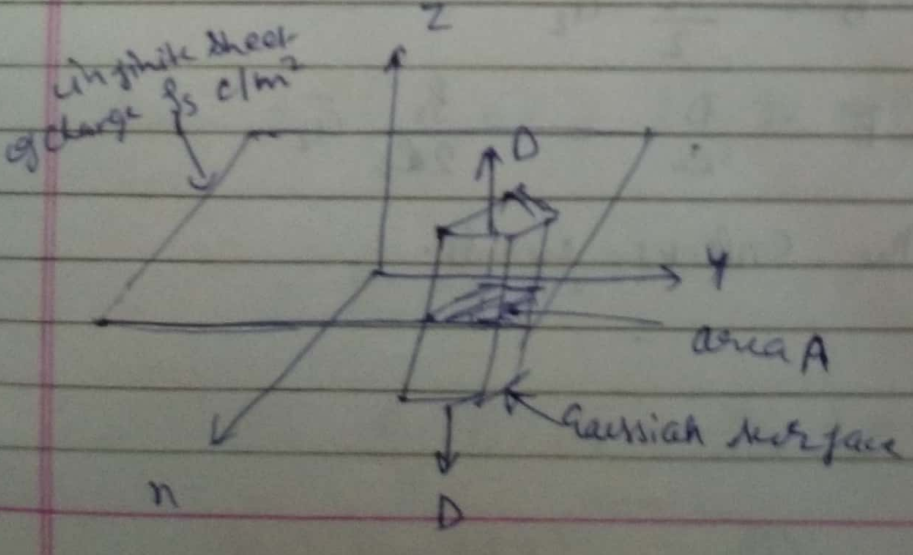
$$2\pi b L \rho_{s(\text{outer})} = -2\pi a L \rho_{s(\text{inner})}$$

$$\rho_{s(\text{outer})} = -\frac{a}{b} \rho_{s(\text{inner})}$$

If the Gaussian surface is considered such that $\rho > b$, then the total charge enclosed will be zero as equal and opposite charge on cylinder will cancel each other.

Similarly inside the inner cylinder, $\rho < a$ also the total charge enclosed will be zero.

(4) infinite Sheet Charge =)



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Consider an infinite sheet of uniform charge ρ_s C/m^2 lying on $z=0$ plane. to determine $\bar{D}$ at point P, we choose a rectangular box that is cut symmetrically by the sheet of charge and has two of it's faces parallel to the sheet as shown in fig.
as $\bar{D}$ is normal to the sheet $\bar{D} = D_z \bar{a}_z$

applying Gauss's law

$$\rho_s \int_s ds \quad Q = \oint_s \bar{D} \cdot d\bar{s} = D_z \left[\int_{top} ds + \int_{bottom} ds \right]$$

note that $\bar{D} \cdot d\bar{s}$ evaluated on the sides of the box is zero because $\bar{D}$ has no component along $\bar{a}_x$ and $\bar{a}_y$. if the top and bottom area of the box, each has A then

$$\rho_s A = D_z (A+A)$$

and thus $D_z = \dfrac{\rho_s}{2}$

$$\bar{D} = \frac{\rho_s}{2} \bar{a}_z$$

$$\bar{E} = \frac{D}{\epsilon_0} = \frac{\rho_s}{2\epsilon_0} \bar{a}_z$$

This is the expected result.

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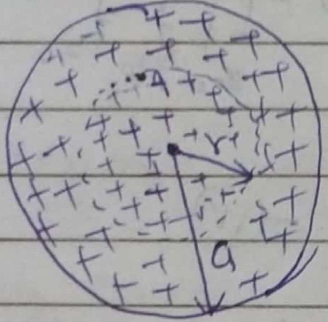
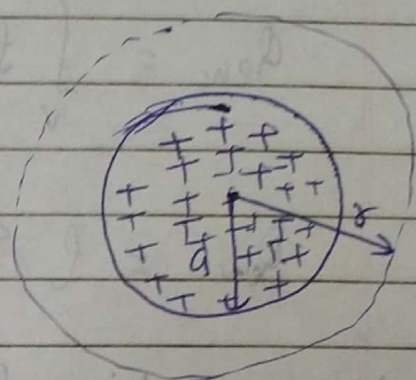
(5) uniformly charged sphere ⇒

Consider a sphere of radius a with a uniform charge ρ_0 C/m^3. To determine $\bar{D}$ everywhere, we construct Gaussian surfaces for case $r \le a$ and $r \ge a$ seperately. Since charge has spherical symmetry, it is obvious that a spherical surface is an appropriate Gaussian surface.



(a)



(b)

fig- Gaussian surface for a uniformly charged sphere when (a) $r \le a$ and (b) $r \ge a$

for $r \le a$, the total charge enclosed by the spherical surface of radius r, is

$$Q_{enc} = \int_v \rho_v \, dv = \rho_0 \int_v dv = \rho_0 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \rho_0 \frac{4}{3} \pi r^3 \quad -(1)$$

and $\Psi = \oint_s \bar{D} \cdot d\bar{s} = D_r \oint_s ds = D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi$

$$\Psi = D_r \, 4\pi r^2 \quad -(11)$$

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hence $\Psi = Q_{enc}$ gives

$$D_r 4\pi r^2 = \frac{4\pi r^3}{3} \rho_o$$

or $$\bar{D} = \frac{r}{3} \rho_o \bar{a}_r \qquad 0 < r \leq a \quad \text{— (III)}$$

for $r \geq a$

$$Q_{enc} = \int_V \rho_v \, dv = \rho_o \int_V dv = \rho_o \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$Q_{enc} = \rho_o \frac{4}{3} \pi a^3 \qquad \text{— (IV)}$$

while $$\Psi = \oint_S \bar{D} \cdot d\bar{S} = D_r 4\pi r^2 \quad \text{— (5)}$$

just as in eq. (II)

$$D_r 4\pi r^2 = \frac{4}{3} \pi a^3 \rho_o$$

$$\bar{D} = \frac{a^3}{3r^2} \rho_o \bar{a}_r \qquad r \geq a \quad \text{— (VI)}$$

thus from eq. (III) & (VI), $\bar{D}$ is everywhere is given by

$$\bar{D} = \begin{cases} \frac{r}{3} \rho_o \bar{a}_r & 0 < r \leq a \\ \frac{a^3}{3r^2} \rho_o \bar{a}_r & r \geq a \end{cases} \quad \text{— (VII)}$$

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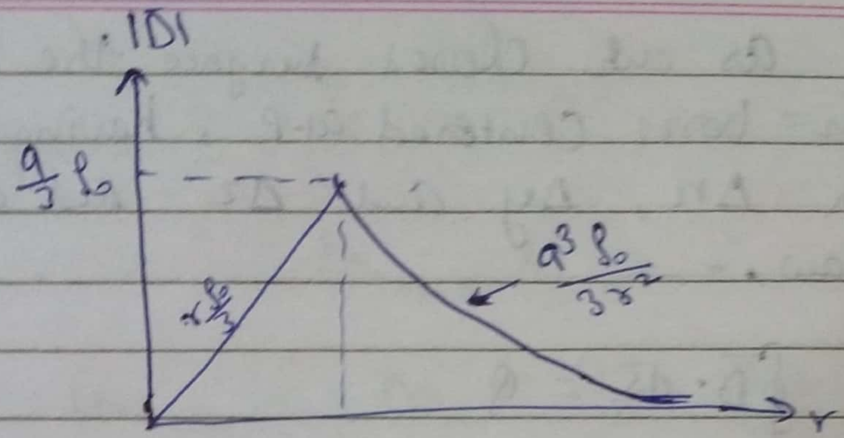
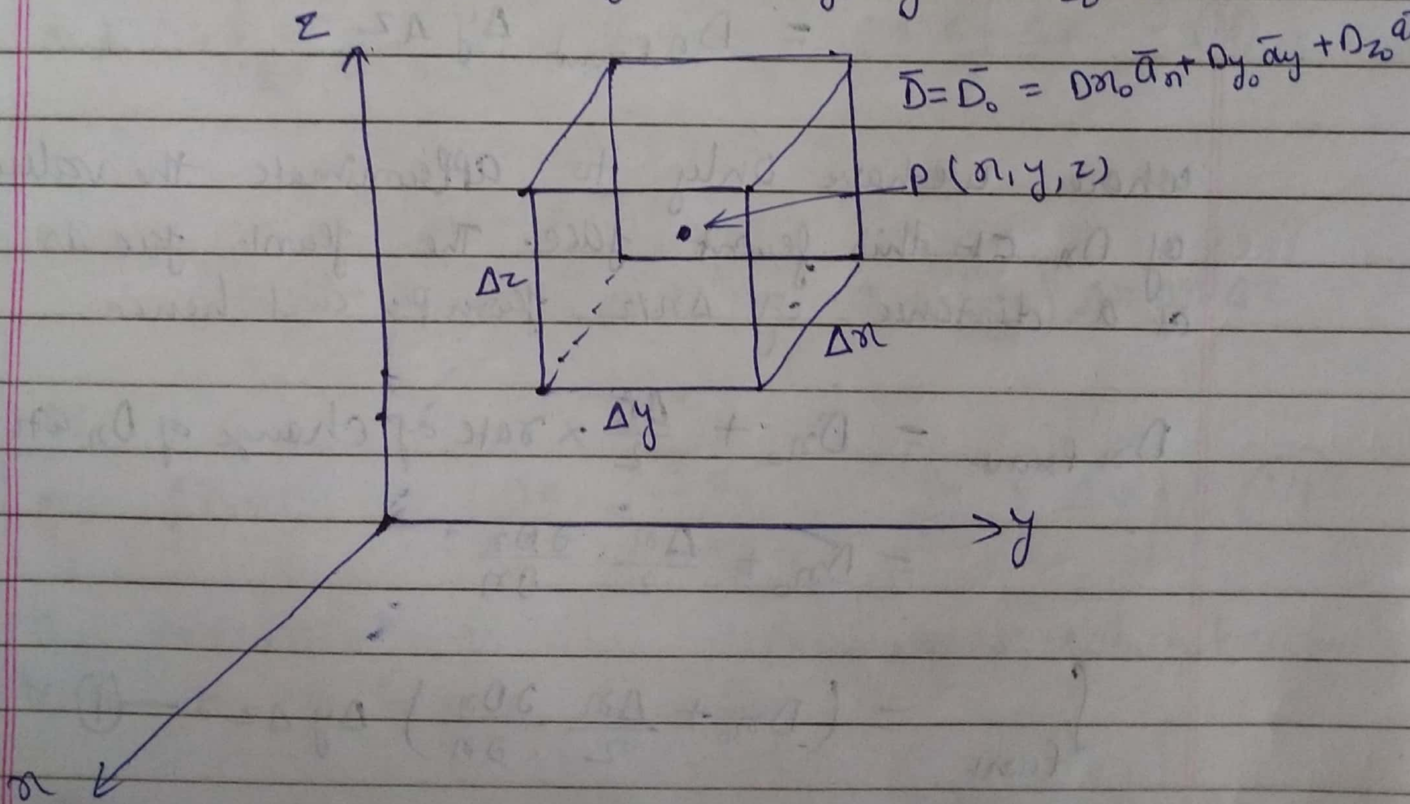


Sketch of $|\bar{D}|$ against r for a uniformly charged sphere

☆ Differential volume element ⇒

Let us consider any Point P, shown in fig, located by a cartesian coordinate system. The value $\bar{D}$ at the Point P may be expressed as -

$$\bar{D}_0 = D_{x_0}\,\bar{a}_x + D_{y_0}\,\bar{a}_y + D_{z_0}\,\bar{a}_z$$



we Choose as our closed surface the small rectangular box, centered at P, having sides of length Δx, Δy and Δz and apply Gauss's law.-

$$\oint \bar{D} \cdot d\bar{S} = Q$$

$$\oint \bar{D} \cdot d\bar{S} = \int_{front} + \int_{back} + \int_{left} + \int_{Right} + \int_{top} + \int_{bottom}$$

Consider the first of these in detail. Since the surface element is very small, $\bar{D}$ is essentially constant.

$$\int_{front} = \bar{D}_{front} \cdot \overline{dS}_{front}$$

$$= \bar{D}_{front} \cdot \Delta y \Delta z \, \bar{a}_x$$

$$= D_{x\,front} \Delta y \Delta z$$

where we have only to approximate the value of D_x at this front face. The front face is at a distance of $\Delta x/2$ from P, and hence

$$D_{x\,front} = D_{x_0} + \frac{\Delta x}{2} \times \text{rate of change of } D_x \text{ with } x$$

$$= D_{x_0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}$$

$$\int_{front} = \left(D_{x_0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right) \Delta y \Delta z \quad \text{--- (1)}$$

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Now, consider -

$$\int_{back} = \bar{D}_{back} \cdot d\bar{S}_{back} = \bar{D}_{back} \cdot (-\Delta y \, \Delta z \, \bar{a}_x)$$

$$= -D_{x\,back} \, \Delta y \, \Delta z$$

and $D_{x\,back} = D_{x_0} - \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}$

$$\int_{back} = \left(-D_{x_0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}\right) \Delta y \, \Delta z \quad \text{— (II)}$$

$$\int_{front} + \int_{back} = \frac{\partial D_x}{\partial x} \Delta x \, \Delta y \, \Delta z \quad \text{— (III)}$$

Similarly

$$\int_{right} + \int_{left} = \frac{\partial D_y}{\partial y} \Delta x \, \Delta y \, \Delta z \quad \text{— (IV)}$$

and $$\int_{top} + \int_{bottom} = \frac{\partial D_z}{\partial z} \Delta x \, \Delta y \, \Delta z \quad \text{— (V)}$$

hence

$$\oint \bar{D} \cdot d\bar{S} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \Delta x \, \Delta y \, \Delta z$$

or

$$\boxed{\phi = \oint \bar{D} \cdot d\bar{S} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \Delta V} \quad \text{(VI)}$$

The expression is an approximation which become better as ΔV becomes smaller.

Charge enclosed in volume $\Delta V = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \times \Delta V$

This leads to the concept of divergence

Divergence ⇒

Applying Gauss's law to the diff. volume element, we have obtained the relation,

$$Q = \left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \Delta V \quad \text{—(I)}$$

This is the charge enclosed in the volume ΔV,

But $Q = \oint_S \bar{D} \cdot d\bar{S}$ By Gauss's law

—(II)

by (I) & (II)

$$\oint \bar{D} \cdot d\bar{S} = \lim_{V \to 0} \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \Delta V$$

$$\therefore \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \lim_{\Delta V \to 0} \frac{Q}{\Delta V} = \lim_{\Delta V \to 0} \frac{\oint \bar{D} \cdot d\bar{S}}{\Delta V}$$

—(III)

Thus in general if $\bar{A}$ is any vector say force, velocity, temp. gradient etc then,

$$\boxed{\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \lim_{V \to 0} \frac{\oint \bar{A} \cdot d\bar{S}}{\Delta V}} \quad \text{—(IV)}$$

This mathematical operation on $\bar{A}$ is called a divergence.

$$\boxed{\operatorname{div} \bar{A} = \lim_{\Delta V \to 0} \frac{\oint_S \bar{A} \cdot d\bar{S}}{\Delta V} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

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Physical meaning of divergence ⇒

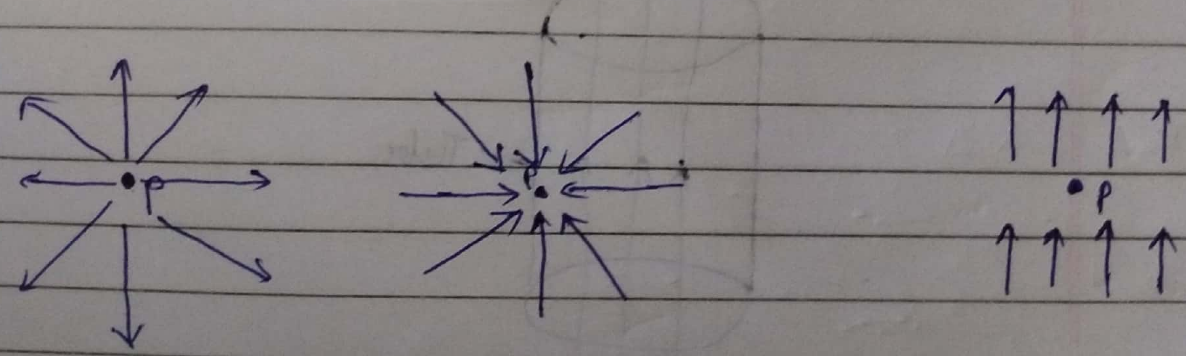
Let $\bar{A}$ be the flux desity vector then,
The divergence of the vector flux density $\bar{A}$ is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero.

Hence the divergence of $\bar{A}$ at a given point is a measure of how much the field represented by $\bar{A}$ diverges or converges from that point.

→ if the field is diverging at point P of vector field $\bar{A}$ shown in fig (a), then divergence of $\bar{A}$ at point P is positive. the field is spreading out point P.

→ if the field is converging at point P as shown in fig (b), then div $\bar{A}$ at point P is negative.

→ if the field at point P is shown in fig (c) so whatever field is converging, same is diverging then the div $\bar{A}$ at point P is zero.



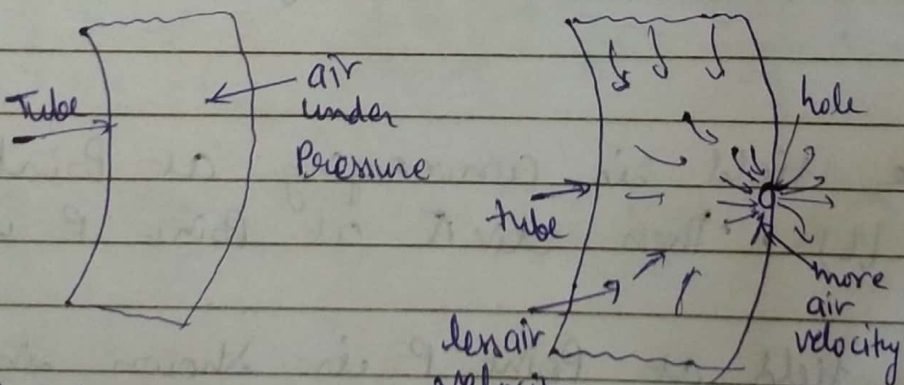
(a) Positive (b) Negative (c) Zero.

Practically consider a tube of a ~~vehicle~~ vehicle in which air is filled at a Pressure. if it is Punctured, then air inside tries to rush out from a tube through a small hole.

Thus the velocity of air at the hole is greatest while away from the hole it is less.

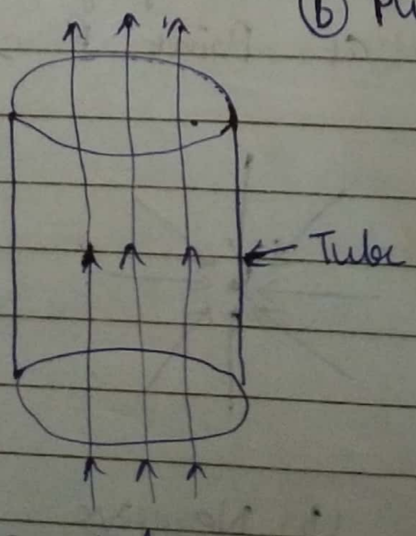
if now any closed surface is considered inside the tube, at one end velocity field is less while from other end it has higher value, as air rushes towards the hole.

Hence the divergence of such velocity inside is Positive. As shown in fig (a) and (b)



ⓐ tube

ⓑ Punctured tube



(C) Hollow tube

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if there is a hollow tube open from both ends the air enters from one end and passes through the tube and leaves from other end. this shown in fig (c). The velocity of air is constant everywhere inside the tube. In this case the divergence of the velocity field is zero, inside the tube.

→ Positive divergence of any vector quantity indicates a source of that vector quantity at that point.
→ Negative divergence of any vector quantity indicates sink of that vector quantity at that point.
→ Zero divergence indicates there is no source or sink exists at that point.

Maxwell's first equation (Electrostatics) ⇒

The divergence of electric flux density $\bar{D}$ is given by.

$$\operatorname{div} \bar{D} = \lim_{\Delta V \to 0} \frac{\oint_S \bar{D} \cdot d\bar{S}}{\Delta V} \quad — (I)$$

According to Gauss's law, we know that -

$$\psi = Q = \oint_S \bar{D} \cdot d\bar{S} \quad — (II)$$

expressing Gauss's law per unit volume basis, and limit $\Delta V \to 0$

$$\lim_{\Delta V \to 0} \frac{Q}{\Delta V} = \lim_{\Delta V \to 0} \frac{\oint \bar{D} \cdot d\bar{S}}{\Delta V} \quad — (III)$$

but $\lim_{\Delta V \to 0} \frac{Q}{\Delta V} = \rho_v$ at that Point (IV)

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So
$$\lim_{\Delta v \to 0} \frac{\oint_s \bar{D} \cdot d\bar{s}}{\Delta v} = \rho_v \quad \text{and}$$

$$\text{div } \bar{D} = \rho_v$$

i.e
$$\boxed{\nabla \cdot \bar{D} = \rho_v} \quad \text{--- (V)}$$

This equation is called Maxwell's first equation applied to electrostatics.
This is also called the Point form of Gauss's law or Gauss's law in differential form.

Divergence theorem ⇒

from Gauss's law,

$$q = \oint_S \bar{D} \cdot d\bar{s} \quad \text{— (I)}$$

while charge enclosed in a volume is given by

$$q = \int_V \rho_v \, dV \quad \text{— (II)}$$

But according to Gauss's law in point form.

$$\nabla \cdot \bar{D} = \rho_v \quad \text{— (III)}$$

So -

$$q = \int_V (\nabla \cdot \bar{D}) \, dV \quad \text{— (IV)}$$

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equating (I) & (IV) we get-

$$\boxed{\oint \bar{D} \cdot d\bar{s} = \int_v (\nabla \cdot \bar{D})\, dv} \quad -(V)$$

This equation is called divergence theorem

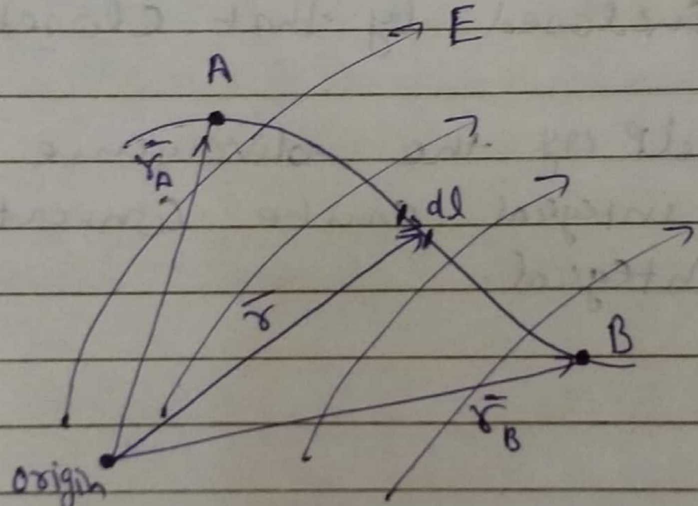
it states that - " The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by that closed surface."

with the help of the divergence theorem, the Surface integral canbe converted into a Volume integral.

Electric Potential ⇒

Another way of obtaining $\bar{E}$ is from the electric scalar Potential V, This way of finding $\bar{E}$ is easier because it is easier to handle scalars than vectors.

Suppose we wish to move a Point charge Q from Point A to Point B in an electric field $\bar{E}$ as shown in fig.



Displacement of Point charge Q in an electrostatic field $\bar{E}$

From Coulomb's law, the force on Q is $\bar{F}$ and

$$\bar{F} = Q\bar{E} \quad -(1)$$

So that the work done in displacing the charge by dl is

$$dW = -\bar{F}\cdot d\bar{l} = -Q\bar{E}\cdot d\bar{l} \quad -(11)$$

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The negative sign indicates that the work is being done by an external agent. Thus the total work done, or Potential energy required, in moving q from A to B, is -

$$\boxed{W = -q\int_A^B \bar{E}\cdot d\bar{l}} \quad \text{— (III)}$$

dividing W by q in eq. (III), gives the Potential energy Per unit charge. This quantity denoted by V_{AB}, and is known as Potential diff. b/w Point A & B thus.

$$\boxed{V_{AB} = \frac{W}{q} = -\int_A^B \bar{E}\cdot d\bar{l}} \quad \text{— (IV)}$$

Note that -

1- In determining V_{AB}, A is initial Point while B is the final Point.

2. if V_{AB} is negative, there is a loss in Potential energy in moving q from A to B; this implies that the work is being done by the field. However, if V_{AB} is Positive, there is a gain in Potential energy in the movement; an external agent Performs the work.

3. V_{AB} is independent of the Path.

4. V_{AB} is measured in Joules/Coulomb, Commonly reffered to as Volts (V)

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⇒ Now, for example-
if the $\bar{E}$ field in given fig is due to a Point charge Q located at the origin, then

$$\bar{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\bar{a}_r \quad \text{—(V)}$$

So from eq. (IV)

$$V_{AB} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\epsilon_0 r^2}\,\bar{a}_r \cdot dr\,\bar{a}_r$$

$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

$$= \frac{Q}{4\pi\epsilon_0 r_B} - \frac{Q}{4\pi\epsilon_0 r_A}$$

$$V_{AB} = V_B - V_A \quad \text{—— (VI)}$$

Where V_B and V_A are Potential (or <u>absolute Potential</u>) at B and A respectively.

Thus the Potential difference V_{AB} may be regarded as the Potential at B with reference to A.
In Problembs involving Point charges, it is customary to choose infinity as reference that is, we assume the Potential at infinity is zero.
Thus if $V_A = 0$ as $r_A \to \infty$, the Potential at Point B ($r_B \to r$) due to Point charge located at origin is

$$\boxed{V = \frac{q}{4\pi\epsilon_0 r}} \quad \text{— (VII)}$$

⇒ Vectors whose line integral does not depend on the path of integration ~~can~~ are called conservative. E is conservative.

⇒ By assuming zero potential at ∞, the potential at a distance r from the point charge is the workdone per unit charge by an external agent in transferring a test charge from infinity to that point. Thus

$$V = -\int_{\infty}^{r} \bar{E} \cdot d\bar{l}$$

if the point charge q in (VII) is not located at the origin but at a point whose position vector is r' the potential V(r) at r becomes

$$V(r) = \frac{q}{4\pi\epsilon_0 |\bar{r} - r'|} \quad \text{— (VIII)}$$

Here we have we have considered electric potential due to point charge. The same basic ideas apply to other type of charge distribution

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$$\rightarrow V(r) = \frac{Q_1}{4\pi\epsilon_0|\bar{r}-\bar{r}_1|} + \frac{Q_2}{4\pi\epsilon_0|\bar{r}-\bar{r}_2|} + \cdots + \frac{Q_n}{4\pi\epsilon_0|\bar{r}-\bar{r}_n|}$$

$$V(r) = \frac{1}{4\pi\epsilon_0}\sum_{k=1}^{n}\frac{Q_k}{|\bar{r}-\bar{r}_k|}$$ (Point charge) — (IX)

$$V(r) = \frac{1}{4\pi\epsilon_0}\int_L \frac{\rho_L(r')\,dl'}{|r-r'|}$$ (line charge) — (X)

$$V(r) = \frac{1}{4\pi\epsilon_0}\int_S \frac{\rho_s(r')\,ds'}{|\bar{r}-\bar{r}'|}$$ (surface charge) — (XI)

$$V(r) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho_v(r')\,dv'}{|\bar{r}-\bar{r}'|}$$ (volume charge) — (XII)

The following Points should be noted -

1. We recall that in obtaining eq. (VIII) to (XII), the zero Potential (reference) Point has been choosen arbitrarily to be at infinity. If any other Point is chosen as ref., eq. (VIII) becomes

$$V = \frac{Q}{4\pi\epsilon_0 r} + C \quad \text{— (XIII)}$$

C = Constant and is determined at the chosen Point of ref.

2. The Potential at a Point can be determined in two ways depending on whether the charge distribution or $\bar{E}$ is known. If charge

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distribution is known, we simply use one of eq. from (VIII) to (XII).

if $\bar{E}$ is known we use

$$V = -\int \bar{E} \cdot d\bar{l} + C$$

the potential diff V_{AB} can be found generally from

$$V_{AB} = V_B - V_A = -\int_A^B \bar{E} \cdot d\bar{l} = \frac{W}{q}$$

Relationship b/w $\bar{E}$ and V ⇒
(maxwell's equation)

As the Potential diff. b/w Points A & B is independent of Path taken. Hence

$$V_{BA} = -V_{AB}$$

that is $V_{BA} + V_{AB} = \oint \bar{E} \cdot d\bar{l} = 0$

or $$\boxed{\oint \bar{E} \cdot d\bar{l} = 0} \quad \text{--- (1)}$$

This shows that the line integral of $\bar{E}$ along a closed Path as shown in fig must be zero.

Physically, this implies that no work ~~done~~ is done in moving a charge along a closed Path in an electrostatic field.

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applying Stoke's theorem to eq. ① we get

$$\oint_L \bar{E} \cdot d\bar{l} = \int_S (\nabla \times \bar{E}) \cdot dS = 0$$

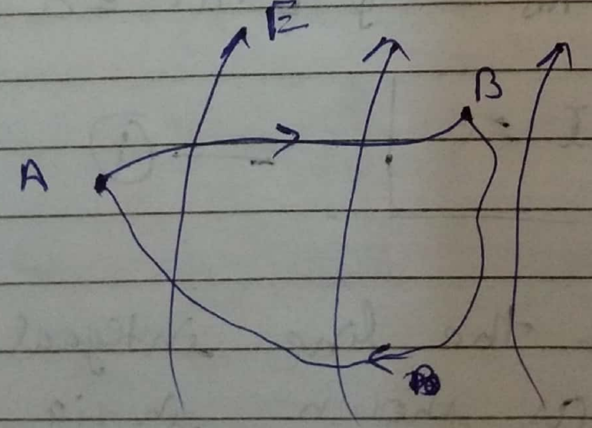
$$\boxed{\nabla \times \bar{E} = 0} \quad \text{— (11)}$$

Any vector field that satisfies eq. ① or (11) is said to be conservative or irrotational.

or → vector whose line integral does not depend on the path of integration are called conservative vectors.

Thus an electrostatic field is a conservative field.

→ eq. ① & (11) is reffered to as maxwell's second equation for static fields.
eq. ① is integral form and (11) is differential form



conservative nature of an electrostatic field.

$$V = -\int \bar{E} \cdot d\bar{l}$$

$$dV = -\bar{E} \cdot d\bar{l} = -E_x dx - E_y dy - E_z dz$$

But from calculus of multivariables, a total change in $V(x,y,z)$ is the sum of partial changes with respect to x, y, z variables.

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$

Comparing two expression for dV, we obtain

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}$$

thus

$$\boxed{\bar{E} = -\nabla V}$$

That is, the electric field intensity is the gradient of V. The negative sign shows that the direction of $\bar{E}$ is opposite to the direction in which V increases; $\bar{E}$ is directed from higher to lower levels of V.

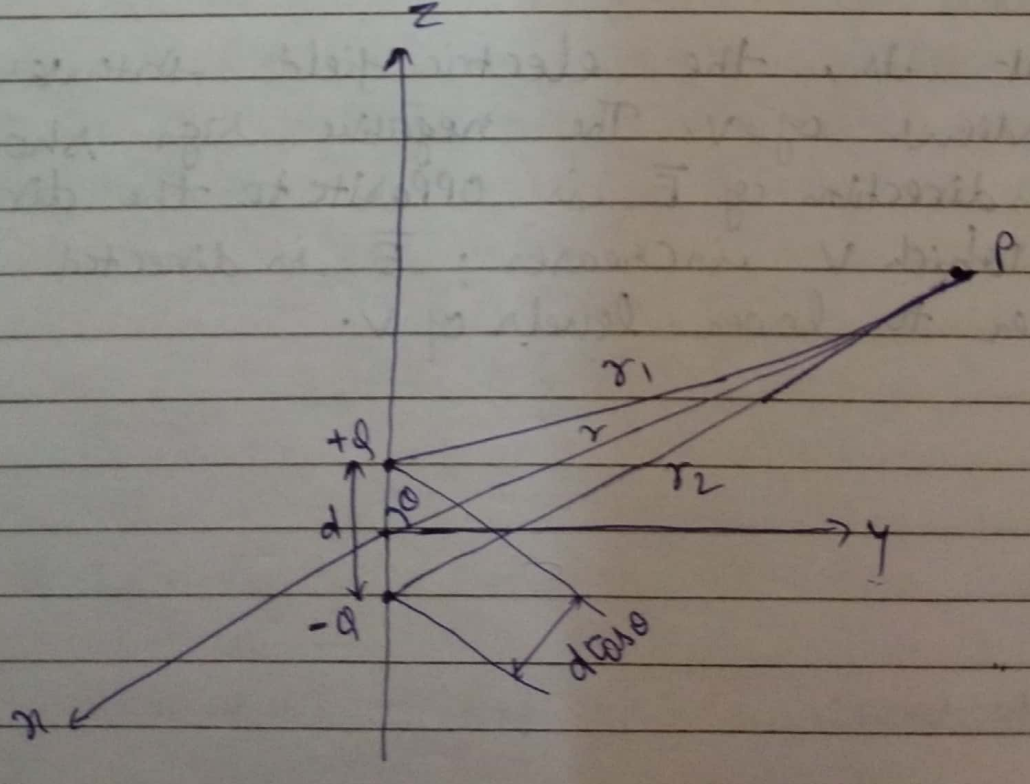
An Electric dipole and flux lines ⇒

An electric dipole is formed when two point charges of equal magnitude but opposite sign are seperated by a small distance.

Consider a dipole shown in fig. The Potential at Point $P(r, \theta, \phi)$ is given by

$$V = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] = \frac{Q}{4\pi\epsilon_0}\left[\frac{r_2 - r_1}{r_1 r_2}\right] \quad \text{--- (1)}$$

where r_1 and r_2 are the distance b/w P and +Q and P and −Q, respectively. If $r >> d$, $r_2 - r_1 \simeq d\cos\theta$, $r_2 r_1 \simeq r^2$



An electric dipole.

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So $V = \frac{Q}{4\pi\epsilon_0} \frac{d\cos\theta}{r^2}$ — (II)

since $d\cos\theta = \bar{d}\cdot\bar{a}_r$, where $\bar{d} = d\,\bar{a}_z$,

if we define $\bar{P} = Q\bar{d}$ — (III) as the dipole moment, eq. (II) may be written as-

$$V = \frac{\bar{P}\cdot\bar{a}_r}{4\pi\epsilon_0 r^2} \quad \text{— (IV)}$$

note that the dipole moment $\bar{P}$ is directed from $-Q$ to $+Q$. if the dipole center is not at origin but at r' eq. (IV) becomes

$$\boxed{V(r) = \frac{P\cdot(\bar{r}-\bar{r}')}{4\pi\epsilon_0 |\bar{r}-\bar{r}'|^3}} \quad \text{— (V)}$$

The electric field due to the dipole with center at the origin, shown in fig, can be obtained radially from eq.

$$\bar{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial\theta}\bar{a}_\theta\right]$$

$$\bar{E} = -\left[\frac{\partial}{\partial r}\left(\frac{Qd\cos\theta}{4\pi\epsilon_0 r^2}\right)\bar{a}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{Qd\cos\theta}{4\pi\epsilon_0 r^2}\right)\bar{a}_\theta\right]$$

$$= \frac{Qd\cos\theta}{4\pi\epsilon_0 r^3}\bar{a}_r + \frac{Qd\sin\theta}{4\pi\epsilon_0 r^3}\bar{a}_\theta$$

or

$$\boxed{\bar{E} = \frac{P}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\bar{a}_r + \sin\theta\,\bar{a}_\theta\right)} \quad \text{— (VI)}$$

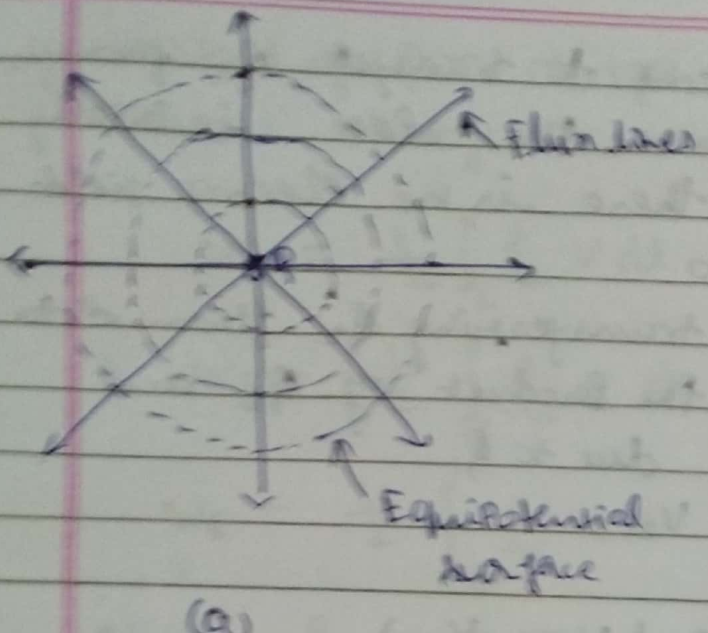
where $P = |\bar{P}| = Qd$

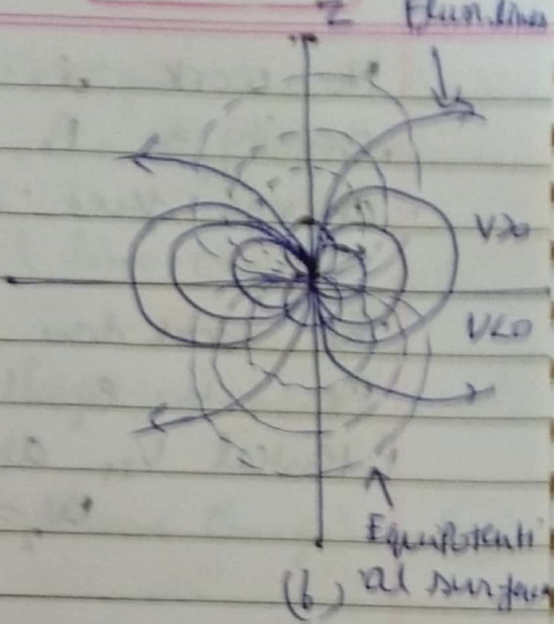
→ Notice that a Point charge is a monopole and it's electric field varies inversely as r^2 while it's Potential field varies inversely as r. Electric field due to a dipole varies inversely as r^3, while it's Potential varies inversely as r^2. The electric fields due to successive higher-order multi Poles (such as a quadrupole consisting of two dipoles or an octupole consisting of two quadrupoles) vary inversely as r^4, r^5, r^6 --- while their corresponding Potential vary inversely as r^3, r^4, r^5, ----..

→ Any surface on which the potential is the same throughout is known as an equipotential surface. The intersection of an equipotential surface and a plane results in a Path of line known as an equipotential line.
No work is done in moving a charge from one Point to another along an equipotential line or surface ($V_A - V_B = 0$) and hence

$$\int_L \bar{E} \cdot d\bar{l} = 0$$

on the line or surface.
Flux lines are always normal to equipotential surfaces.
ex. by equipotential surfaces for Point charge and a dipole are shown in fig

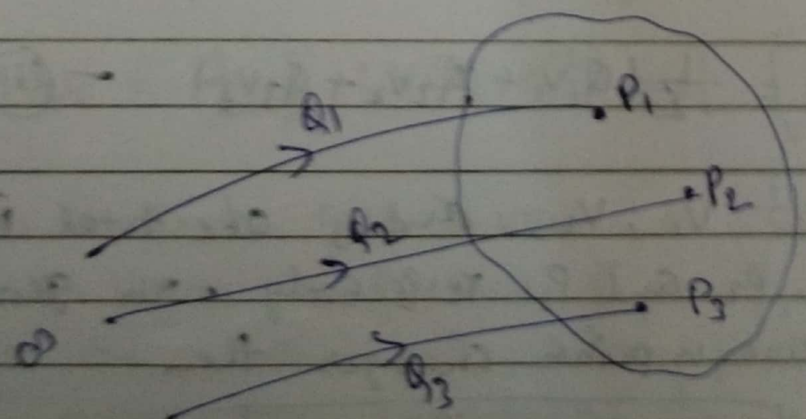




Equipotential surfaces for (a) Point charge and (b) an electric field dipole.

Energy density in Electrostatic field ⇒

To determine the energy present in an assembly of charges, we must first determine the amount of work necessary to assemble them. Suppose we wish to position three point charges Q_1, Q_2, and Q_3 in an initially empty space shown in fig.



Assembling of charges.

No work is required to transfer q_1 from infinity to P_1 because the space is initially charge free and there is no electric field.

$$W_1 = 0$$

The work done in transferring q_2 from ∞ to P_2 is equal to the product of q_2 and potential V_{21} at P_2 due to q_1

$$W_2 = q_2 V_{21}$$

Similarly $\quad W_3 = q_3(V_{31} + V_{32})$

$$W_E = W_1 + W_2 + W_3$$
$$= 0 + q_2 V_{21} + q_3(V_{31} + V_{32}) \quad \text{— (I)}$$

if the charges were positioned in reverse order

$$W_E = W_3 + W_2 + W_1$$
$$= 0 + q_2 V_{23} + q_1(V_{12} + V_{13}) \quad \text{— (II)}$$

by (I) + (II)

$$2W_E = q_1(V_{12} + V_{13}) + q_2(V_{21} + V_{23}) + q_3(V_{31} + V_{32})$$

$$= q_1 V_1 + q_2 V_2 + q_3 V_3$$

or

$$W_E = \frac{1}{2}(q_1 V_1 + q_2 V_2 + q_3 V_3) \quad \text{— (III)}$$

where V_1, V_2, and V_3 are total potentials at P_1, P_2 and P_3 respectively. In general if there are n point charges then

$$\boxed{W_E = \frac{1}{2}\sum_{k=1}^{n} q_k V_k} \quad \text{— (IV)}$$

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If instead of point charges, the region has a continuous charge distribution, the eq. (IV) becomes

$$W_E = \frac{1}{2}\int_L \rho_L V\, dl \quad \text{(line charge)} \quad -(V)$$

$$W_E = \frac{1}{2}\int_S \rho_S V\, ds \quad \text{(surface charge)} \quad -(VI)$$

$$W_E = \frac{1}{2}\int_V \rho_V V\, dv \quad \text{(volume charge)} \quad -(VII)$$

since $\rho_v = \nabla \cdot \bar{D}$

$$W_E = \frac{1}{2}\int_V (\nabla \cdot \bar{D}) V \, dv \quad -(VIII)$$

But for any vector $\bar{A}$ and scalar V, the identity

$$\nabla \cdot V\bar{A} = \bar{A} \cdot \nabla V + V(\nabla \cdot \bar{A})$$

or $$(\nabla \cdot \bar{A})V = \nabla \cdot V\bar{A} - \bar{A} \cdot \nabla V$$

$$W_E = \frac{1}{2}\int_V (\nabla \cdot V\bar{D})\, dv - \frac{1}{2}\int_V (\bar{D} \cdot \nabla V)\, dv \quad -(IX)$$

By applying divergence theorem to the first term on the right-hand side of this equation, we have

$$W_E = \frac{1}{2}\oint_S (V\bar{D}) \cdot d\bar{S} - \frac{1}{2}\int_V (\bar{D} \cdot \nabla V)\, dv$$

As V varies as $\frac{1}{r}$ and $\bar{D}$ varies as $\frac{1}{r^2}$ for point-charges; V varies as $\frac{1}{r^2}$ and $\bar{D}$ as $\frac{1}{r^3}$ for dipoles, and so on. Hence $V\bar{D}$ in the first term on RHS must vary atleast as $\frac{1}{r^3}$ while ds varies as r^2. Consequently, the first integral in above eq. must tends to zero as surface S becomes large.

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hence $W_E = -\frac{1}{2}\int_V (\bar{D}\cdot\nabla V)\,dv = \frac{1}{2}\int_V (\bar{D}\cdot\bar{E})\,dv$ — (X)

as $\bar{E} = -\nabla V$ and $\bar{D} = \epsilon_0\bar{E}$

$$\boxed{W_E = \frac{1}{2}\int_V \bar{D}\cdot\bar{E}\,dv = \frac{1}{2}\int_V \epsilon_0 E^2\,dv} \quad \text{— (XI)}$$

from this we can define electrostatics energy density w_E (in J/m^3) as

$$w_E = \frac{dW_E}{dv} = \frac{1}{2}\bar{D}\cdot\bar{E} = \frac{1}{2}\epsilon_0 E^2 = \frac{D^2}{2\epsilon_0}$$

eq. (X) is written as

$$W_E = \int_V w_E\,dv \quad \text{— (XII)}$$

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EXAMPLE 3.7

Determine **D** at (4, 0, 3) if there is a point charge -5π mC at (4, 0, 0) and a line charge 3π mC/m along the *y*-axis.

Solution:

Let $\mathbf{D} = \mathbf{D}_Q + \mathbf{D}_L$, where $\mathbf{D}_Q$ and $\mathbf{D}_L$ are flux densities due to the point charge and line charge, respectively, as shown in Figure 3.11:

$$\mathbf{D}_Q = \varepsilon_o \mathbf{E} = \frac{Q}{4\pi R^2}\,\mathbf{a}_R = \frac{Q\,(\mathbf{r} - \mathbf{r}')}{4\pi|\mathbf{r} - \mathbf{r}'|^3}$$

where $\mathbf{r} - \mathbf{r}' = (4, 0, 3) - (4, 0, 0) = (0, 0, 3)$. Hence,

$$\mathbf{D}_Q = \frac{-5\pi \cdot 10^{-3}(0, 0, 3)}{4\pi|(0, 0, 3)|^3} = -0.138\,\mathbf{a}_z \text{ mC/m}^2$$

Also

$$\mathbf{D}_L = \frac{\rho_L}{2\pi\rho}\,\mathbf{a}_\rho$$

In this case

$$\mathbf{a}_\rho = \frac{(4, 0, 3) - (0, 0, 0)}{|(4, 0, 3) - (0, 0, 0)|} = \frac{(4, 0, 3)}{5}$$

$$\rho = |(4, 0, 3) - (0, 0, 0)| = 5$$

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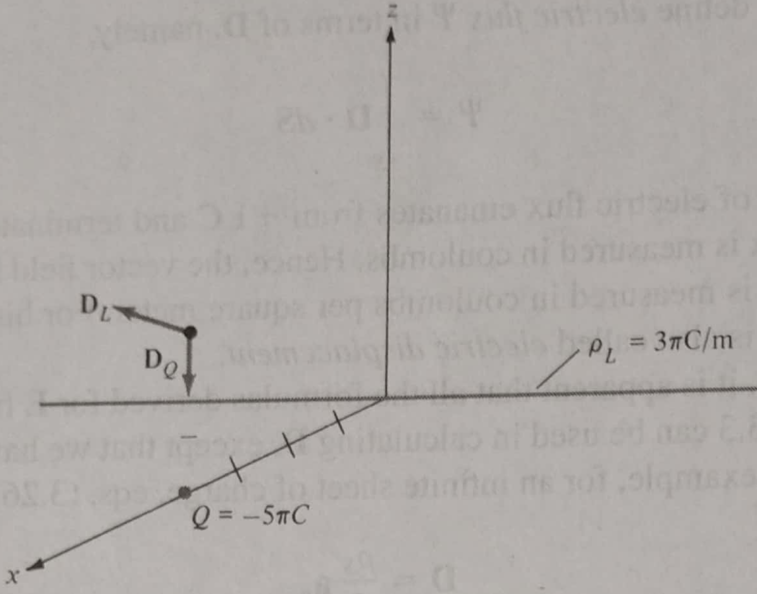

Figure 3.11 Flux density **D** due to a point charge and an infinite line charge.

Hence,

$$\mathbf{D}_L = \frac{3\pi}{2\pi(25)}(4\mathbf{a}_x + 3\mathbf{a}_z) = 0.24\mathbf{a}_x + 0.18\mathbf{a}_z \text{ mC/m}^2$$

Thus

$$\mathbf{D} = \mathbf{D}_Q + \mathbf{D}_L$$

$$= 240\mathbf{a}_x + 42\mathbf{a}_z \ \mu\text{C/m}^2$$

PRACTICE EXERCISE 3.7

A point charge of 30 nC is located at the origin, while plane $y = 3$ carries charge 10 nC/m^2. Find **D** at (0, 4, 3).

Answer: $5.076\mathbf{a}_y + 0.0573\mathbf{a}_z$ nC/m^2.

ρ_v

EXAMPLE 3.8

Given that $\mathbf{D} = z\rho \cos^2\phi \, \mathbf{a}_z$ C/m^2, calculate the charge density at $(1, \pi/4, 3)$ and the total charge enclosed by the cylinder of radius 1 m with $-2 \leq z \leq 2$ m.

Solution:

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{\partial D_z}{\partial z} = \rho \cos^2 \phi$$

At $(1, \pi/4, 3)$, $\rho_v = 1 \cdot \cos^2(\pi/4) = 0.5$ C/m^3. The total charge enclosed by the cylinder can be found in two different ways.

Method 1: This method is based directly on the definition of the total volume charge.

$$Q = \int_v \rho_v \, dv = \int_v \rho \cos^2 \phi \, \underbrace{\rho \, d\phi \, d\rho \, dz}_{dS}$$

$$= \int_{z=-2}^{2} dz \int_{\phi=0}^{2\pi} \cos^2 \phi \, d\phi \int_{\rho=0}^{1} \rho^2 \, d\rho = 4(\pi)(1/3)$$

$$= \frac{4\pi}{3} \text{ C}$$

Method 2: Alternatively, we can use Gauss's law

$$Q = \Psi = \oint \mathbf{D} \cdot d\mathbf{S} = \left[\int_s + \int_t + \int_b \right] \mathbf{D} \cdot d\mathbf{S}$$

$$= \Psi_s + \Psi_t + \Psi_b$$

where Ψ_s, Ψ_t, and Ψ_b are the flux through the sides, the top surface, and the bottom surface of the cylinder, respectively (see Figure 3.17). Since $\mathbf{D}$ does not have component along $\mathbf{a}_\rho$, $\Psi_s = 0$, for Ψ_t, $d\mathbf{S} = \rho \, d\phi \, d\rho \, \mathbf{a}_z$ so

$$\Psi_t = \int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} z\rho \cos^2 \phi \, \rho \, d\phi \, d\rho \bigg|_{z=2} = 2 \int_0^1 \rho^2 d\rho \int_0^{2\pi} \cos^2 \phi \, d\phi$$

$$= 2\left(\frac{1}{3}\right)\pi = \frac{2\pi}{3}$$

and for Ψ_b, $d\mathbf{S} = -\rho \, d\phi \, d\rho \, \mathbf{a}_z$, so

$$\Psi_b = -\int_{\rho=0}^{1} \int_{\phi=0}^{2\pi} z\rho \cos^2 \phi \, \rho \, d\phi \, d\rho \bigg|_{z=-2} = 2 \int_0^1 \rho^2 \, d\rho \int_0^{2\pi} \cos^2 \phi \, d\phi$$

$$= \frac{2\pi}{3}$$

Thus

$$Q = \Psi = 0 + \frac{2\pi}{3} + \frac{2\pi}{3} = \frac{4\pi}{3} \text{ C}$$

as obtained earlier.

PRACTICE EXERCISE 3.8

$4x$ -4 C/m³

If $\mathbf{D} = (2y^2 + z)\mathbf{a}_x + 4xy\mathbf{a}_y + x\mathbf{a}_z$ C/m², find

$\nabla =$

(a) The volume charge density at $(-1, 0, 3)$

(b) The flux through the cube defined by $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$

(c) The total charge enclosed by the cube $(2y^2 + x)\,dx\,dy$

Answer: (a) -4 C/m³, (b) 2 C, (c) 2 C.

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EXAMPLE 3.9

A charge distribution with spherical symmetry has density

$$\rho_v = \begin{cases} \dfrac{\rho_o r}{R}, & 0 \leq r \leq R \\ 0, & r > R \end{cases}$$

Determine **E** everywhere.

Solution:

The charge distribution is similar to that in Figure 3.16. Since symmetry exists, we can apply Gauss's law to find **E**.

$$\varepsilon_o \oint_S \mathbf{E} \cdot d\mathbf{S} = Q_{\text{enc}} = \int_v \rho_v \, dv$$

(a) For $r < R$

symmetrical property

$$\varepsilon_o E_r \, 4\pi r^2 = Q_{\text{enc}} = \int_0^r \int_0^\pi \int_0^{2\pi} \rho_v \, r^2 \sin\theta \, d\phi \, d\theta \, dr$$

$$= \int_0^r 4\pi r^2 \frac{\rho_o r}{R} \, dr = \frac{\rho_o \pi r^4}{R}$$

or

$$\mathbf{E} = \frac{\rho_o r^2}{4\varepsilon_o R} \mathbf{a}_r$$

(b) For $r > R$,

$$\varepsilon_o E_r 4\pi r^2 = Q_{\text{enc}} = \int_0^r \int_0^\pi \int_0^{2\pi} \rho_v r^2 \sin\theta \, d\phi \, d\theta \, dr$$

$$= \int_0^R \frac{\rho_o r}{R} 4\pi r^2 \, dr + \int_R^r 0 \cdot 4\pi r^2 \, dr$$

$$= \pi \rho_o R^3$$

or

$$\mathbf{E} = \frac{\rho_o R^3}{4\varepsilon_o r^2} \mathbf{a}_r$$

PRACTICE EXERCISE 3.9

A charge distribution in free space has $\rho_v = 2r \text{ nC/m}^3$ for $0 \leq r \leq 10$ m and zero otherwise. Determine **E** at $r = 2$ m and $r = 12$ m.

Answer: $226\mathbf{a}_r$ V/m, $3.927\mathbf{a}_r$ kV/m.

EXAMPLE 3.10

Two point charges $-4\ \mu C$ and $5\ \mu C$ are located at $(2, -1, 3)$ and $(0, 4, -2)$, respectively. Find the potential at $(1, 0, 1)$, assuming zero potential at infinity.

Solution:

Let

$$Q_1 = -4\ \mu C, \qquad Q_2 = 5\ \mu C$$

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_1|} + \frac{Q_2}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_2|} + C_o$$

If $V(\infty) = 0$, $C_o = 0$,

$$|\mathbf{r} - \mathbf{r}_1| = |(1, 0, 1) - (2, -1, 3)| = |(-1, 1, -2)| = \sqrt{6}$$

$$|\mathbf{r} - \mathbf{r}_2| = |(1, 0, 1) - (0, 4, -2)| = |(1, -4, 3)| = \sqrt{26}$$

Hence

$$V(1, 0, 1) = \frac{10^{-6}}{4\pi \times \dfrac{10^{-9}}{36\pi}}\left[\frac{-4}{\sqrt{6}} + \frac{5}{\sqrt{26}}\right]$$

$$= 9 \times 10^3 (-1.633 + 0.9806)$$

$$= -5.872\ \text{kV}$$

PRACTICE EXERCISE 3.10

If point charge $3\ \mu C$ is located at the origin in addition to the two charges of Example 3.10, find the potential at $(-1, 5, 2)$, assuming $V(\infty) = 0$.

Answer: 10.23 kV.

EXAMPLE 3.11

A point charge of 5 nC is located at $(-3, 4, 0)$, while line $y = 1$, $z = 1$ carries uniform charge 2 nC/m.

(a) If $V = 0$ V at $O(0, 0, 0)$, find V at $A(5, 0, 1)$.
(b) If $V = 100$ V at $B(1, 2, 1)$, find V at $C(-2, 5, 3)$.
(c) If $V = -5$ V at O, find V_{BC}.

Solution:

Let the potential at any point be

$$V = V_Q + V_L$$

where V_Q and V_L are the contributions to V at that point due to the point charge and the line charge, respectively. For the point charge,

$$V_Q = -\int \mathbf{E} \cdot d\mathbf{l} = -\int \frac{Q}{4\pi\varepsilon_o r^2} \mathbf{a}_r \cdot dr\, \mathbf{a}_r$$
$$= \frac{Q}{4\pi\varepsilon_o r} + C_1$$

For the infinite line charge,

$$V_L = -\int \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\rho_L}{2\pi\varepsilon_o \rho} \mathbf{a}_\rho \cdot d\rho\, \mathbf{a}_\rho$$
$$= -\frac{\rho_L}{2\pi\varepsilon_o} \ln \rho + C_2$$

Hence,

$$V = -\frac{\rho_L}{2\pi\varepsilon_o} \ln \rho + \frac{Q}{4\pi\varepsilon_o r} + C$$

where $C = C_1 + C_2 =$ constant, ρ is the perpendicular distance from the line $y = 1$, $z = 1$ to the field point, and r is the distance from the point charge to the field point.

(a) If $V = 0$ at $O(0, 0, 0)$, and V at $A(5, 0, 1)$ is to be determined, we must first determine the values of ρ and r at O and A. Finding r is easy; we use eq. (2.31). To find ρ for any point (x, y, z), we utilize the fact that ρ is the perpendicular distance from (x, y, z) to line $y = 1$, $z = 1$, which is parallel to the x-axis. Hence ρ is the distance between (x, y, z) and $(x, 1, 1)$ because the distance vector between the two points is perpendicular to $\mathbf{a}_x$. Thus

$$\rho = |(x, y, z) - (x, 1, 1)| = \sqrt{(y-1)^2 + (z-1)^2}$$

Applying this for ρ and eq. (2.31) for r at points O and A, we obtain

$$\rho_O = |(0, 0, 0) - (0, 1, 1)| = \sqrt{2}$$
$$r_O = |(0, 0, 0) - (-3, 4, 0)| = 5$$
$$\rho_A = |(5, 0, 1) - (5, 1, 1)| = 1$$
$$r_A = |(5, 0, 1) - (-3, 4, 0)| = 9$$

Hence,

$$V_O - V_A = -\frac{\rho_L}{2\pi\varepsilon_o} \ln \frac{\rho_O}{\rho_A} + \frac{Q}{4\pi\varepsilon_o} \left[\frac{1}{r_O} - \frac{1}{r_A}\right]$$
$$= \frac{-2 \cdot 10^{-9}}{2\pi \cdot \frac{10^{-9}}{36\pi}} \ln \frac{\sqrt{2}}{1} + \frac{5 \cdot 10^{-9}}{4\pi \cdot \frac{10^{-9}}{36\pi}} \left[\frac{1}{5} - \frac{1}{9}\right]$$

$$0 - V_A = -36 \ln \sqrt{2} + 45 \left(\frac{1}{5} - \frac{1}{9}\right)$$

or

$$V_A = 36 \ln \sqrt{2} - 4 = 8.477 \text{ V}$$

Notice that we have avoided calculating the constant C by subtracting one potential from another and that it does not matter which one is subtracted from which.

(b) If $V = 100$ at $B(1, 2, 1)$ and V at $C(-2, 5, 3)$ is to be determined, we find

$$\rho_B = |(1, 2, 1) - (1, 1, 1)| = 1$$
$$r_B = |(1, 2, 1) - (-3, 4, 0)| = \sqrt{21}$$
$$\rho_C = |(-2, 5, 3) - (-2, 1, 1)| = \sqrt{20}$$
$$r_C = |(-2, 5, 3) - (-3, 4, 0)| = \sqrt{11}$$
$$V_C - V_B = -\frac{\rho_L}{2\pi\varepsilon_o} \ln \frac{\rho_C}{\rho_B} + \frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{r_C} - \frac{1}{r_B}\right]$$
$$V_C - 100 = -36 \ln \frac{\sqrt{20}}{1} + 45 \cdot \left[\frac{1}{\sqrt{11}} - \frac{1}{\sqrt{21}}\right]$$
$$= -50.175 \text{ V}$$

or

$$V_C = 49.825 \text{ V}$$

(c) To find the potential difference between two points, we do not need a potential reference if a common reference is assumed.

$$V_{BC} = V_C - V_B = 49.825 - 100$$
$$= -50.175 \text{ V}$$

as obtained in part (b).

PRACTICE EXERCISE 3.11

A point charge of 5 nC is located at the origin. If $V = 2$ V at $(0, 6, -8)$, find

(a) The potential at $A(-3, 2, 6)$

(b) The potential at $B(1, 5, 7)$

(c) The potential difference V_{AB}

Answer: (a) 3.929 V, (b) 2.696 V, (c) −1.233 V.

EXAMPLE 3.12 Given the potential $V = \frac{10}{r^2} \sin\theta \cos\phi$,

(a) Find the electric flux density **D** at $(2, \pi/2, 0)$.

(b) Calculate the work done in moving a 10 μC charge from point $A(1, 30°, 120°)$ to $B(4, 90°, 60°)$.

Solution:

(a) $\mathbf{D} = \varepsilon_o \mathbf{E}$

But

$$\mathbf{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi\right]$$

$$= \frac{20}{r^3}\sin\theta\cos\phi\,\mathbf{a}_r - \frac{10}{r^3}\cos\theta\cos\phi\,\mathbf{a}_\theta + \frac{10}{r^3}\sin\phi\,\mathbf{a}_\phi$$

$\varepsilon_0 = 8.854 \times 10^{-}$

At $(2, \pi/2, 0)$,

$$\mathbf{D} = \varepsilon_o \mathbf{E}\,(r = 2, \theta = \pi/2, \phi = 0) = \varepsilon_o\left(\frac{20}{8}\mathbf{a}_r - 0\mathbf{a}_\theta + 0\mathbf{a}_\phi\right)$$

$$= 2.5\varepsilon_o \mathbf{a}_r \text{ C/m}^2 = 22.1\ \mathbf{a}_r \text{ pC/m}^2$$

(b) The work done can be found in two ways, using either **E** or V.

Method 1:

$$W = -Q \int_L \mathbf{E} \cdot d\mathbf{l} \quad \text{or} \quad -\frac{W}{Q} = \int_L \mathbf{E} \cdot d\mathbf{l}$$

not imp
Unimportant
Understand the line

and because the electrostatic field is conservative, the path of integration is immaterial. Hence the work done in moving Q from $A(1, 30°, 120°)$ to $B(4, 90°, 60°)$ is the same as that in moving Q from A to A', from A' to B', and from ~~B' to~~ B, where

$$\begin{array}{ccc} A(1, 30°, 120°) & & B(4, 90°, 60°) \\ \downarrow\ d\mathbf{l} = dr\,\mathbf{a}_r & d\mathbf{l} = r\,d\theta\,\mathbf{a}_\theta & \uparrow\ d\mathbf{l} = r\sin\theta\,d\phi\,\mathbf{a}_\phi \\ A'(4, 30°, 120°) & \rightarrow & B'(4, 90°, 120°) \end{array}$$

first in r direction
there in θ
there in ϕ

That is, instead of being moved directly from A and B, Q is moved from $A \rightarrow A'$, $A' \rightarrow B'$, $B' \rightarrow B$, so that only one variable is changed at a time. This makes the line integral much easier to evaluate. Thus

$$\frac{-W}{Q} = -\frac{1}{Q}(W_{AA'} + W_{A'B'} + W_{B'B})$$

$$= \left(\int_{AA'} + \int_{A'B'} + \int_{B'B}\right) \mathbf{E} \cdot d\mathbf{l}$$

$\phi \le \theta$

$$= \int_{r=1}^{4} \frac{20\sin\theta\cos\phi}{r^3}\,dr\,\bigg|_{\theta=30°,\ \phi=120°}$$

$$+ \int_{\theta=30°}^{90°} \frac{-10\cos\theta\cos\phi}{r^3}\,r\,d\theta\,\bigg|_{r=4,\ \phi=120°}$$

$-\nabla V$

$$+ \int_{\phi=120°}^{60°} \frac{10\sin\phi}{r^3}\,r\sin\theta\,d\phi\,\bigg|_{r=4,\ \theta=90°}$$

$$= 20\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left[-\frac{1}{2r^2}\bigg|_{r=1}^{4}\right]$$

$$-\frac{10}{16}\frac{(-1)}{2}\sin\theta\bigg|_{30°}^{90°} + \frac{10}{16}(1)\left[-\cos\phi\bigg|_{120°}^{60°}\right]$$

$$-\frac{W}{Q} = \frac{-75}{32} + \frac{5}{32} - \frac{10}{16}$$

or

$$W = \frac{45}{16} Q = 28.125 \ \mu\text{J}$$

Method 2:

Since V is known, this method is much easier.

$$\begin{aligned} W &= -Q \int_A^B \mathbf{E} \cdot d\mathbf{l} = QV_{AB} \\ &= Q(V_B - V_A) \\ &= 10 \left(\frac{10}{16} \sin 90^\circ \cos 60^\circ - \frac{10}{1} \sin 30^\circ \cos 120^\circ \right) \cdot 10^{-6} \\ &= 10 \left(\frac{10}{32} - \frac{-5}{2} \right) \cdot 10^{-6} \\ &= 28.125 \ \mu\text{J as obtained before} \end{aligned}$$

PRACTICE EXERCISE 3.12

Given that $\mathbf{E} = (3x^2 + y)\mathbf{a}_x + x\mathbf{a}_y$ kV/m, find the work done in moving a $-2\ \mu$C charge from (0, 5, 0) to (2, −1, 0) by taking the straight-line path

(a) $(0, 5, 0) \rightarrow (2, 5, 0) \rightarrow (2, -1, 0)$

(b) $y = 5 - 3x$

Answer: (a) 12 mJ, (b) 12 mJ.

AMPLE 3.13

Two dipoles with dipole moments $-5\mathbf{a}_z$ nC/m and $9\mathbf{a}_z$ nC/m are located at points $(0, 0, -2)$ and $(0, 0, 3)$, respectively. Find the potential at the origin.

Solution:

$$V = \sum_{k=1}^{2} \frac{\mathbf{p}_k \cdot \mathbf{r}_k}{4\pi\varepsilon_o r_k^3} = \frac{1}{4\pi\varepsilon_o}\left[\frac{\mathbf{p}_1 \cdot \mathbf{r}_1}{r_1^3} + \frac{\mathbf{p}_2 \cdot \mathbf{r}_2}{r_2^3}\right]$$

where

$$\mathbf{p}_1 = -5\mathbf{a}_z, \quad \mathbf{r}_1 = (0,0,0) - (0,0,-2) = 2\mathbf{a}_z, \quad r_1 = |\mathbf{r}_1| = 2$$

$$\mathbf{p}_2 = 9\mathbf{a}_z, \quad \mathbf{r}_2 = (0,0,0) - (0,0,3) = -3\mathbf{a}_z, \quad r_2 = |\mathbf{r}_2| = 3$$

Hence,

$$V = \frac{1}{4\pi \cdot \dfrac{10^{-9}}{36\pi}}\left[\frac{-10}{2^3} - \frac{27}{3^3}\right] \cdot 10^{-9} = -20.25 \text{ V}$$

PRACTICE EXERCISE 3.13

An electric dipole of $100\ \mathbf{a}_z$ pC · m is located at the origin. Find V and $\mathbf{E}$ at points

(a) $(0, 0, 10)$

(b) $(1, \pi/3, \pi/2)$

Answer: (a) 9 mV, $1.8\mathbf{a}_r$ mV/m, (b) 0.45 V, $0.9\mathbf{a}_r + 0.7794\mathbf{a}_\theta$ V/m.

EXAMPLE 3.14

The point charges -1 nC, 4 nC, and 3 nC are located at (0, 0, 0), (0, 0, 1), and (1, 0, 0), respectively. Find the energy in the system.

Solution:

$$W = W_1 + W_2 + W_3$$

$$= 0 + Q_2V_{21} + Q_3(V_{31} + V_{32})$$

$$= Q_2 \cdot \frac{Q_1}{4\pi\varepsilon_o\,|(0, 0, 1) - (0, 0, 0)|}$$

$$+ \frac{Q_3}{4\pi\varepsilon_o}\left[\frac{Q_1}{|(1,0,0) - (0,0,0)|} + \frac{Q_2}{|(1,0,0) - (0,0,1)|}\right]$$

$$= \frac{1}{4\pi\varepsilon_o}\left(Q_1Q_2 + Q_1Q_3 + \frac{Q_2Q_3}{\sqrt{2}}\right)$$

$$= \frac{1}{4\pi \cdot \dfrac{10^{-9}}{36\pi}}\left(-4 - 3 + \frac{12}{\sqrt{2}}\right) \cdot 10^{-18}$$

$$= 9\left(\frac{12}{\sqrt{2}} - 7\right)\text{ nJ} = 13.37\text{ nJ}$$

Alternatively,

$$W = \frac{1}{2}\sum_{k=1}^{3} Q_kV_k = \frac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3)$$

$$= \frac{Q_1}{2}\left[\frac{Q_2}{4\pi\varepsilon_o(1)} + \frac{Q_3}{4\pi\varepsilon_o(1)}\right] + \frac{Q_2}{2}\left[\frac{Q_1}{4\pi\varepsilon_o(1)} + \frac{Q_3}{4\pi\varepsilon_o(\sqrt{2})}\right]$$

$$+ \frac{Q_3}{2}\left[\frac{Q_1}{4\pi\varepsilon_o(1)} + \frac{Q_2}{4\pi\varepsilon_o(\sqrt{2})}\right]$$

$$= \frac{1}{4\pi\varepsilon_o}\left(Q_1Q_2 + Q_1Q_3 + \frac{Q_2Q_3}{\sqrt{2}}\right)$$

$$= 9\left(\frac{12}{\sqrt{2}} - 7\right)\text{ nJ} = 13.37\text{ nJ}$$

as obtained in the first solution.

PRACTICE EXERCISE 3.14

Point charges $Q_1 = 1$ nC, $Q_2 = -2$ nC, $Q_3 = 3$ nC, and $Q_4 = -4$ nC are positioned one at a time and in that order at (0, 0, 0), (1, 0, 0), (0, 0, −1), and (0, 0, 1), respectively. Calculate the energy in the system after each charge is positioned.

Answer: 0, −18 nJ, −29.18 nJ, −68.27 nJ.

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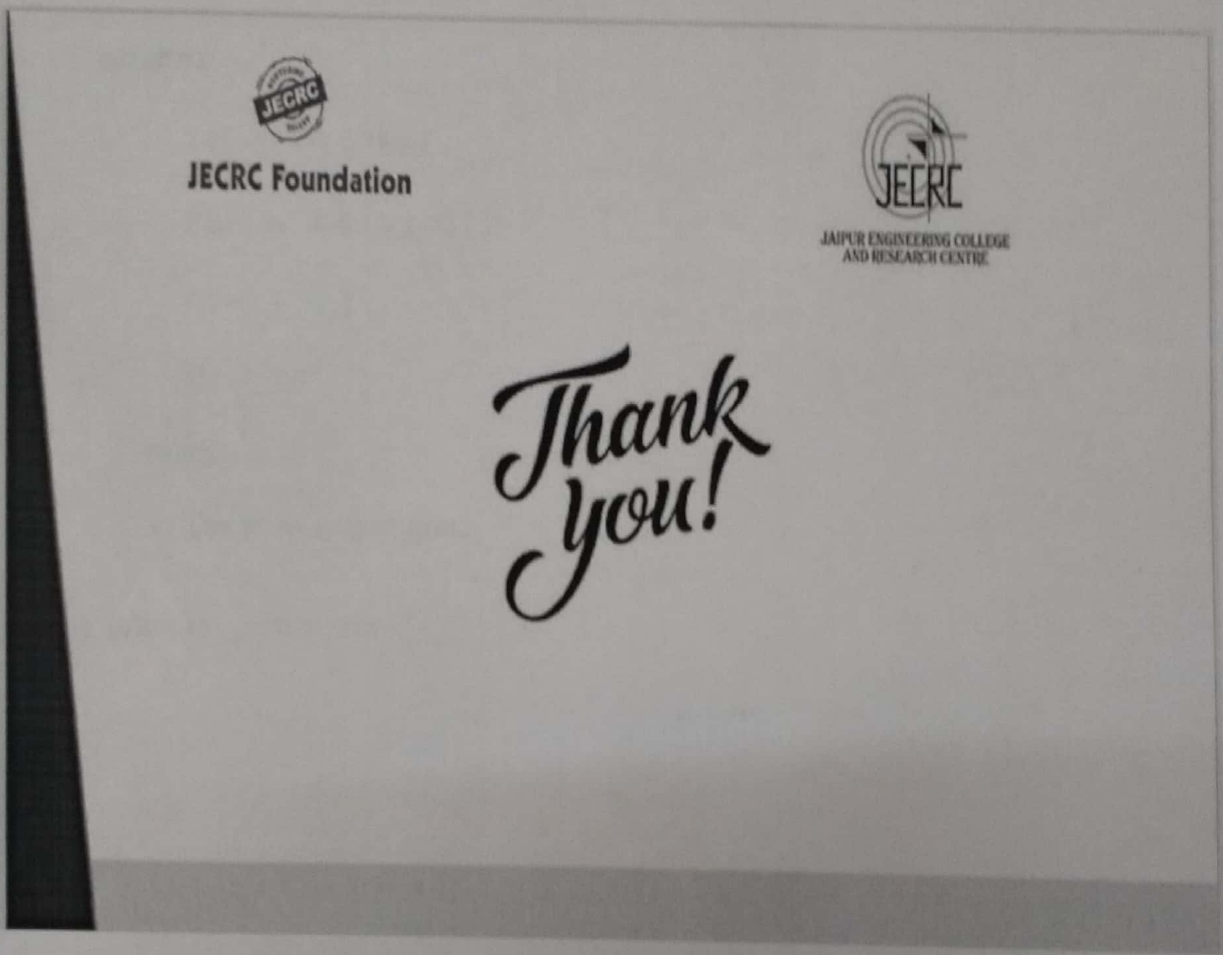
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