Examples with solutions

Q.1 *Given* $\bar{A} = 5\bar{a}_x$ *and* $\bar{B} = 4\bar{a}_x + B_y\bar{a}_y$ *then find* B_y *such that angle between* $\bar{A}$ *and* $\bar{B}$ *is* 45°. *If* $\bar{B}$ *also has a term* $B_z\,\bar{a}_z$, *what relationship must exist between* B_y *and* B_z ?

Solution : $\bar{A} = 5\,\bar{a}_x$ and $\bar{B} = 4\,\bar{a}_x + B_y\,\bar{a}_y$, $\theta_{AB} = 45°$

Now $\quad \bar{A}\cdot\bar{B} = A_x B_x + A_y B_y + A_z B_z$

$$= (5\times 4)+(0)+(0) = 20$$

But $\quad \bar{A}\cdot\bar{B} = |\bar{A}|\;|\bar{B}|\cos\theta_{AB}$

$$\therefore \quad 20 = \sqrt{(5)^2}\times\sqrt{(4)^2+(B_y)^2}\times\cos 45°$$

$$\therefore \quad \sqrt{16+B_y^2} = 5.6568$$

$$\therefore \quad B_y^2 = 16$$

$$\therefore \quad B_y = \pm 4$$

Now $\quad \bar{B} = 4\bar{a}_x + B_y\,\bar{a}_y + B_z\,\bar{a}_z$

Still $\quad \bar{A}\cdot\bar{B} = 20$

$$\therefore \quad 20 = \sqrt{(5)^2}\times\sqrt{(4)^2+(B_y)^2+(B_z)^2}\times\cos 45°$$

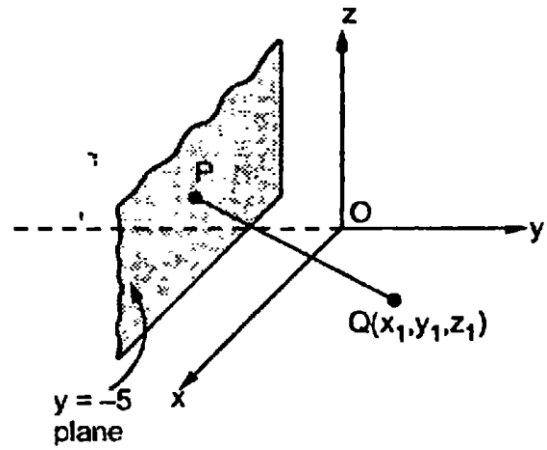
$$\therefore \quad \sqrt{16+B_y^2+B_z^2} = 5.6568$$

$$\therefore \quad B_y^2 + B_z^2 = 16$$

This is the required relation between B_y and B_z.

Q.2 – *Find the unit vector directed towards the point* (x_1, y_1, z_1) *from an arbitrary point in the plane* $y = -5$.

Solution : The plane y = – 5 is parallel to xz plane as shown in the Fig. 1.53.



The coordinates of point P are (x, – 5, z) as $y = -5$ is constant. While Q is arbitrary point having co-ordinates (x_1, y_1, z_1). To find unit vector along the direction PQ.

$$\bar{a}_{PQ} = \frac{\overline{PQ}}{|\overline{PQ}|}$$

where $\overline{PQ} = \overline{Q} - \overline{P}$

$$\overline{PQ} = (x_1 - x)\bar{a}_x + (y_1 - (-5))\bar{a}_y + (z_1 - z)\bar{a}_z$$

$$\therefore \quad |\overline{PQ}| = \sqrt{(x_1 - x)^2 + (y_1 + 5)^2 + (z_1 - z)^2}$$

$$\therefore \quad \bar{a}_{PQ} = \frac{(x_1 - x)\,\bar{a}_x + (y_1 + 5)\,\bar{a}_y + (z_1 - z)\,\bar{a}_z}{\sqrt{(x_1 - x)^2 + (y_1 + 5)^2 + (z_1 - z)^2}}$$

Q.3- *Given points* $P(r = 5, \phi = 60°, z = 2)$ *and* $Q(r = 2, \phi = 110°, z = -1)$ *in cylindrical co-ordinate system. Find*

i) Unit vector in cartesian co-ordinates at P directed towards Q

ii) Unit vector in cylindrical co-ordinates at P directed towards Q.

Solution : Let us obtain the cartesian co-ordinates of P and Q.

It is known that $x = r\cos\phi$, $y = r\sin\phi$ and $z = z$

$\therefore$ P (2.5, 4.33, 2) and Q (– 0.684, 1.8793, – 1)

i) The unit vector from P to Q is,

$$\bar{a}_{PQ} = \frac{\overline{PQ}}{|\overline{PQ}|} = \frac{\overline{Q} - \overline{P}}{|\overline{PQ}|}$$ where $\overline{P}$ and $\overline{Q}$ are position vectors

$$= \frac{(-0.684 - 2.5)\bar{a}_x + (1.8793 - 4.33)\bar{a}_y + (-1 - 2)\bar{a}_z}{|\overline{PQ}|}$$

$$= \frac{-3.184\bar{a}_x - 2.4507\bar{a}_y - 3\bar{a}_z}{\sqrt{(-3.184)^2 + (-2.4507)^2 + (-3)^2}}$$

$$\therefore \qquad \bar{a}_{PQ} = \mathbf{-0.6349\ \bar{a}_x - 0.4887\ \bar{a}_y - 0.5983\ \bar{a}_z}$$

ii) The vector $\overline{PQ} = -3.184\ \bar{a}_x - 2.4507\ \bar{a}_y - 3\bar{a}_z$ ... As obtained earlier.

Let us transform this into cylindrical coordinates.

$$(PQ)_r = \overline{PQ} \cdot \bar{a}_r = -3.184\bar{a}_x \cdot \bar{a}_r - 2.4507\bar{a}_y \cdot \bar{a}_r - 3\bar{a}_z \cdot \bar{a}_r$$

$$= -3.184 \cos\phi - 2.4507(-\sin\phi) + 0 \qquad \text{... Refer Table 1.2}$$

At point P, $\phi = 60°$

$$\therefore \qquad (PQ)_r = -3.184 \times 0.5 - 2.4507(-0.866) = 0.5303$$

$$(PQ)_\phi = \overline{PQ} \cdot \bar{a}_\phi = -3.184\ \bar{a}_x \cdot \bar{a}_\phi - 2.4507\ \bar{a}_y \cdot \bar{a}_\phi - 3\bar{a}_z \cdot \bar{a}_\phi$$

$$= -3.184(-\sin\phi) - 2.4507 \cos\phi$$

$$\therefore \qquad (PQ)_\phi = -3.184(-0.866) - 2.4507 \times 0.5 = 1.5319$$

and $\qquad (PQ)_z = \overline{PQ} \cdot \bar{a}_z = -3 \qquad$ ... $\bar{a}_x \cdot \bar{a}_z = \bar{a}_y \cdot \bar{a}_z = 0$

$$\therefore \qquad \overline{PQ} = 0.5303\ \bar{a}_r + 1.5319\ \bar{a}_\phi - 3\bar{a}_z$$

$$\therefore \qquad \bar{a}_{PQ} = \frac{\overline{PQ}}{|\overline{PQ}|} = \frac{0.5303\bar{a}_r + 1.5319\bar{a}_\phi - 3\bar{a}_z}{\sqrt{(0.5303)^2 + (1.5319)^2 + (-3)^2}}$$

$$= \mathbf{0.155\ \bar{a}_r + 0.449\ \bar{a}_\phi - 0.88\ \bar{a}_z}$$

Q.4-

Find the area of the curved surface using the cylindrical co-ordinates which lies on the right circular cylinder of radius 2 m, height 8 m and $40° \le \phi \le 90°$.

Solution : The surface is shown in the Fig. 1.55.

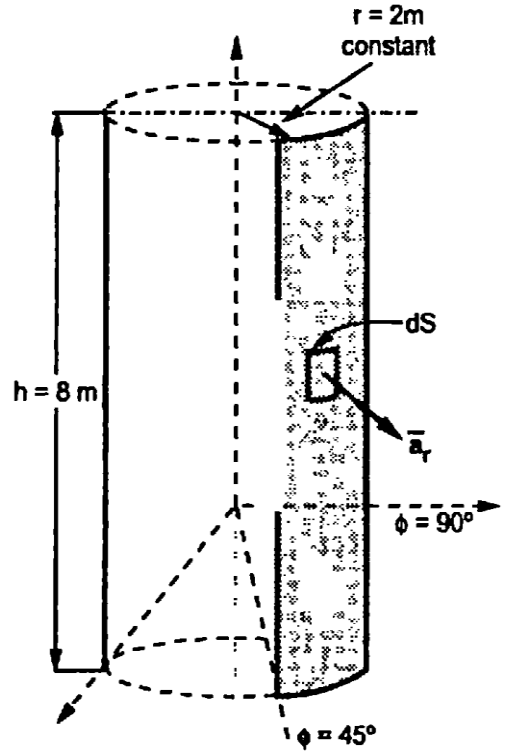


Fig. 1.55

The differential area normal to $\bar{a}_r$ is,

$$d\bar{S} = r\, d\phi\, dz\, \bar{a}_r$$

The surface is constant r surface and normal to it is unit vector $\bar{a}_r$.

$$\therefore \qquad S = \int dS = \iint r\, d\phi\, dz$$

$$= \int_{z=0}^{8} \int_{\phi=45°}^{90°} r\, d\phi\, dz \qquad \ldots r = 2 \text{ m}$$

$$= r\,[\phi]_{45°}^{90°}\,[z]_0^8$$

$$= 2\times[90°-45°]\times\frac{\pi}{180°}\times[8-0] \qquad \ldots\text{Use } \phi \text{ in radians}$$

$$= \frac{2\times 45°\times\pi\times 8}{180°} = \mathbf{12.5663\ m^2}$$

Q.5- *Convert point P (1,3,5) from Cartesian to cylindrical and spherical co-ordinates.*

Solution : P(1, 3, 5) i.e. x = 1, y = 3, z = 5

In cylindrical system

$$r = \sqrt{x^2+y^2} = \sqrt{1+3^2} = 3.1622$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{3}{1} = 71.56°$$

$$z = z = 5$$

$\therefore$ **P(3.1622, 71.56°, 5)** in cylindrical

In spherical system :

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{1^2+3^2+5^2} = 5.916$$

$$\theta = \tan^{-1}\frac{z}{r} = \cos^{-1}\frac{5}{5.916} = 32.31°$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{3}{1} = 71.56°$$

$\therefore$ **P(5.916, 32.31°, 71.56°)** in spherical.

Q.6- *Given a vector function*

$$\overline{A} = (3x+c_1z)\ \overline{a}_x + (c_2x-5z)\ \overline{a}_y + (4x-c_3y+c_4z)\ \overline{a}_z$$

Calculate c_1, c_2, c_3 and c_4 if A is irrotational and solenoidal.

Solution : For $\overline{A}$ to be irrotational, $\nabla\times\overline{A} = 0$

$$\nabla\times\overline{A} = \left[\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right]\overline{a}_x + \left[\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right]\overline{a}_y + \left[\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right]\overline{a}_z$$

$A_x = 3x+c_1z, \quad A_y = c_2x-5z \quad A_z = 4x-c_3y+c_4z$

$\therefore \quad \nabla\times\overline{A} = [-c_3+5]\,\overline{a}_x + [c_1-4]\overline{a}_y + [c_2-0]\overline{a}_z = 0$

$\therefore \quad c_3 = 5, \quad c_1 = 4, \quad c_2 = 0$ ... For $\overline{A}$ to be irrotational

For $\overline{A}$ to be solenoidal, $\nabla\bullet\overline{A} = 0$

$$\therefore \quad \nabla \bullet \overline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0$$

$$\therefore \quad 3 + 0 + c_4 = 0$$

$$\therefore \quad c_4 = -3$$ ... For $\overline{A}$ to be irrotational

Q.7-

Verify that vector field $\overline{A} = yz\,\overline{a}_x + zx\,\overline{a}_y + xy\,\overline{a}_z$ is irrotational and solenoidal.

Solution : For $\overline{A}$ to be irrotational, $\nabla \times \overline{A} = 0$

$$\nabla \times \overline{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\overline{a}_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\overline{a}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\overline{a}_z$$

$A_x = yz$, $A_y = zx$ and $A_z = xy$ ...Given

$$\therefore \frac{\partial A_x}{\partial y} = z, \frac{\partial A_x}{\partial z} = y, \frac{\partial A_y}{\partial x} = z, \frac{\partial A_y}{\partial z} = x, \frac{\partial A_z}{\partial x} = y, \frac{\partial A_z}{\partial y} = x$$

$$\therefore \quad \nabla \times \overline{A} = [x - x]\,\overline{a}_x + [y - y]\,\overline{a}_y + [z - z]\,\overline{a}_z = 0$$

Thus $\overline{A}$ **is irrotational.**

For $\overline{A}$ to be solenoidal, $\nabla \bullet \overline{A} = 0$

$$\nabla \bullet \overline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0$$

$$\frac{\partial A_x}{\partial x} = 0, \quad \frac{\partial A_y}{\partial y} = 0, \quad \frac{\partial A_z}{\partial z} = 0$$

$\therefore \quad \nabla \bullet \overline{A} = 0$ hence $\overline{A}$ **is solenoidal**

Q.8- *If $\overline{A} = \alpha\,\overline{a}_x + 2\,\overline{a}_y + 10\,\overline{a}_z$ and $\overline{B} = 4\alpha\,\overline{a}_x + 8\,\overline{a}_y - 2\alpha\,\overline{a}_z$, find out the value of α for which the two vectors become perpendicular.*

Solution : $\overline{A} = \alpha\,\overline{a}_x + 2\,a_y + 10\,\overline{a}_z$, $\quad \overline{B} = 4\alpha\,\overline{a}_x + 8\,\overline{a}_y - 2\alpha\,\overline{a}_z$

For perpendicular vectors, $\overline{A} \bullet \overline{B} = 0$

$$\therefore (\alpha)(4\alpha) + (2)(8) + 10(-2\alpha) = 0$$

$$\therefore \quad 4\alpha^2 - 20\alpha + 16 = 0$$

$$\therefore \quad \alpha = 4 \text{ or } 1$$

Q.9- *Given points A(x = 2, y = 3, z = − 1) and B(ρ = 4, ϕ = − 50°, z = 2) find the distance A to B.*

Solution : A (x = 2, y = 3, z = − 1), B(ρ = 4, ϕ = − 50°, z = 2)

Converting point B to cartesian system,

$$x = \rho \cos \phi = 4 \cos(-50°) = 2.57115$$

$$y = \rho \sin \phi = 4 \sin(-50°) = -3.0641$$

$$z = z = 2$$

$$\therefore \quad d_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

$$= \sqrt{(2.57115-2)^2 + (-3.0641-3)^2 + [2-(-1)]^2}$$

$$= \sqrt{0.326212 + 36.77331 + 9} = \mathbf{6.7896}$$

Q.10- *Show that the vector fields*

$$\overline{A} = \bar{a}_r \frac{\sin 2\theta}{r^2} + 2\bar{a}_\theta \frac{(\sin\theta)}{r^2} \text{ and}$$

$\overline{B} = r \cos\theta \, \bar{a}_r + r\,\bar{a}_\theta$ *are every where parallel to each other.*

Solution : For parallel vectors, $\overline{A} \times \overline{B} = 0$

Given $\overline{A}$ and $\overline{B}$ are in spherical co-ordinates.

$$\overline{A} \times \overline{B} = \begin{vmatrix} \bar{a}_r & \bar{a}_\theta & \bar{a}_\phi \\ A_r & A_\theta & A_\phi \\ B_r & B_\theta & B_\phi \end{vmatrix}$$

$$A_r = \frac{\sin 2\theta}{r^2}, \quad A_\theta = \frac{2\sin\theta}{r^2}, \quad A_\phi = 0$$

$$B = r\cos\theta, \quad B_\theta = r, \quad B_\phi = 0$$

$$\therefore \quad \overline{A} \times \overline{B} = \bar{a}_r[A_\theta B_\phi - B_\theta A_\phi] - \bar{a}_\theta[A_r B_\phi - B_r A_\phi] + \bar{a}_\phi[A_r B_\theta - B_r A_\theta]$$

$$= 0\,\bar{a}_r - 0\,\bar{a}_\theta + \left[\frac{\sin 2\theta}{r^2} \times r - \frac{r\cos\theta \times 2\sin\theta}{r^2}\right]\bar{a}_\phi$$

$$= \left[\frac{2\sin\theta\cos\theta}{r} - \frac{2\sin\theta\cos\theta}{r}\right]\bar{a}_\phi = 0$$

As $\overline{A} \times \overline{B} = 0$, the two vector fields are parallel to each other.

Q.11-

Express the field $\bar{E} = \frac{A}{r^2}\bar{a}_r$ in (i) rectangular components, ii) cylindrical components.

Solution :

$$\bar{E} = \frac{A\bar{a}_r}{r^2} \qquad \text{...In spherical co-ordinates}$$

i) Spherical to rectangular

$$\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A/r^2 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \qquad E_x = \frac{A}{r^2}\sin\theta\cos\phi, \quad E_y = \frac{A}{r^2}\sin\theta\sin\phi, \quad E_z = \frac{A\cos\theta}{r^2}$$

$$\therefore \qquad \bar{E} = \frac{A}{r^2}\sin\theta\cos\phi\ \bar{a}_x + \frac{A}{r^2}\sin\theta\sin\phi\ \bar{a}_y + \frac{A\cos\theta}{r^2}\bar{a}_z$$

But $$r = \sqrt{x^2+y^2+z^2}, \quad \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}, \quad \phi = \tan^{-1}\frac{y}{x}$$

$$\therefore \qquad \sin\theta = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}, \quad \cos\theta\ \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\sin\phi = \frac{y}{\sqrt{x^2+y^2}}, \quad \cos\phi = \frac{x}{\sqrt{x^2+y^2}}$$

$$\therefore \qquad \bar{E} = \frac{A}{x^2+y^2+z^2}\times\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\times\frac{x}{\sqrt{x^2+y^2}}\bar{a}_x$$

$$+ \frac{A}{x^2+y^2+z^2}\times\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\times\frac{y}{\sqrt{x^2+y^2}}\bar{a}_y$$

$$+ \frac{A}{x^2+y^2+z^2}\times\frac{z}{\sqrt{x^2+y^2+z^2}}\bar{a}_z$$

$$\therefore \qquad \bar{E} = \frac{Ax}{(x^2+y^2+z^2)^{3/2}}\ \bar{a}_x + \frac{Ay}{(x^2+y^2+z^2)^{3/2}}\ \bar{a}_y + \frac{Az}{(x^2+y^2+z^2)^{3/2}}\ \bar{a}_z$$

ii) Spherical to cylindrical

$$\begin{bmatrix} E_\rho \\ E_\phi \\ E_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A/r^2 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \qquad E_\rho = \frac{A\sin\theta}{r^2}, \quad E_\phi = 0, \quad E_z = \frac{A\cos\theta}{r^2}$$

$$r = \sqrt{\rho^2 + z^2}, \quad \theta = \tan^{-1}\frac{\rho}{z}$$

$$\therefore \qquad \sin\theta = \frac{\rho}{\sqrt{\rho^2+z^2}}, \quad \cos\theta = \frac{z}{\sqrt{\rho^2+z^2}}$$

$$\therefore \qquad \bar{E} = \frac{A\rho}{(\rho^2+z^2)^{3/2}}\,\bar{a}_\rho + \frac{Az}{(\rho^2+z^2)^{3/2}}\,\bar{a}_z$$

Q.12- *Find the divergence and curl of the following function :*

$\bar{A} = 2xy\,\bar{a}_x + x^2 z\,\bar{a}_y + z^3\bar{a}_z$

Solution : $\bar{A} = 2xy\,\bar{a}_x + x^2 z\,\bar{a}_y + z^3\bar{a}_z$

$$\nabla\cdot\bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 2y + 0 + 3z^2 = 2y + 3z^2$$

$$\nabla\times\bar{A} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x^2 z & z^3 \end{vmatrix}$$

$$= \bar{a}_x\left[\frac{\partial z^3}{\partial y} - \frac{\partial x^2 z}{\partial z}\right] - \bar{a}_y\left[\frac{\partial z^3}{\partial x} - \frac{\partial 2xy}{\partial z}\right] + \bar{a}_z\left[\frac{\partial x^2 z}{\partial x} - \frac{\partial 2xy}{\partial y}\right]$$

$$= \bar{a}_x[0 - x^2] - \bar{a}_y[0] + \bar{a}_z[2xz - 2x]$$

$$= -x^2\bar{a}_x + 2x(z-1)\bar{a}_z$$