JAIPUR ENGINEERING COLLEGE AND RESEARCH CENTRE

Year & Sem - B.Tech II year, sem- III

Subject - Electromagnetic Fields

Unit - II

Presented by - Ms. Ritu Soni

Designation - Assistant Professor

Department- Electrical Engineering

Vision and Mission of Institute

Vision of institute

To become a renowned centre of outcome based learning, and work towards, professional, cultural and social enrichment of the lives of individuals and communities.

Mission of institute

- **M1**.Focus on evaluation of learning outcomes and motivate students to inculcate research aptitude by project based learning.
- **M2**.Identify ,based on informed perception of Indian, regional and global needs, the areas of focus and provide platform to gain knowledge and solutions.
- **M3**.Offer opportunities for interaction between academia and industry.
- **M4**.Develop human potential to its fullest extent so that intellectually capable and imaginatively gifted leaders may emerge in a range of professions

Vision and Mission of Electrical Engineering Department

Vision of department

The Electrical Engineering department strives to recognized globally for outcome based technical knowledge and produce quality human being who can manage the advance technologies and contribute to society.

Mission Of department

M1. To impart quality technical knowledge to the learners to make them globally competitive Electrical Engineers.

M2. To provide the learners ethical guidelines along with excellent academic environment for a long productive career.

M3. To promote industry- institute relationship.

Syllabus of Electromagnetic fields

unit 1- Review of Vector Calculus

Vector algebra- addition, subtraction, components of vectors, scalar and vector multiplications, triple products, three orthogonal coordinate systems (rectangular, cylindrical and spherical). Vector calculus differentiation, partial differentiation, integration, vector operator del, gradient, divergence and curl; integral theorems of vectors. Conversion of a vector from one coordinate system to another.

Unit 2- Static Electric Field

Coulomb’s law, Electric field intensity, Electrical field due to point charges. Line, Surface and Volume charge distributions. Gauss law and its applications. Absolute Electric potential, Potential difference, Calculation of potential differences for different configurations. Electric dipole, Electrostatic Energy and Energy density.

Unit 3- Conductors, Dielectrics and Capacitance

Current and current density, Ohms Law in Point form, Continuity of current, Boundary conditions of perfect dielectric materials. Permittivity of dielectric materials, Capacitance, Capacitance of a two wire line, Poisson’s equation, Laplace’s equation, Solution of Laplace and Poisson’s equation, Application of Laplace’s and Poisson’s equations.

unit 4- Static Magnetic Fields

Biot-Savart Law, Ampere Law, Magnetic flux and magnetic flux density, Scalar and Vector Magnetic potentials. Steady magnetic fields produced by current carrying conductors.

Unit5- Magnetic Forces, Materials and Inductance

Force on a moving charge, Force on a differential current element, Force between differential current elements, Nature of magnetic materials, Magnetization and permeability, Magnetic boundary conditions, Magnetic circuits, inductances and mutual inductances.

Unit 6- Time Varying Fields and Maxwell's Equations

Faraday's law for Electromagnetic induction, Displacement current, Point form of Maxwell's equation, Integral form of Maxwell's equations, Motional Electromotive forces. Boundary Conditions.

Unit 7- Electromagnetic Waves

Derivation of Wave Equation, Uniform Plane Waves, Maxwell's equation in Phasor form, Wave equation in Phasor form, Plane waves in free space and in a homogenous material. Wave equation for a conducting medium, Plane waves in lossy dielectrics, Propagation in good conductors, Skin effect. Poynting theorem.

Course outcomes for Electromagnetic fields

CO1-Acquire basic understanding of vectors , their representation and conversion in different coordinate systems.

CO2-Able to compute the force, fields & energy of the electrostatic & magneto static fields. Able to analyze the materials, conductors, dielectrics, inductances and capacitances.

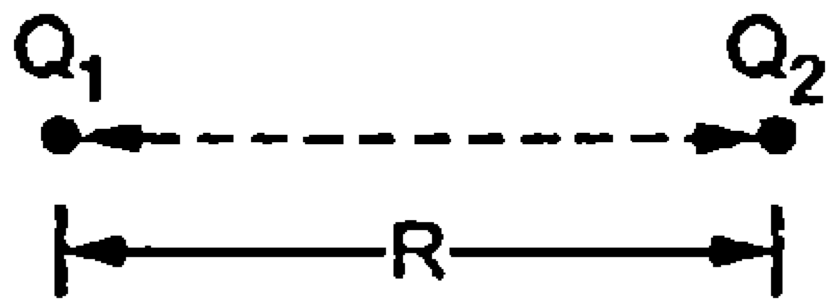
CO3-Understand the concept of time varying field and able to solve electromagnetic relation using Maxwell equations. Also able to analyze the electromagnetic waves.

Unit II Static Electric field

Coulomb's Law

The Coulomb's law states that force between the two point charges Q_1 and Q_2,

1. Acts along the line joining the two point charges.
2. Is directly proportional to the product (Q_1Q_2) of the two charges.
3. Is inversely proportional to the square of the distance between them.



Consider the two point charges Q_1 and Q_2 as shown in the Fig. separated by the distance R. The charge Q_1 exerts a force on Q_2 while Q_2 also exerts a force on Q_1. The force acts along the line joining Q_1 and Q_2. The force exerted between them is repulsive if the charges are of same polarity while it is attractive if the charges are of different polarity.

Mathematically the force F between the charges can be expressed as,

$$F \propto \frac{Q_1 Q_2}{R^2}$$

where $Q_1 Q_2$ = Product of the two charges

R = Distance between the two charges

The Coulomb's law also states that this force depends on the medium in which the point charges are located. The effect of medium is introduced in the equation of force as a constant of proportionality denoted as k.

$$F = k\frac{Q_1 Q_2}{R^2}$$

where k = Constant of proportionality

Constant of proportionality K

The constant of proportionality takes into account the effect of medium, in which charges are located. In the International System of Units (SI), the charges Q_1 and Q_2 are expressed in Coulombs (C), the distance R in metres (m) and the force F in newtons (N). Then to satisfy Coulomb's law, the constant of proportionality is defined as,

$$k = \frac{1}{4\pi\varepsilon}$$

where ε = Permittivity of the medium in which charges are located

The units of ε are farads/metre (F/m).

In general ε is expressed as,

$$\varepsilon = \varepsilon_0 \varepsilon_r$$

where ε_0 = Permittivity of the free space or vacuum

ε_r = Relative permittivity or dielectric constant of the medium with respect to free space

ε = Absolute permittivity

For the free space or vacuum, the relative permittivity $\varepsilon_r = 1$, hence

$$\varepsilon = \varepsilon_0$$

$$\therefore \quad F = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{R^2}$$

The value of permittivity of free space ε_0 is,

$$\boxed{\varepsilon_0 = \frac{1}{36\pi}\times 10^{-9} = 8.854\times 10^{-12} \text{ F/m}}$$

$$\therefore \quad k = \frac{1}{4\pi\varepsilon_0} = \frac{1}{4\pi\times 8.854\times 10^{-12}} = 8.98\times 10^{9} = 9\times 10^{9} \text{ m/F}$$

Hence the Coulomb's law can be expressed as,

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}$$

This is the force between the two point charges located in free space or vacuum.

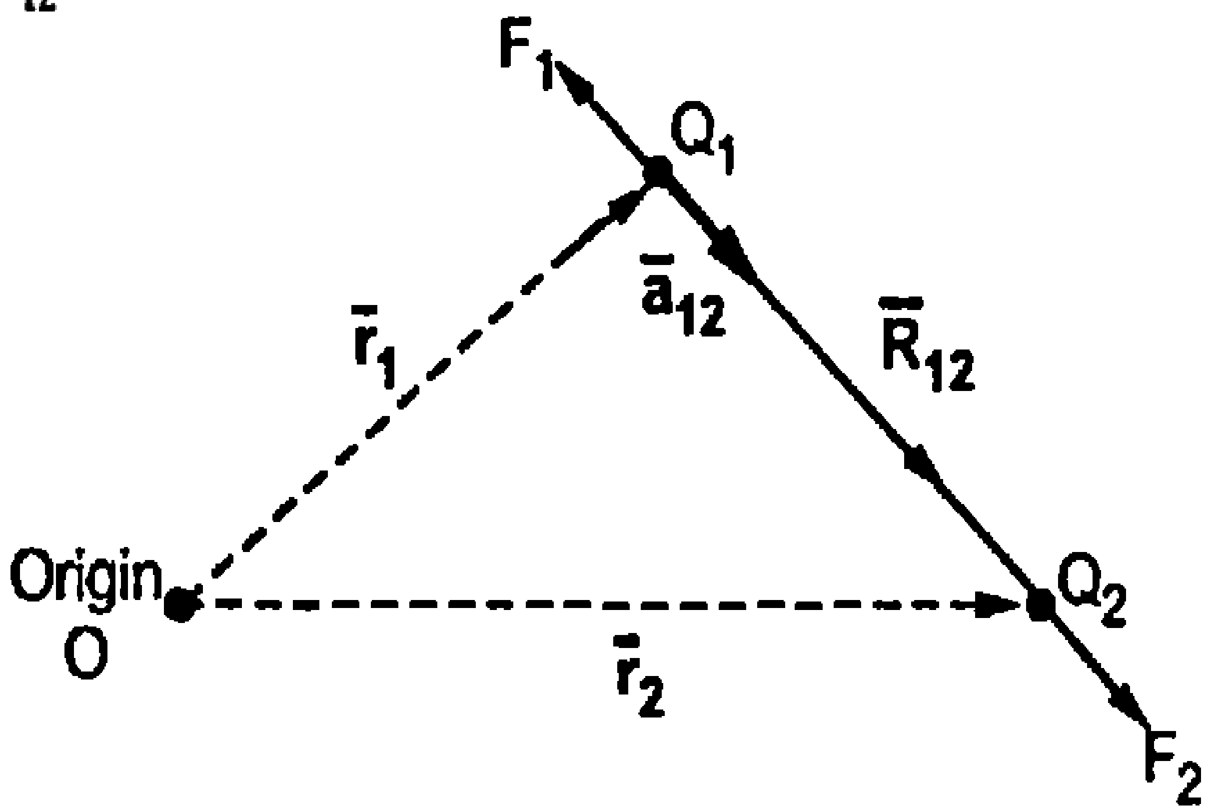
Vector Form of coulomb's Law

The force exerted between the two point charges has a fixed direction which is a straight line joining the two charges. Hence the force exerted between the two charges can be expressed in a vector form.

Consider the two point charges Q_1 and Q_2 located at the points having position vectors $\bar{r}_1$ and $\bar{r}_2$ as shown in the Fig.

Then the force exerted by Q_1 on Q_2 acts along the direction $\bar{R}_{12}$ where $\bar{a}_{12}$ is unit vector along $\bar{R}_{12}$. Hence the force in the vector form can be expressed as,

$$\bar{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\bar{a}_{12}$$



where $\bar{a}_{12}$ = Unit vector along $\bar{R}_{12} = \dfrac{\text{Vector}}{\text{Magnitude of vector}}$

$$\therefore \quad \bar{a}_{12} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|} = \frac{\bar{r}_2 - \bar{r}_1}{|\bar{R}_{12}|} = \frac{\bar{r}_2 - \bar{r}_1}{|\bar{r}_2 - \bar{r}_1|}$$

where $|\bar{R}_{12}|$ = R = distance between the two charges

The following observations are important :

1. As shown in the Fig. the force F_1 is the force exerted on Q_1 due to Q_2. It can be expressed as,

$$\bar{F}_1 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{21}^2}\bar{a}_{21} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{21}^2} \times \frac{\bar{r}_1 - \bar{r}_2}{|\bar{r}_1 - \bar{r}_2|}$$

But $\quad \bar{r}_1 - \bar{r}_2 = -[\bar{r}_2 - \bar{r}_1]$

$\therefore \quad \bar{a}_{21} = -\bar{a}_{12}$

$$\bar{F}_1 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{21}^2}(-\bar{a}_{21}) = -\bar{F}_2$$

Hence force exerted by the two charges on each other is equal but opposite in direction.

2. The like charges repel each other while the unlike charges attract each other.

These are experiement conclusions though not reflected in the mathematical expression.

3. It is necessary that the two charges are the point charges and stationary in nature.

4. The two point charges may be positive or negative. Hence their signs must be considered

5. The Coulomb's law is linear which shows that if any one charge is increased 'n' times then the force exerted also increass by n times.

$\therefore \quad \bar{F}_2 = -\bar{F}_1$ then $n\bar{F}_2 = -n\bar{F}_1$

where n = Scalar

It is also true that the force on a charge in the presence of several other charges is the sum of the forces on that charge due to each of the other charges acting alone.

Example *A charge $Q_1 = -20\ \mu C$ is located at P (– 6, 4, 6) and a charge $Q_2 = 50\ \mu C$ is located at R (5, 8, – 2) in a free space. Find the force exerted on Q_2 by Q_1 in vector form. The distances given are in metres.*

Solution : From the co-ordinates of P and R , the respective position vectors are –

$$\bar{P} = -6\bar{a}_x + 4\bar{a}_y + 6\bar{a}_z \qquad \bar{R} = 5\bar{a}_x + 8\bar{a}_y - 2\bar{a}_z$$

The force on Q_2 is given by,

$$\bar{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\bar{a}_{12}$$

$$\bar{R}_{12} = \bar{R}_{PR} = \bar{R} - \bar{P} = [5-(-6)]\ \bar{a}_x + (8-4)\ \bar{a}_y + [-2-(6)\bar{a}_z]$$

$$= 11\bar{a}_x + 4\bar{a}_y - 8\bar{a}_z$$

$$\therefore \qquad |R_{12}| = \sqrt{(11)^2 + (4)^2 + (-8)^2} = 14.1774$$

$$\bar{a}_{12} = \frac{\bar{R}_{12}}{|\bar{R}_{12}|} = \frac{11\bar{a}_x + 4\bar{a}_y - 8\bar{a}_z}{14.1774} = 0.7758\ \bar{a}_x + 0.2821\ \bar{a}_y - 0.5642\ \bar{a}_z$$

$$\bar{F}_2 = \frac{-20\times 10^{-6}\times 50\times 10^{-6}}{4\pi\times 8.854\times 10^{-12}\times(14.1774)^2}[\bar{a}_{12}]$$

$$= -0.0447\,[0.7758\;\bar{a}_x + 0.2821\;\bar{a}_y - 0.5642\;\bar{a}_z]$$

$$= -0.0346\;\bar{a}_x - 0.01261\;\bar{a}_y + 0.02522\;\bar{a}_z \;\text{ N}$$

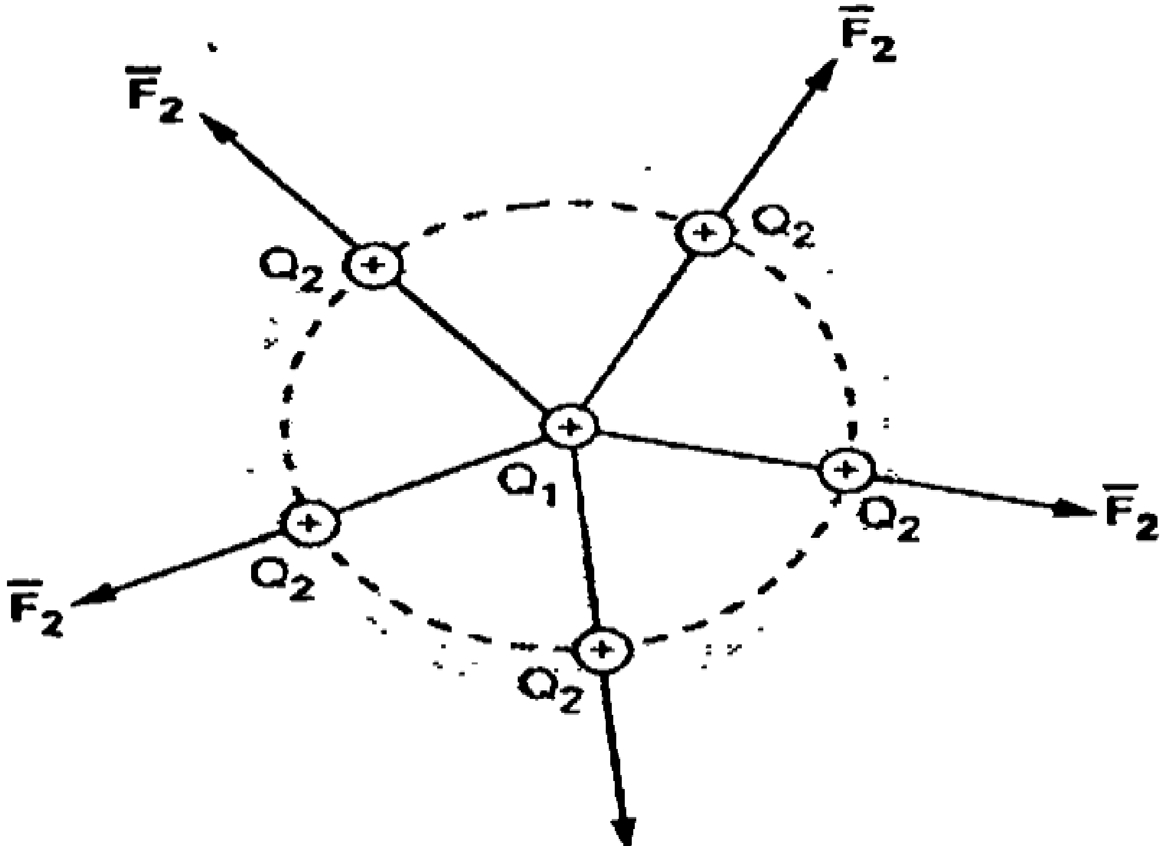
This is the required force exerted on Q_2 by Q_1.

The magnitude of the force is,

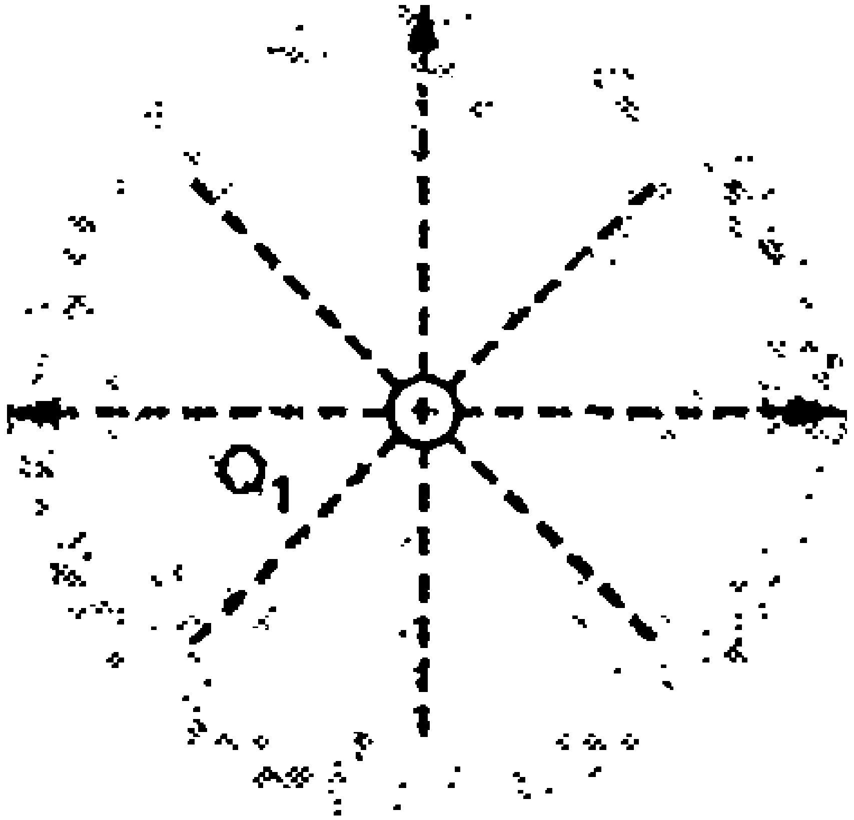
$$|\bar{F}_2| = \sqrt{(0.0346)^2 + (0.01261)^2 + (-0.02522)^2} = 44.634 \text{ mN}$$

Electric field intensity

Consider a point charge Q_1 as shown in Fig. (a)



(a)



(b)

If any other similar charge Q_2 is brought near it, Q_2 experiences a force. Infact if Q_2 is moved around Q_1, still Q_2 experiences a force as shown in the Fig. (a)

Thus there exists a region around a charge in which it exerts a force on any other charge. This region where a particular charge exerts a force on any other charge located in that region is called **electric field** of that charge. The electric field of Q_1 is shown in the Fig. (b)

The force experienced by the charge Q_2 due to Q_1 is given by Coulomb's law as,

$$\bar{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2} \bar{a}_{12}$$

Thus force per unit charge can be written as, $\frac{\bar{F}_2}{Q_2} = \frac{Q_1}{4\pi\varepsilon_0 R_{12}^2} \bar{a}_{12}$ ... (1)

This force exerted per unit charge is called **electric field intensity** or **electric field strength**. It is a **vector quantity** and is directed along a segment from the charge Q_1 to the position of any other charge. It is denoted as $\bar{E}$.

$$\bar{E} = \frac{Q_1}{4\pi\varepsilon_0 R_{1p}^2} \bar{a}_{1p} \quad \text{... (2)}$$

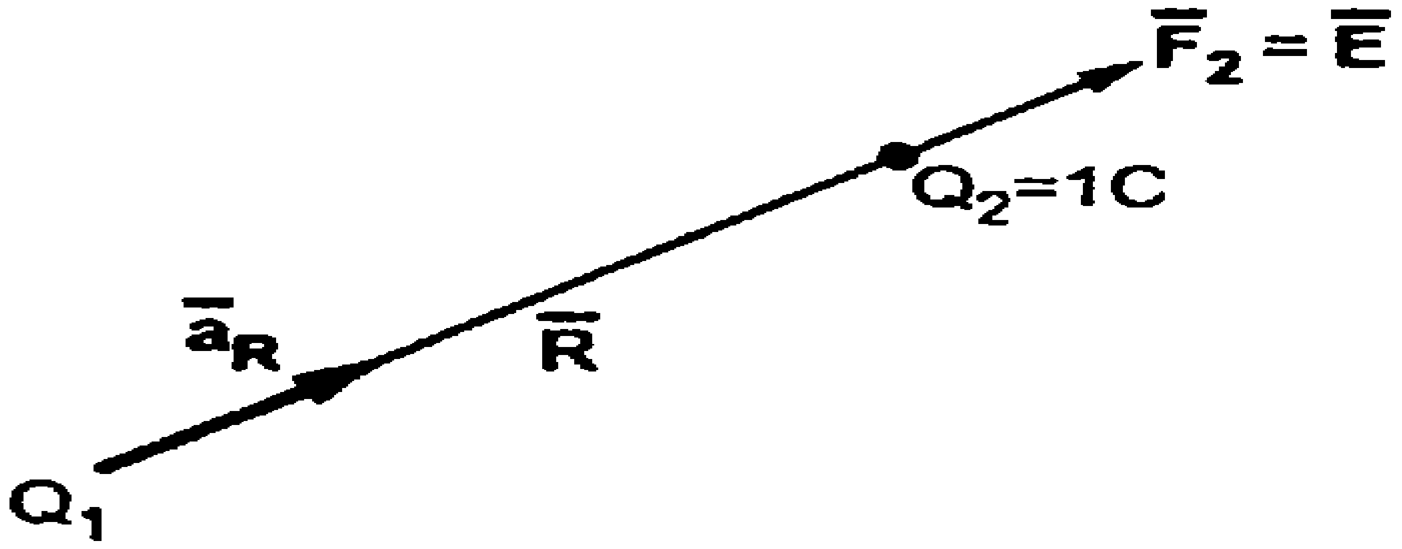
where $\quad p = $ Position of any other charge around Q_1

The equation (2) is the electric field intensity due to a single point charge Q_1 in a free space or vacuum.

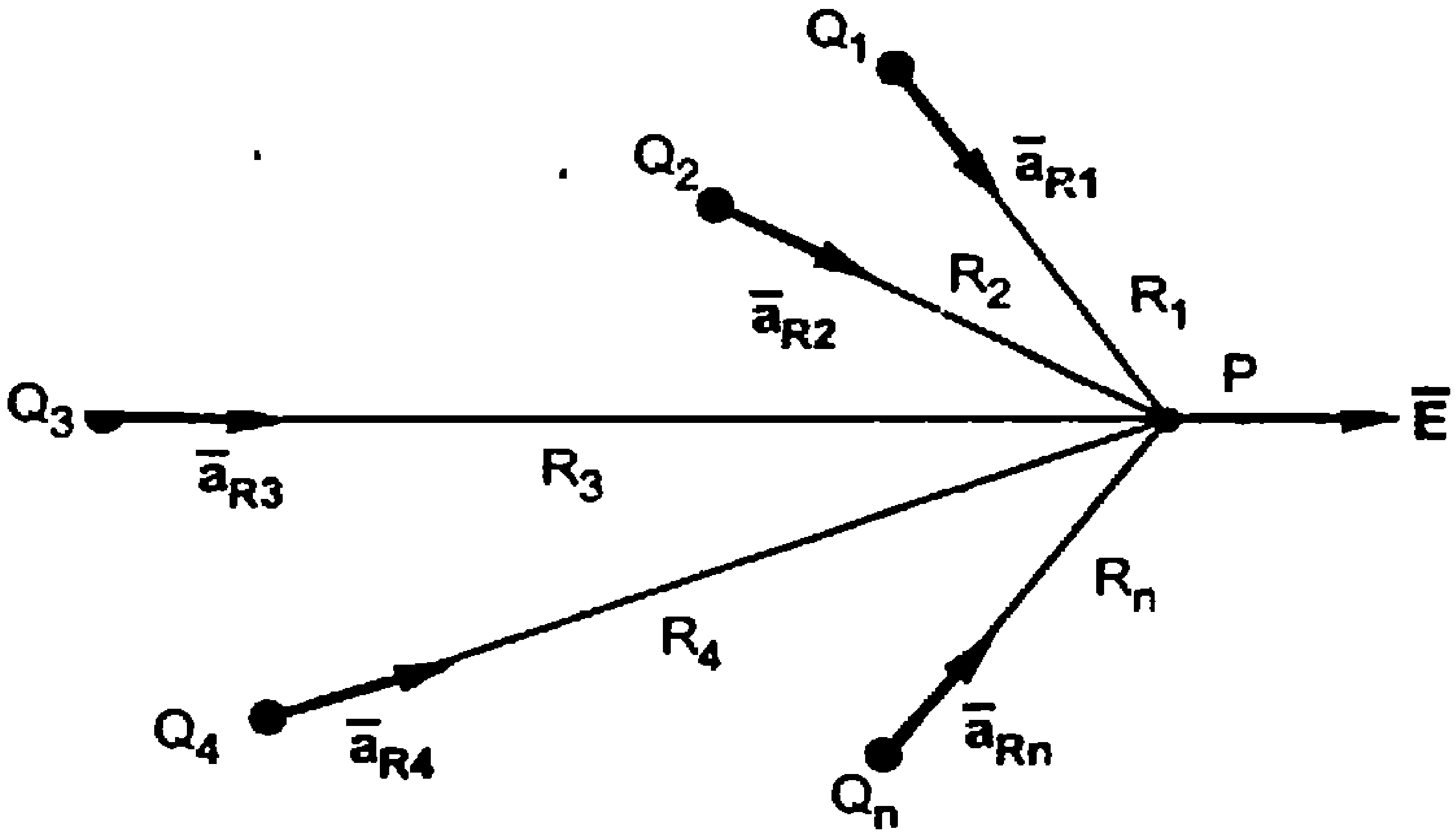
Another definition of electric field intensity is the force experienced by a unit positive test charge i.e. $Q_2 = 1C$.

Consider a charge Q_1 as shown in the Fig. The unit positive charge $Q_2 = 1C$ is placed at a distance R from Q_1. Then the force acting on Q_2 due to Q_1 is along the unit vector $\bar{a}_R$. As the charge Q_2 is **unit charge**, the force exerted on Q_2 is nothing but electric field intensity $\bar{E}$ of Q_1 at the point where unit charge is placed.

$$\bar{E} = \frac{Q_1}{4\pi\varepsilon_0 R^2}\bar{a}_R$$



Consider n charges $Q_1, Q_2 \ldots Q_n$ as shown in the Fig. The combined electric field intensity is to be obtained at point P.



$\bar{E}$ **due to n number of charges** the total electric field intensity at point P is the vector sum of the individual field intensities produced by the various charges at the point P.

$$\bar{E} = \bar{E}_1 + \bar{E}_2 + \bar{E}_3 + \ldots + \bar{E}_n$$

$$= \frac{Q_1}{4\pi\varepsilon_0 R_1^2}\bar{a}_{R1} + \frac{Q_2}{4\pi\varepsilon_0 R_2^2}\bar{a}_{R2} + \ldots + \frac{Q_n}{4\pi\varepsilon_0 R_n^2}\bar{a}_{Rn}$$

$$\bar{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{Q_i}{R_i^2}\bar{a}_{Ri}$$

$$\bar{a}_{Ri} = \frac{\bar{r}_P - \bar{r}_i}{|\bar{r}_P - \bar{r}_i|}$$

where $\bar{r}_P$ = Position vector of point P

$\bar{r}_i$ = Position vector of point where charge Q_i is placed.

Units of $\bar{E}$

The definition of electric field intensity is,

$$\bar{E} = \frac{\text{Force}}{\text{Unit charge}} = \frac{\text{(N) Newtons}}{\text{(C) Coulomb}}$$

Hence units of $\bar{E}$ is N/C. But the electric potential has units J/C i.e. Nm/C and hence $\bar{E}$ is also measured in units **V/m (volts per metre)**. This unit is used practically to express $\bar{E}$.

Example *Determine the electric field intensity at P(– 0.2, 0, – 2.3) m due to a point charge of +5 nC at Q(0.2, 0.1, – 2.5) m in air.*

Solution :

$$\bar{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\,\bar{a}_R$$

$$\bar{a}_R = \frac{\bar{R}_{QP}}{|\bar{R}_{QP}|}$$

$$= \frac{\bar{P}-\bar{Q}}{|\bar{P}-\bar{Q}|}$$

$$\bar{P}-\bar{Q} = (-0.2-0.2)\bar{a}_x + (0-0.1)\bar{a}_y + [-2.3-(-2.5)]\bar{a}_z$$

$$= -0.4\,\bar{a}_x - 0.1\,\bar{a}_y + 0.2\,\bar{a}_z$$

$$\therefore \quad \bar{a}_R = \frac{-0.4\,\bar{a}_x - 0.1\,\bar{a}_y + 0.2\,\bar{a}_z}{\sqrt{(-0.4)^2 + (0.1)^2 + (0.2)^2}}$$

$$= \frac{-0.4\,\bar{a}_x - 0.1\,\bar{a}_y + 0.2\,\bar{a}_z}{0.45825}$$

$$= -0.8728\,\bar{a}_x - 0.2182\,\bar{a}_y + 0.4364\,\bar{a}_z$$

$$\therefore \quad R = \left|\bar{P} - \bar{Q}\right| = 0.45825$$

$$\therefore \quad \bar{E} = \frac{5\times 10^{-9}}{4\pi\times 8.854\times 10^{-12}\times (0.45825)^2}\,[\bar{a}_R] = 214\,\bar{a}_R$$

Substituting value of $\bar{a}_R$,

$$\bar{E} = -186.779\,\bar{a}_x - 46.694\,\bar{a}_y + 93.389\,\bar{a}_z \ \text{V/m}$$

This is electric field intensity at point P.

Example *A charge of 1 C is at (2, 0, 0). What charge must be placed at (– 2, 0, 0) which will make y component of total $\bar{E}$ zero at the point (1, 2, 2) ?*

Solution :

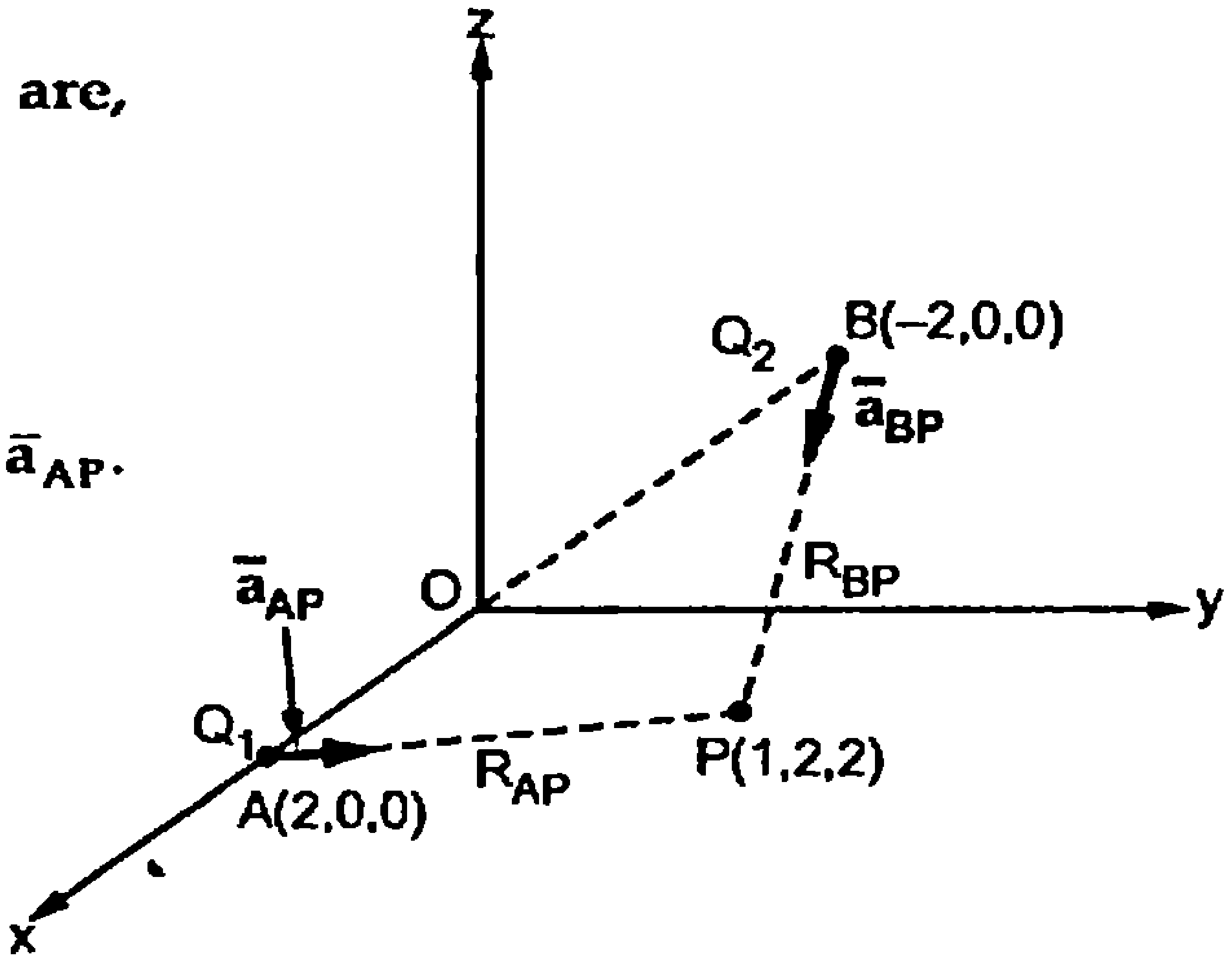
The position vectors of points A, B and P are,

$$\bar{A} = 2\bar{a}_x , \quad \bar{B} = -2\bar{a}_x$$

$$\bar{P} = \bar{a}_x + 2\bar{a}_y + 2\bar{a}_z$$

$\bar{E}_A$ is field at P due to Q_1, and will act along $\bar{a}_{AP}$.

$\bar{E}_B$ is field at P due to Q_2 and will act along $\bar{a}_{BP}$.



$$\bar{E}_A = \frac{Q_1}{4\pi\varepsilon_0 R_{AP}^2}\bar{a}_{AP} = \frac{Q_1}{4\pi\varepsilon_0 R_{AP}^2} \times \frac{\bar{P}-\bar{A}}{|\bar{P}-\bar{A}|}$$

$$\bar{E}_B = \frac{Q_2}{4\pi\varepsilon_0 R_{BP}^2}\,\bar{a}_{BP} = \frac{Q_2}{4\pi\varepsilon_0 R_{BP}^2} \times \frac{\bar{P}-\bar{B}}{|\bar{P}-\bar{B}|}$$

$$\therefore\ \bar{E} \text{ at } P = \bar{E}_A + \bar{E}_B = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q_1}{R_{AP}^2}\frac{\bar{P}-\bar{A}}{|\bar{P}-\bar{A}|} + \frac{Q_2}{R_{BP}^2}\frac{\bar{P}-\bar{B}}{|\bar{P}-\bar{B}|}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{1[-\bar{a}_x + 2\bar{a}_y + 2\bar{a}_z]}{(\sqrt{9})^2\sqrt{(1)^2+(2)^2+(2)^2}} + \frac{Q_2[3\bar{a}_x + 2\bar{a}_y + 2\bar{a}_z]}{(\sqrt{17})^2\sqrt{(3)^2+(2)^2+(2)^2}}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{-\bar{a}_x + 2\bar{a}_y + 2\bar{a}_z}{27} + \frac{Q_2[3\bar{a}_x + 2\bar{a}_y + 2\bar{a}_z]}{70.0927}\right]$$

The y component of $\bar{E}$ must be zero.

$$\therefore \quad \frac{2}{27} + \frac{2\,Q_2}{70.0927} = 0$$

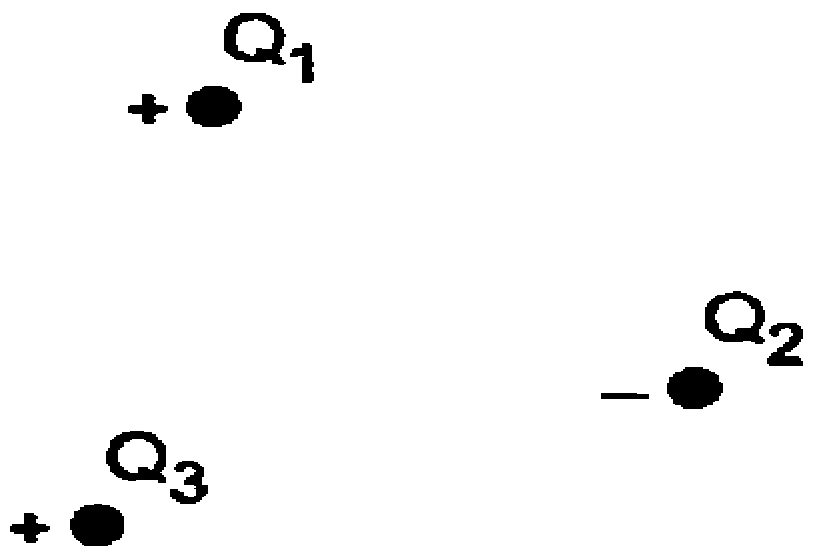
$$Q_2 = -\frac{2}{27} \times \frac{70.0927}{2} = -2.596\ \text{C}$$

This is the required charge Q_2 to be placed at $(-2, 0, 0)$ which will make y component of $\bar{E}$ zero at point P.

Types of Charge Distributions

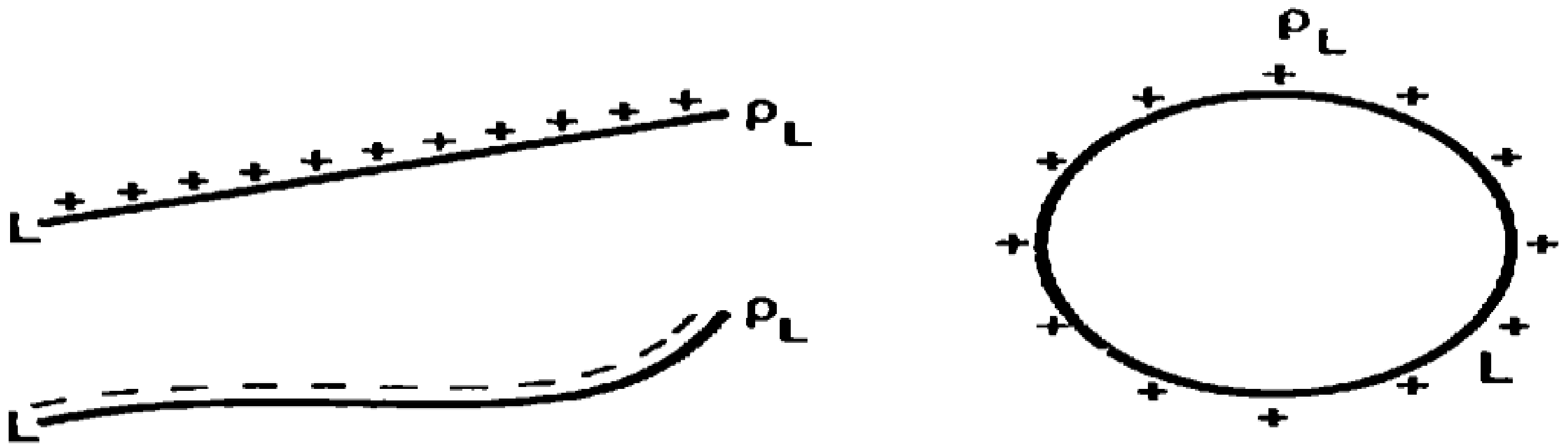
1. Point charge 2. Line charge 3. Surface charge 4. Volume charge

Point Charge



(a) Point charges

Line Charge



(b) Line charges

The charge density of the line charge is denoted as ρ_L and defined as charge per unit length.

$$\rho_L = \frac{\text{Total charge in coulomb}}{\text{Total length in metres}} \quad (\text{C/m})$$

Thus ρ_L is measured in C/m. The ρ_L is constant all along the length L of the line carrying the charge.

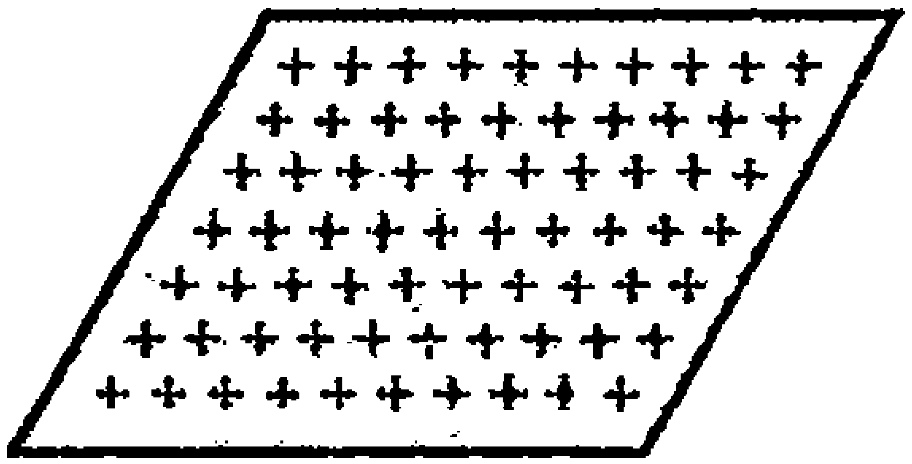
$dQ = \rho_L \, dl$ = charge on differential length dl

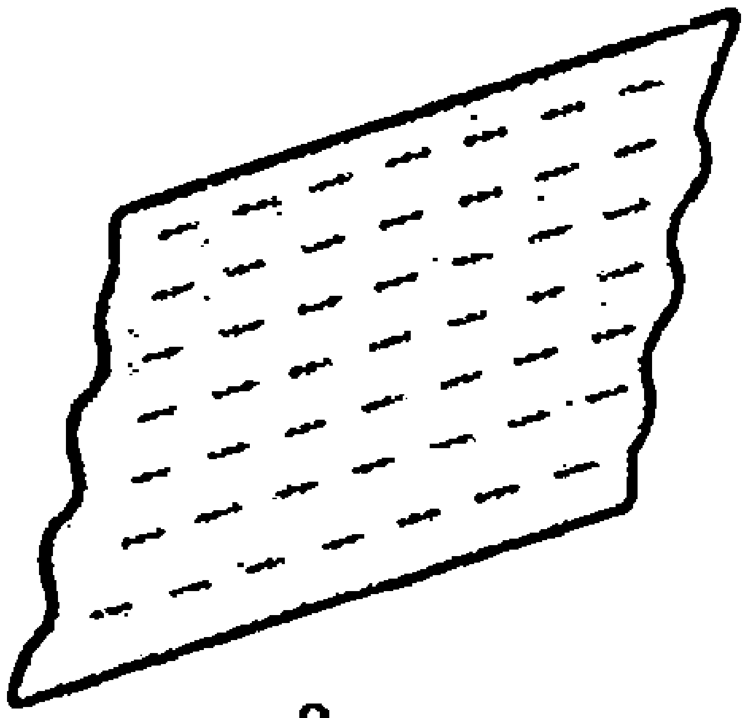
$$Q = \int_L dQ = \int_L \rho_L \, dl$$

$$d\bar{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\bar{a}_R = \frac{\rho_L \, dl}{4\pi\varepsilon_0 R^2}\bar{a}_R$$

$$\bar{E} = \int_L \frac{\rho_L \, dl}{4\pi\varepsilon_0 R^2}\bar{a}_R$$

Surface Charge





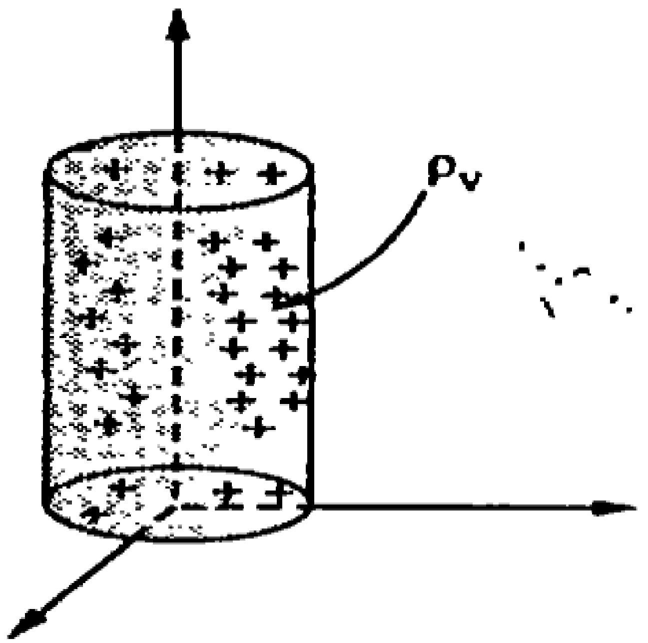
the surface charge density is denoted as ρ_S and defined as the charge per unit surface area.

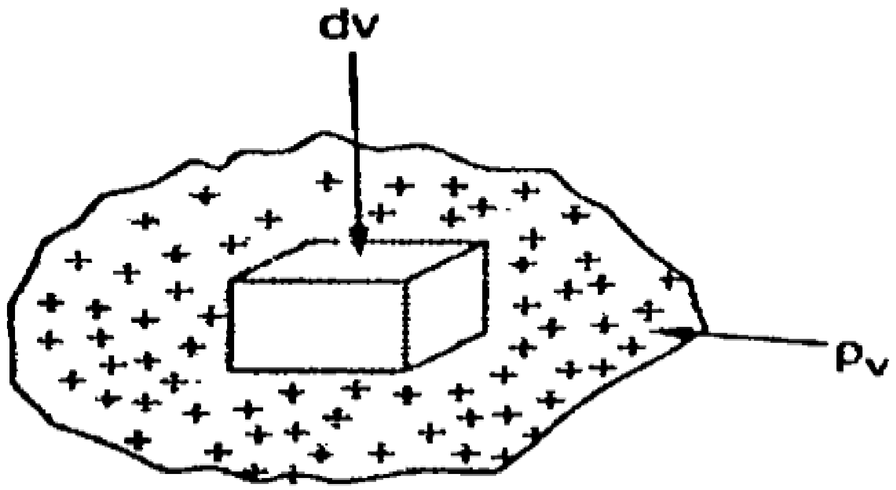
$$\rho_S = \frac{\text{Total charge in coulomb}}{\text{Total area in square metres}} \; (C/m^2)$$

Thus ρ_S is expressed in C/m^2. The ρ_S is constant over the surface carrying the charge.

$$Q = \int_S dQ = \int_S \rho_S \, dS \qquad \bar{E} = \int_S \frac{\rho_S \, dS}{4 \pi \varepsilon_0 R^2} \bar{a}_R$$

Volume Charge





The **volume charge density** is denoted as ρ_v and defined as the charge per unit volume.

$$\rho_v = \frac{\text{Total charge in coulomb}}{\text{Total volume in cubic metres}} \left(\frac{C}{m^3}\right)$$

Thus ρ_v is expressed in C/m^3.

$$Q = \int_{vol} \rho_v \, dv \qquad \bar{E} = \int_{Vol} \frac{\rho_v \, dv}{4\pi\varepsilon_0 R^2} \bar{a}_R$$

Electric field due to infinite line charge

Consider an infinitely long straight line carrying uniform line charge having density ρ_L C/m. Let this line lies along z-axis from $-\infty$ to ∞ and hence called **infinite line charge** Let point P is on y-axis at which electric field intensity is to be determined. The distance of point P from the origin is 'r' as shown in the Fig.

Consider a small differential length d*l* carrying a charge dQ, along the line as shown in the Fig. It is along z axis hence d*l* = dz.

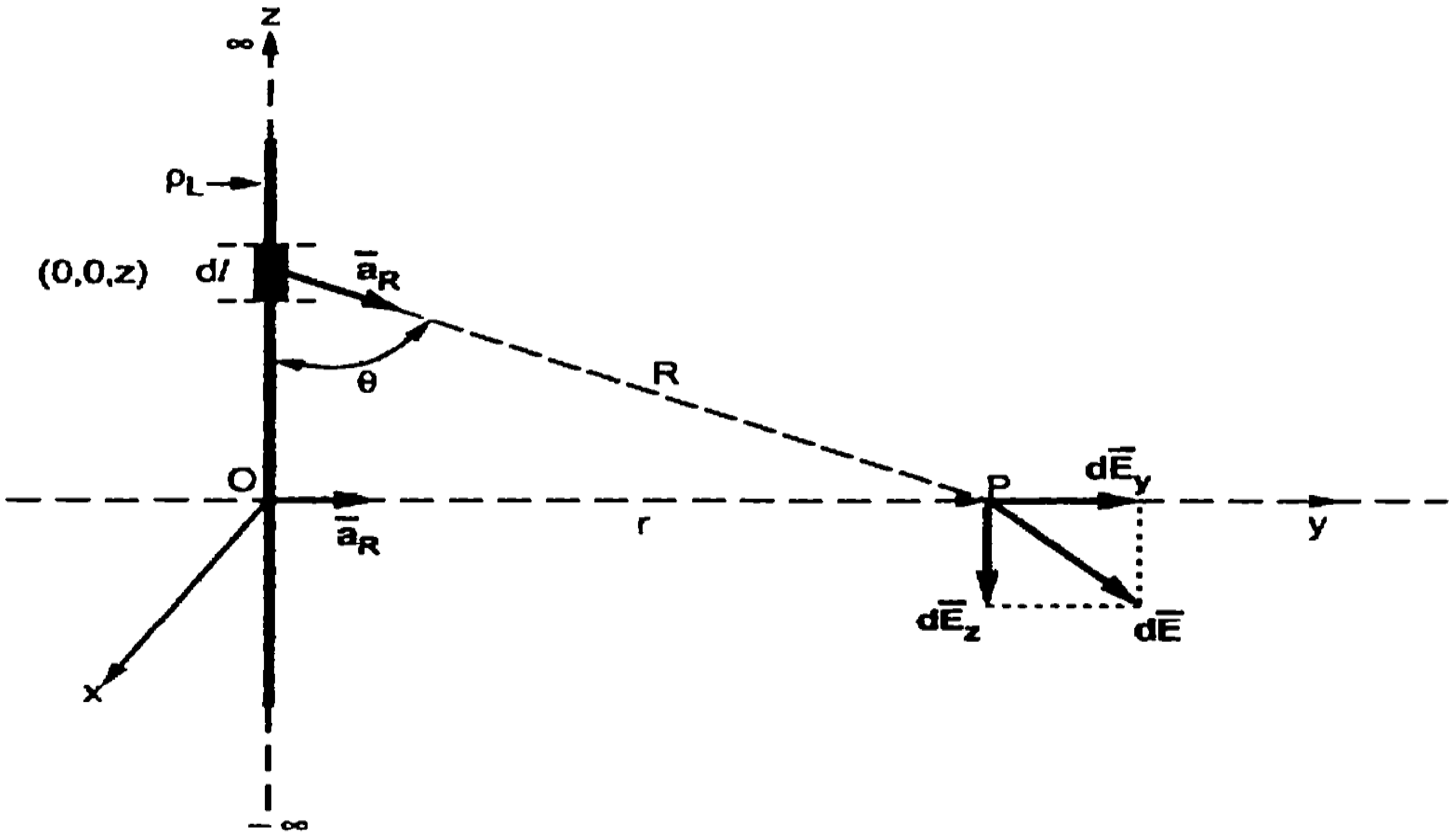
$$\therefore dQ = \rho_L \, dl = \rho_L \, dz \quad \text{... (1)}$$

The co-ordinates of dQ are (0, 0, z) while the co-ordinates of point P are (0, r, 0). Hence the distance vector $\bar{R}$ can be written as,

$$\bar{R} = \bar{r}_P - \bar{r}_{dl} = [r\,\bar{a}_y - z\,\bar{a}_z]$$

$$\therefore |\bar{R}| = \sqrt{r^2 + z^2}$$

$$\therefore \bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{r\,\bar{a}_y - z\,\bar{a}_z}{\sqrt{r^2 + z^2}} \quad \text{... (2)}$$



Field due to infinite line charge

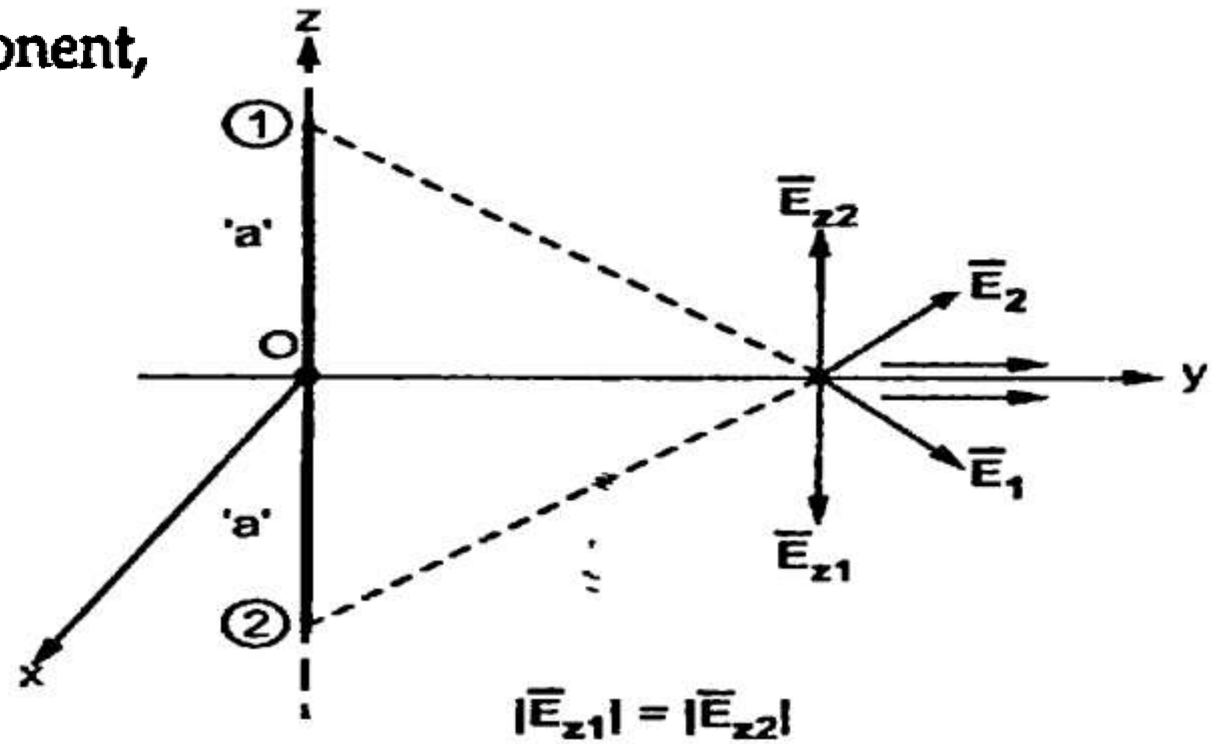
$$\therefore d\,\bar{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\,\bar{a}_R$$

$$= \frac{\rho_L\, dz}{4\pi\varepsilon_0\left(\sqrt{r^2+z^2}\right)^2}\left[\frac{r\bar{a}_y - z\bar{a}_z}{\sqrt{r^2+z^2}}\right] \quad \ldots (3)$$

Note : For every charge on positive z-axis there is equal charge present on negative z-axis. Hence the z component of electric field intensities produced by such charges at point P will cancel each other. Hence effectively there will not be any z component of $\bar{E}$ at P.

Hence the equation of $d\bar{E}$ can be written by eliminating $\bar{a}_z$ component,

$$d\bar{E} = \frac{\rho_L\, dz}{4\pi\varepsilon_0\left(\sqrt{r^2+z^2}\right)^2}\,\frac{r\bar{a}_y}{\sqrt{r^2+z^2}} \quad \ldots (4)$$



Now by integrating $d\bar{E}$ over the z-axis from $-\infty$ to ∞ we can obtain total $\bar{E}$ at point P.

$$\bar{E} = \int_{-\infty}^{\infty} \frac{\rho_L}{4\pi\varepsilon_0\left(r^2+z^2\right)^{3/2}}\, r\, dz\, \bar{a}_y$$

Note : For such an integration, use the substitution

$$z = r \tan\theta \quad \text{i.e.} \quad r = \frac{z}{\tan\theta}$$

$$\therefore \qquad dz = r \sec^2\theta \, d\theta$$

Here r is not the variable of integration.

For $z = -\infty$, $\theta = \tan^{-1}(-\infty) = -\pi/2 = -90°$

For $z = +\infty$, $\theta = \tan^{-1}(\infty) = \pi/2 = +90°$

} Changing the limits

$$\therefore \qquad \bar{E} = \int_{\theta=-\pi/2}^{\pi/2} \frac{\rho_L}{4\pi\varepsilon_0 [r^2 + r^2 \tan^2\theta]^{3/2}} \, r \times r \sec^2\theta \, d\theta \, \bar{a}_y$$

$$= \frac{\rho_L}{4\pi\varepsilon_0} \int_{-\pi/2}^{\pi/2} \frac{r^2 \sec^2\theta \, d\theta}{r^3 [1 + \tan^2\theta]^{3/2}} \, \bar{a}_y$$

But $\qquad 1 + \tan^2\theta = \sec^2\theta$

$$\bar{E} = \frac{\rho_L}{4\pi\varepsilon_0} \int_{-\pi/2}^{\pi/2} \frac{\sec^2\theta \, d\theta}{r \sec^3\theta} \, \bar{a}_y$$

$$= \frac{\rho_L}{4\pi\varepsilon_0 r} \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta \, \bar{a}_y \qquad \ldots \sec\theta = \frac{1}{\cos\theta}$$

$$= \frac{\rho_L}{4\pi\varepsilon_0 r} [\sin\theta]_{-\pi/2}^{\pi/2} \bar{a}_y = \frac{\rho_L}{4\pi\varepsilon_0 r} \left[\sin\frac{\pi}{2} - \sin\left(\frac{-\pi}{2}\right)\right] \bar{a}_y$$

$$= \frac{\rho_L}{4\pi\varepsilon_0 r} [1-(-1)] \bar{a}_y = \frac{\rho_L}{4\pi\varepsilon_0 r} \times 2 \, \bar{a}_y$$

$$\boxed{\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0 r} \bar{a}_y \quad V/m} \qquad \ldots (5)$$

The result of equation (5) which is specifically in cartesian system can be generalized. The $\bar{a}_y$ is unit vector along the distance r which is perpendicular distance of point P from the line charge. Thus in general $\bar{a}_y = \bar{a}_r$.

Hence the result of $\bar{E}$ can be expressed as,

$$\boxed{\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0 r} \bar{a}_r \quad V/m} \qquad \ldots (6)$$

where r = Perpendicular distance of point P from the line charge

$\bar{a}_r$ = Unit vector in the direction of the perpendicular distance of point P from the line charge

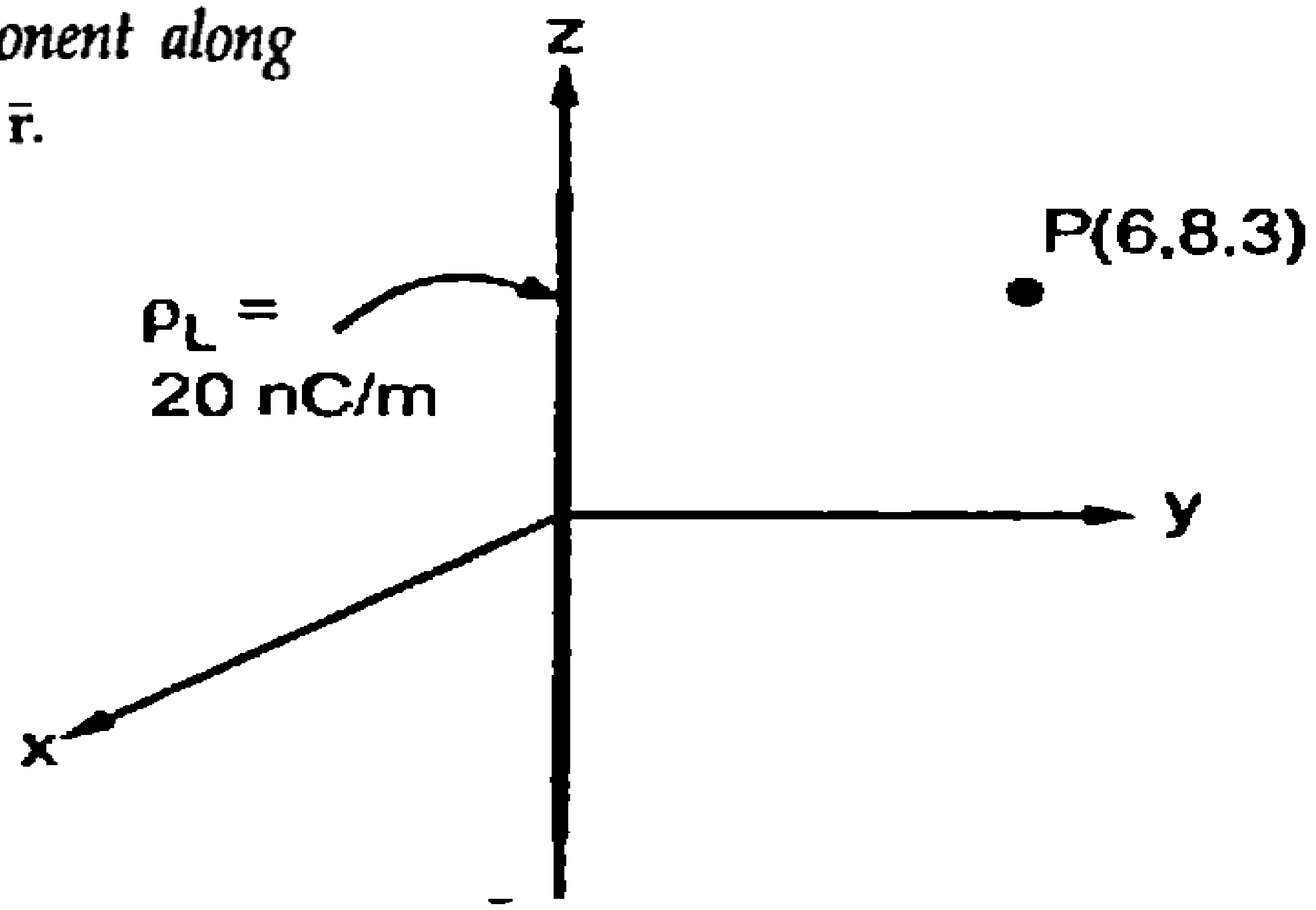
Very important notes : 1. The field intensity $\bar{E}$ at any point has no component in the direction parallel to the line along which the charge is located and the charge is infinite. For example if line charge is parallel to z axis, $\bar{E}$ can not have $\bar{a}_z$ component, if line charge is parallel to y axis, $\bar{E}$ can not have a_y component. This makes the integration calculations easy.

2. The above equation consists r and $\bar{a}_r$ which do not have meanings of cylindrical co-ordinate system. The distance r is to be obtained by distance formula while $\bar{a}_r$ is unit vector in the direction of $\bar{r}$.

Example *A uniform line charge, infinite in extent with $\rho_L = 20$ nC/m lies along the z axis. Find the $\bar{E}$ at (6,8,3) m.*

Solution : The line charge is shown in the Fig. Any point on the line is (0,0,z).

As line charge is along z axis, $\bar{E}$ can not have any component along z direction. So do not consider z co-ordinate while calculating $\bar{r}$.



$$\bar{r} = (6-0)\bar{a}_x + (8-0)\bar{a}_y$$

$$\bar{a}_r = \frac{\bar{r}}{|\bar{r}|} = \frac{6\,\bar{a}_x + 8\,\bar{a}_y}{\sqrt{6^2+8^2}} = \frac{6\,\bar{a}_x + 8\,\bar{a}_y}{10}$$

$$= 0.6\bar{a}_x + 0.8\,\bar{a}_y$$

Thus, $$\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r$$

$$= \frac{20\times 10^{-9}}{2\pi\times 8.854\times 10^{-12}\times 10}[0.6\bar{a}_x + 0.8\,\bar{a}_y] = 10.7853\,\bar{a}_x + 14.38\,\bar{a}_y \text{ V/m}$$

Electric field due to charged circular ring

Consider a charged circular ring of radius r placed in xy plane with centre at origin, carrying a charge uniformly along its circumference. The charge density is ρ_L C/m . The point P is at a perpendicular distance 'z' from the ring as shown in the Fig.

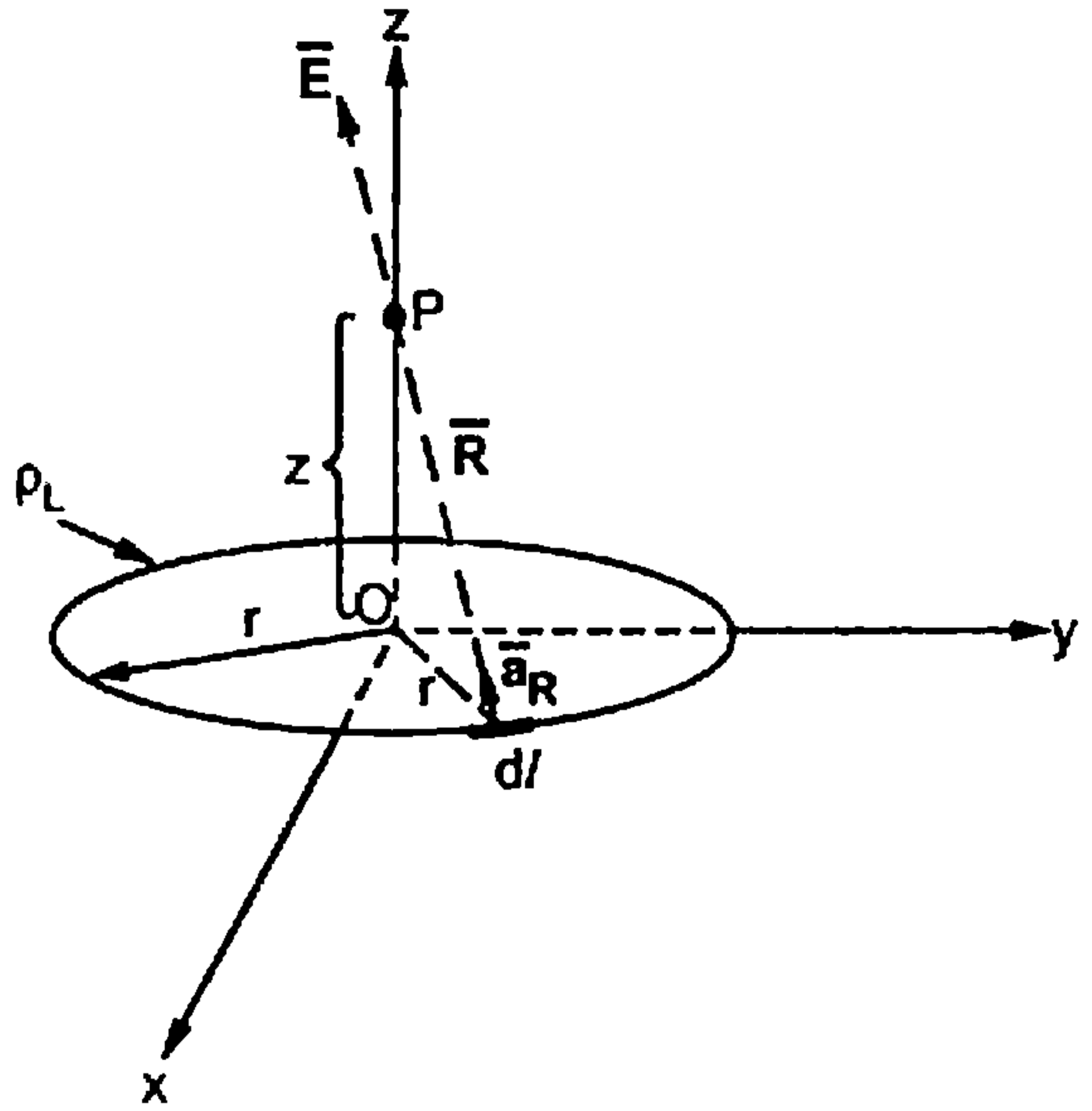
Consider a small differential length dl on this ring. The charge on it is dQ.

$$\therefore \quad dQ = \rho_L \, dl$$

$$\therefore \quad d\bar{E} = \frac{\rho_L \, dl}{4\pi\varepsilon_0 R^2} \bar{a}_R \qquad \dots (1)$$

where R = Distance of point P from dl.

Consider the cylindrical co-ordinate system. For dl we are moving in ϕ direction where $dl = r \, d\phi$.



$$dl = r\,d\phi \quad \dots (2)$$

$$R^2 = r^2 + z^2$$

While $\bar{R}$ can be obtained from its two components, in cylindrical system as shown in the Fig. The two components are,

1) Distance r in the direction of $-\bar{a}_r$, radially inwards i.e. $-r\bar{a}_r$.

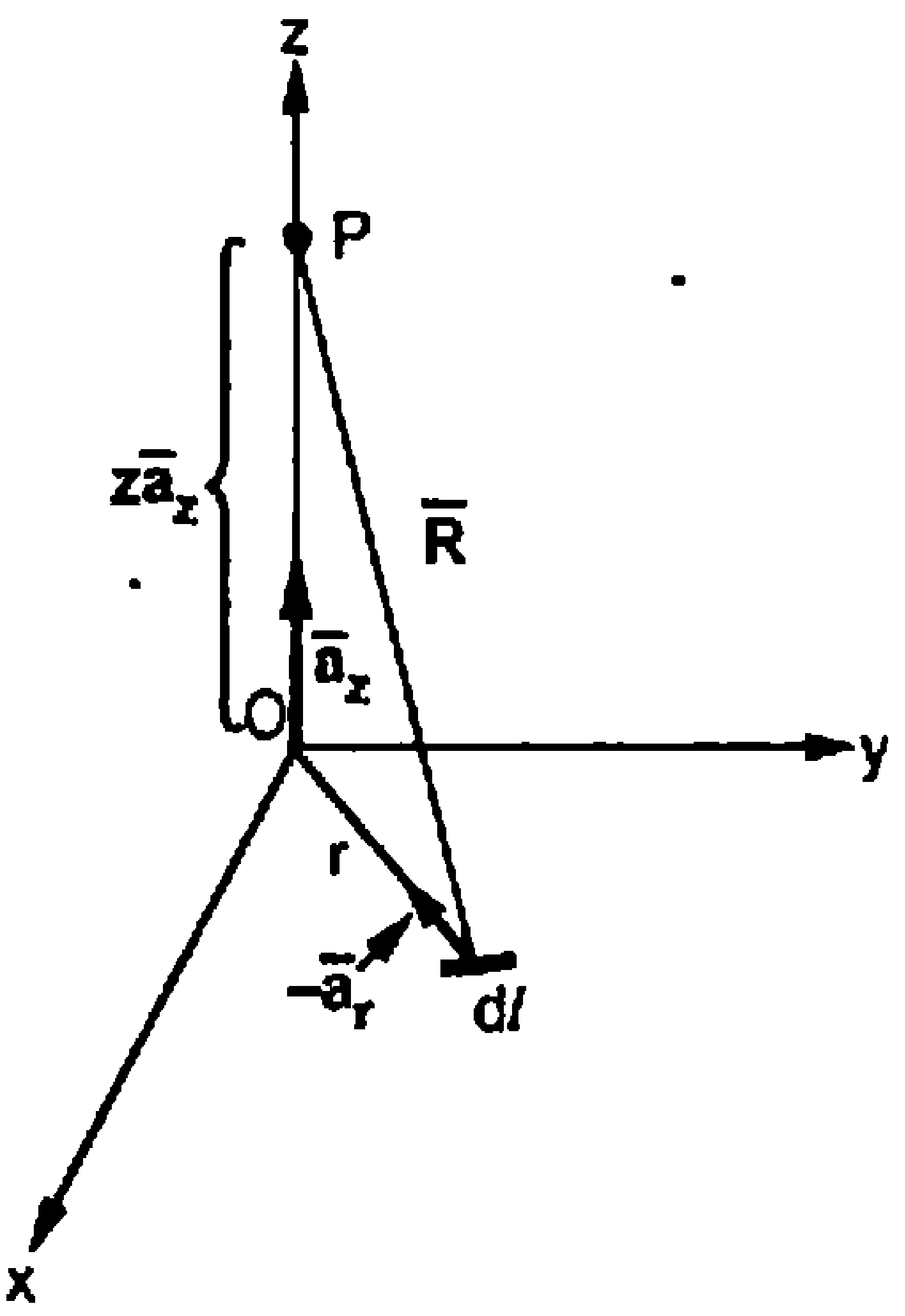
2) Distance z in the direction of $\bar{a}_z$ i.e. $z\,\bar{a}_z$

$$\therefore \quad \bar{R} = -r\bar{a}_r + z\bar{a}_z \quad \dots (3)$$

$$|\bar{R}| = \sqrt{(-r)^2 + (z)^2} = \sqrt{r^2 + z^2} \quad \dots (4)$$

$$\bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{-r\bar{a}_r + z\bar{a}_z}{\sqrt{r^2 + z^2}} \quad \dots (5)$$

$$d\bar{E} = \frac{\rho_L\, dl}{4\pi\varepsilon_0 \left(\sqrt{r^2 + z^2}\right)^2} \times \frac{-r\bar{a}_r + z\bar{a}_z}{\sqrt{r^2 + z^2}}$$



$$d\bar{E} = \frac{\rho_L (r\, d\phi)}{4\pi\varepsilon_0 \left(r^2 + z^2\right)^{3/2}} [-r\bar{a}_r + z\bar{a}_z] \quad \text{... (6)}$$

The radial components of $\bar{E}$ at point P will be symmetrically placed in the plane parallel to xy plane and are going to cancel each other.

Hence neglecting $\bar{a}_r$ component from dE we get,

$$d\bar{E} = \frac{\rho_L (r\, d\phi)}{4\pi\varepsilon_0 \left(r^2 + z^2\right)^{3/2}} z\bar{a}_z \quad \text{... (7)}$$

$$\therefore \quad \bar{E} = \int_{\phi=0}^{2\pi} \frac{\rho_L\, r\, d\phi}{4\pi\varepsilon_0 \left(r^2 + z^2\right)^{3/2}} z\bar{a}_z$$

$$= \frac{\rho_L\, r}{4\pi\varepsilon_0 \left(r^2 + z^2\right)^{3/2}} z\, \bar{a}_z\, [\phi]_0^{2\pi}$$

... Integration w.r.t. ϕ

$$\therefore \quad \boxed{\bar{E} = \frac{\rho_L\, r z}{2\varepsilon_0 \left(r^2 + z^2\right)^{3/2}} \bar{a}_z} \quad \text{... (8)}$$

where r = Radius of the ring ;

z = Perpendicular distance of point P from the ring along the axis of the ring

Electric field due to infinite sheet charge

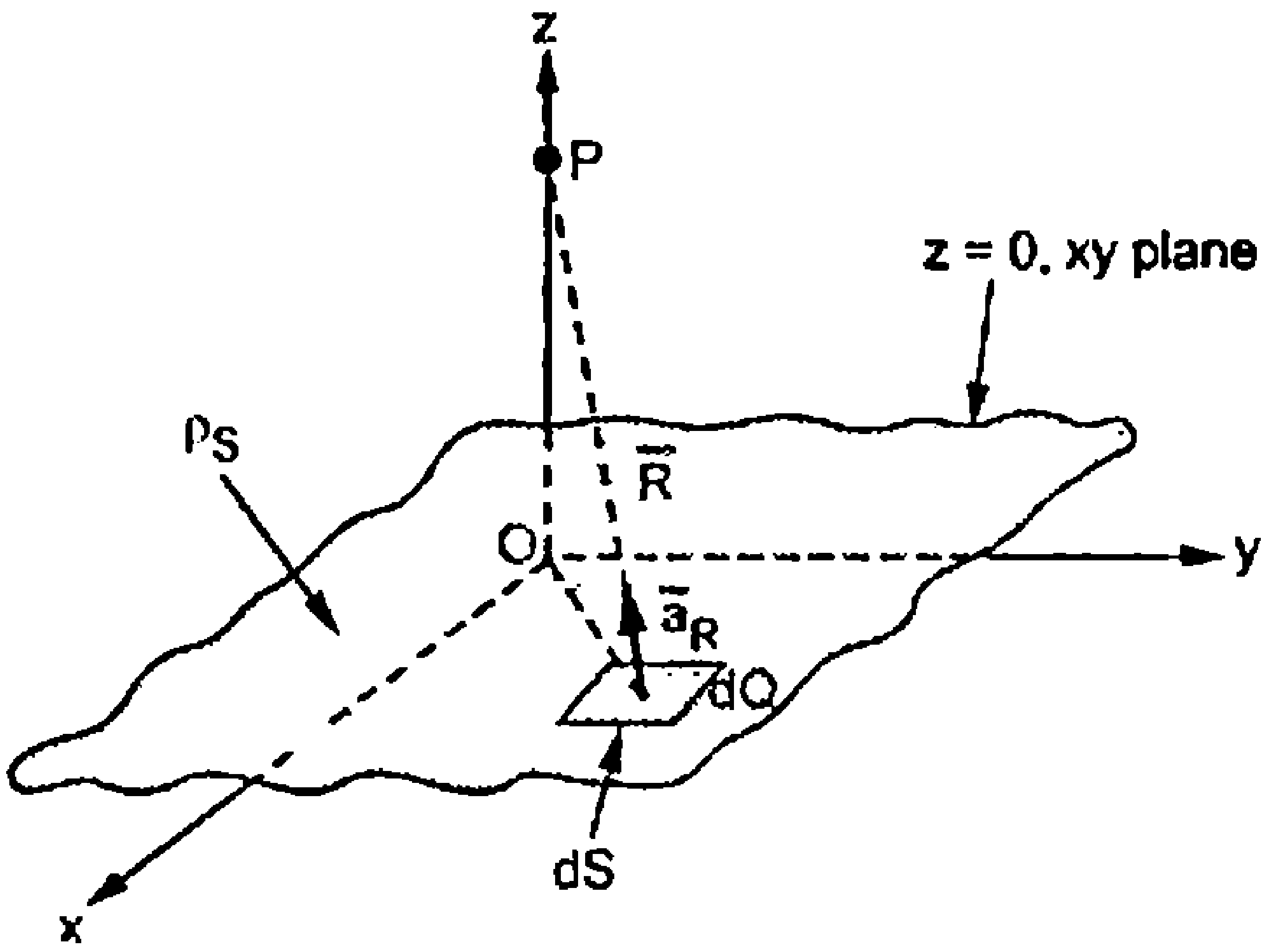
Consider an infinite sheet of charge having uniform charge density $\rho_S\ C/m^2$, placed in xy plane as shown in the Fig. Let us use cylindrical coordinates.

The point P at which $\bar{E}$ to be calculated is on z axis.

Consider the differential surface area dS carrying a charge dQ. The normal direction to dS is z direction hence dS normal to z direction is $r\ dr\ d\phi$.

Now $\quad dQ = \rho_S\, dS = \rho_S\ r\ dr\ d\phi$

Hence, $\quad d\bar{E} = \dfrac{dQ}{4\pi\varepsilon_0 R^2}\,\bar{a}_R = \dfrac{\rho_S\, r\, dr\, d\phi}{4\pi\varepsilon_0 R^2}\,\bar{a}_R$



The distance vector $\bar{R}$ has two components

1. The radial component r along $-\bar{a}_r$ i.e. $-r\,\bar{a}_r$.
2. The component z along $\bar{a}_z$ i.e. $z\,\bar{a}_z$.

With these two components $\bar{R}$ can be obtained from the differential area towards point P as,

$$\bar{R} = -r\bar{a}_r + z\bar{a}_z$$

$$|\bar{R}| = \sqrt{(-r)^2 + (z)^2} = \sqrt{r^2 + z^2}$$

$$\bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{-r\bar{a}_r + z\bar{a}_z}{\sqrt{r^2 + z^2}}$$

$$d\bar{E} = \frac{\rho_S\, r\, dr\, d\phi}{4\pi\varepsilon_0 \left(\sqrt{r^2 + z^2}\right)^2}\left[\frac{-r\bar{a}_r + z\bar{a}_z}{\sqrt{r^2 + z^2}}\right]$$

For infinite sheet in xy plane, r varies from 0 to ∞ while ϕ varies from 0 to 2π.

As there is symmetry about z axis from all radial direction, all $\bar{a}_r$ components of $\bar{E}$ are going to cancell each other and net $\bar{E}$ will not have any radial component.

Hence while integrating $d\bar{E}$ there is no need to consider $\bar{a}_r$ component. Though if considered, after integration procedure, it will get mathematically cancelled.

$$\therefore \quad \bar{E} = \int_{\phi=0}^{2\pi} \int_{r=0}^{\infty} d\bar{E} = \int_{0}^{2\pi} \int_{0}^{\infty} \frac{\rho_S \, r \, dr \, d\phi}{4\pi\varepsilon_0 (r^2 + z^2)^{3/2}} (z\bar{a}_z)$$

Put $r^2 + z^2 = u^2$ hence $2r\,dr = 2u\,du$

For $r = 0$, $u = z$ and $r = \infty$, $u = \infty$ ... Changing limits

$$\bar{E} = \int_{0}^{2\pi} \int_{u=z}^{\infty} \frac{\rho_S}{4\pi\varepsilon_0} \frac{u\,du}{(u^2)^{3/2}} d\phi \, z\,\bar{a}_z$$

$$= \int_{0}^{2\pi} \int_{u=z}^{\infty} \frac{\rho_S}{4\pi\varepsilon_0} \frac{du}{u^2} d\phi \, (z\,\bar{a}_z)$$

$$= \int_0^{2\pi} \frac{\rho_S}{4\pi\varepsilon_0} d\phi \, z \, \bar{a}_z \left[-\frac{1}{u}\right]_z^{\infty} \qquad \text{... as} \int \frac{1}{u^2} = \int u^{-2} = \frac{u^{-1}}{-1} = -\frac{1}{u}$$

$$= \frac{\rho_S}{4\pi\varepsilon_0} [\phi]_0^{2\pi} \, (z\bar{a}_z) \left[-\frac{1}{\infty} - \left(-\frac{1}{z}\right)\right] = \frac{\rho_S}{4\pi\varepsilon_0} (2\pi) \, \bar{a}_z$$

$$\bar{E} = \frac{\rho_S}{2\varepsilon_0} \bar{a}_z \quad \text{V/m} \qquad \text{... For points above xy plane}$$

Now $\bar{a}_z$ is direction normal to differential surface area dS considered. Hence in general if $\bar{a}_n$ is direction normal to the surface containing charge, the above result can be generalized as,

$$\boxed{\bar{E} = \frac{\rho_S}{2\varepsilon_0} \bar{a}_n \quad \text{V/m}}$$

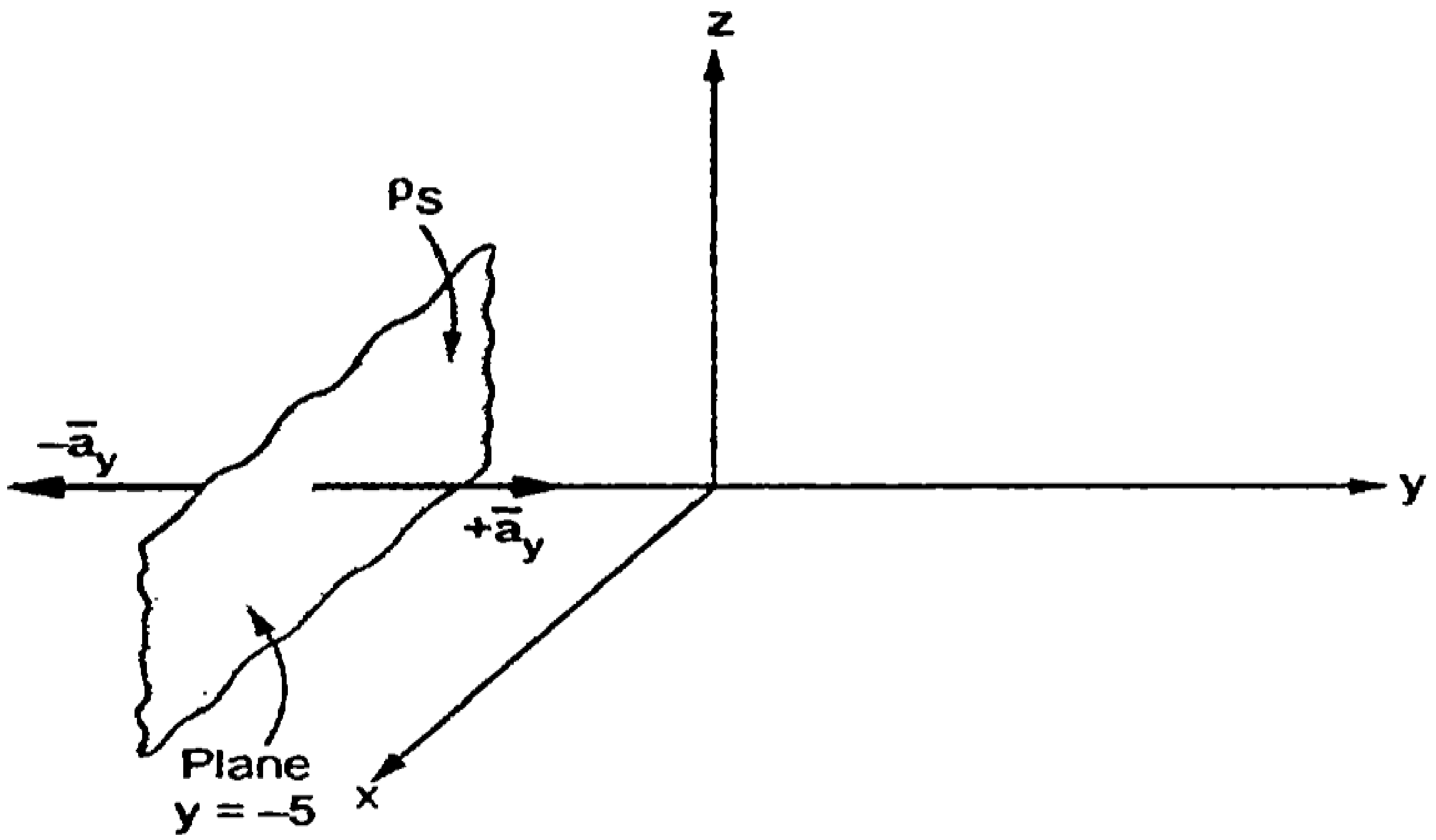
where $\bar{a}_n$ = Direction normal to the surface charge

Thus for the points below xy plane, $\bar{a}_n = -\bar{a}_z$ hence,

$$\therefore \qquad \bar{E} = -\frac{\rho_S}{2\varepsilon_0} \bar{a}_z \quad \text{V/m} \qquad \text{... For points below xy plane.}$$

Example *Charge lies in y = – 5m plane in the form of an infinite square sheet with a uniform charge density of $\rho_S = 20\ nC/m^2$. Determine $\bar{E}$ at all the points.*

Solution : The plane y = – 5 m constant is parallel to xz plane as shown in the Fig.



For $y > -5$, the $\bar{E}$ component will be along $+\bar{a}_y$ as normal direction to the plane $y = -5$ m is $\bar{a}_y$.

$$\therefore \qquad \bar{a}_n = \bar{a}_y$$

$$\therefore \qquad \bar{E} = \frac{\rho_S}{2\varepsilon_0}\bar{a}_n = \frac{\rho_S}{2\varepsilon_0}\bar{a}_y$$

$$= \frac{20\times10^{-9}}{2\times8.854\times10^{-12}}\bar{a}_y = 1129.43\ \bar{a}_y \ \ \text{V/m}$$

For $y < -5$, the $\bar{E}$ component will be along $-\bar{a}_y$ direction, with same magnitude.

$$\therefore \qquad \bar{E} = \frac{\rho_S}{2\varepsilon_0}(-\bar{a}_y) = -1129.43\ \bar{a}_y \ \ \text{V/m}$$

At any point to the left or right of the plane, $|\bar{E}|$ is constant and acts normal to the plane.

Example *Find $\bar{E}$ at P (1, 5, 2) m in free space if a point charge of 6 μC is located at (0,0,1), the uniform line charge density ρ_L = 180 nC/m along x axis and uniform sheet of charge with $\rho_S = 25$ nC/m^2 over the plane z = −1.*

Solution : Case 1 : Point charge Q_1 = 6 μC at A (0, 0, 1) and P (1, 5, 2)

$$\therefore \quad \bar{E}_1 = \frac{Q_1}{4\pi\varepsilon_0 R_{AP}^2}\bar{a}_{AP} = \frac{Q_1}{4\pi\varepsilon_0 R_{AP}^2}\left[\frac{\bar{R}_{AP}}{|\bar{R}_{AP}|}\right]$$

$$\bar{R}_{AP} = (1-0)\bar{a}_x + (5-0)\bar{a}_y + (2-1)\bar{a}_z = \bar{a}_x + 5\bar{a}_y + \bar{a}_z$$

$$\therefore \quad |\bar{R}_{AP}| = \sqrt{(1)^2 + (5)^2 + (1)^2} = \sqrt{27}$$

$$\therefore \quad \bar{E}_1 = \frac{6\times 10^{-6}}{4\pi\times 8.854\times 10^{-12}\times(\sqrt{27})^2}\left[\frac{\bar{a}_x + 5\bar{a}_y + \bar{a}_z}{\sqrt{27}}\right]$$

$$\therefore \quad \bar{E}_1 = 384.375\,\bar{a}_x + 1921.879\,\bar{a}_y + 384.375\,\bar{a}_z \quad \text{V/m}$$

Case 2 : Line charge ρ_L along x axis.

It is infinite hence using standard result,

$$\bar{E}_2 = \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r = \frac{\rho_L}{2\pi\varepsilon_0 r}\frac{\bar{r}}{|\bar{r}|}$$

Consider any point on line charge i.e. (x, 0, 0) while P (1, 5, 2). But as line is along x axis, no component of $\bar{E}$ will be along $\bar{a}_x$ direction. Hence while calculating $\bar{r}$ and $\bar{a}_r$, do not consider x co-ordinates of the points.

$$\therefore \quad \bar{r} = (5-0)\bar{a}_y + (2-0)\bar{a}_z = 5\bar{a}_y + 2\bar{a}_z$$

$$\therefore \quad |\bar{r}| = \sqrt{(5)^2 + (2)^2} = \sqrt{29}$$

$$\therefore \quad \bar{E}_2 = \frac{\rho_l}{2\pi\varepsilon_0 \times \sqrt{29}} \left[\frac{5\bar{a}_y + 2\bar{a}_z}{\sqrt{29}}\right] = \frac{180\times10^{-9}\,[5\bar{a}_y + 2\bar{a}_z]}{2\pi\times 8.854\times10^{-12}\times 29}$$

$$= 557.859\ \bar{a}_y + 223.144\ \bar{a}_z \quad \text{V/m}$$

Case 3 : Surface charge ρ_S over the plane z = − 1. The plane is parallel to xy plane and normal direction to the plane is $\bar{a}_n = \bar{a}_z$, as point P is above the plane. At all the points above z = − 1 plane the $\bar{E}$ is constant along $\bar{a}_z$ direction.

$$\bar{E}_3 = \frac{\rho_S}{2\varepsilon_0}\bar{a}_n = \frac{25\times10^{-9}}{2\times8.854\times10^{-12}}\bar{a}_z = 1411.7913\ \bar{a}_z \text{V/m}$$

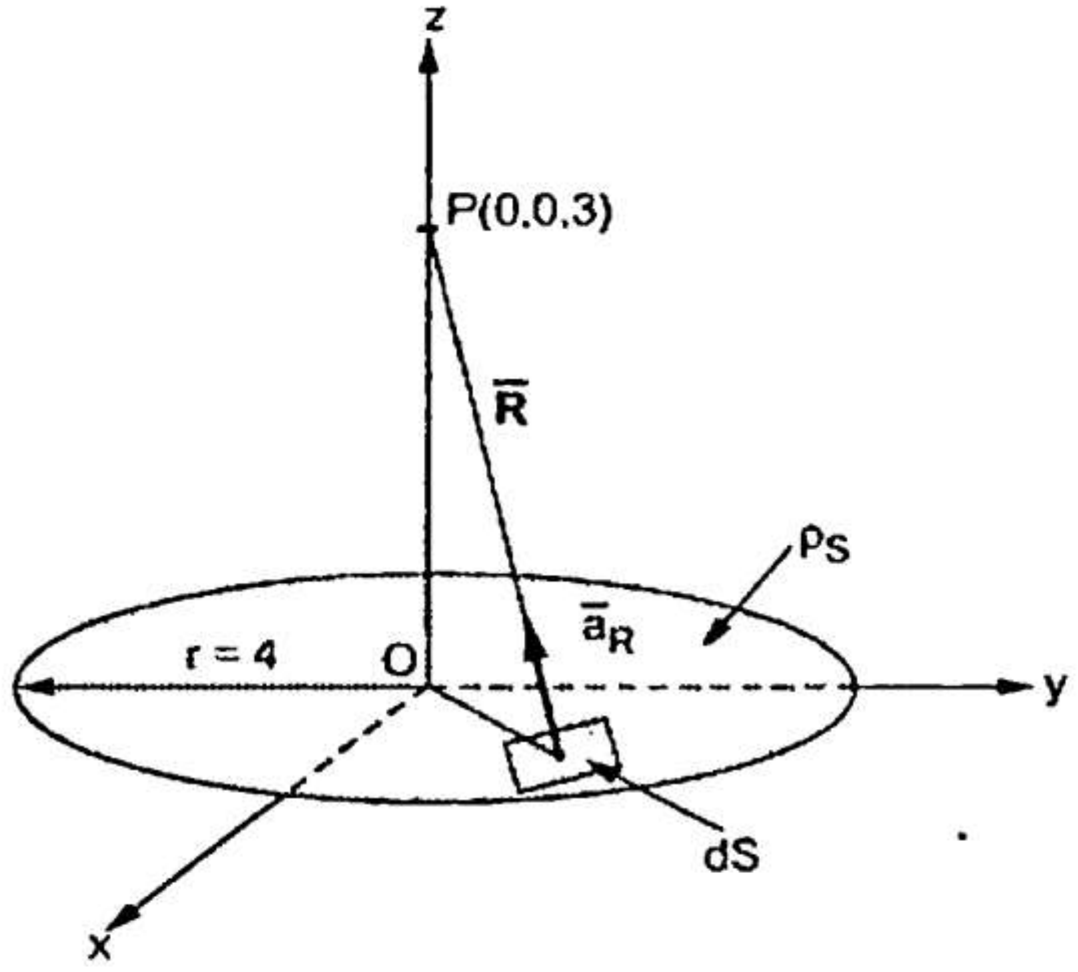
Hence the net $\bar{E}$ at point P is,

$$\bar{E} = \bar{E}_1 + \bar{E}_2 + \bar{E}_3 = 384.375\ \bar{a}_x + 1921.879\ \bar{a}_y + 384.375\ \bar{a}_z + 557.859\ \bar{a}_y + 223.144\ \bar{a}_z + 1411.7913\ \bar{a}_z$$

$$= 384.375\ \bar{a}_x + 2479.738\ \bar{a}_y + 2019.3103\ \bar{a}_z \quad \text{V/m}$$

Example *The charge lies on the circular disc $r \leq 4$ m, $z = 0$, with density $\rho_S = [10^{-4}/r]$ C/m². Determine $\bar{E}$ at $r = 0$, $z = 3$ m.*

Solution : The sheet of charge is shown in the Fig.



Consider the differential area dS carrying the charge dQ. The normal direction to dS is $\bar{a}_z$ hence $dS_z = r\,dr\,d\phi$.

$$dQ = \rho_S\, dS = \rho_S\, r\, dr\, d\phi$$

$$= \frac{10^{-4}}{r} \cdot r\, dr\, d\phi$$

$$dQ = 10^{-4}\ dr\, d\phi$$

$$d\bar{E} = \frac{10^{-4}\, dr\, d\phi}{4\pi\varepsilon_0 R^2}\, \bar{a}_R$$

Consider $\bar{R}$ as shown in the Fig. which has two components in cylindrical system,

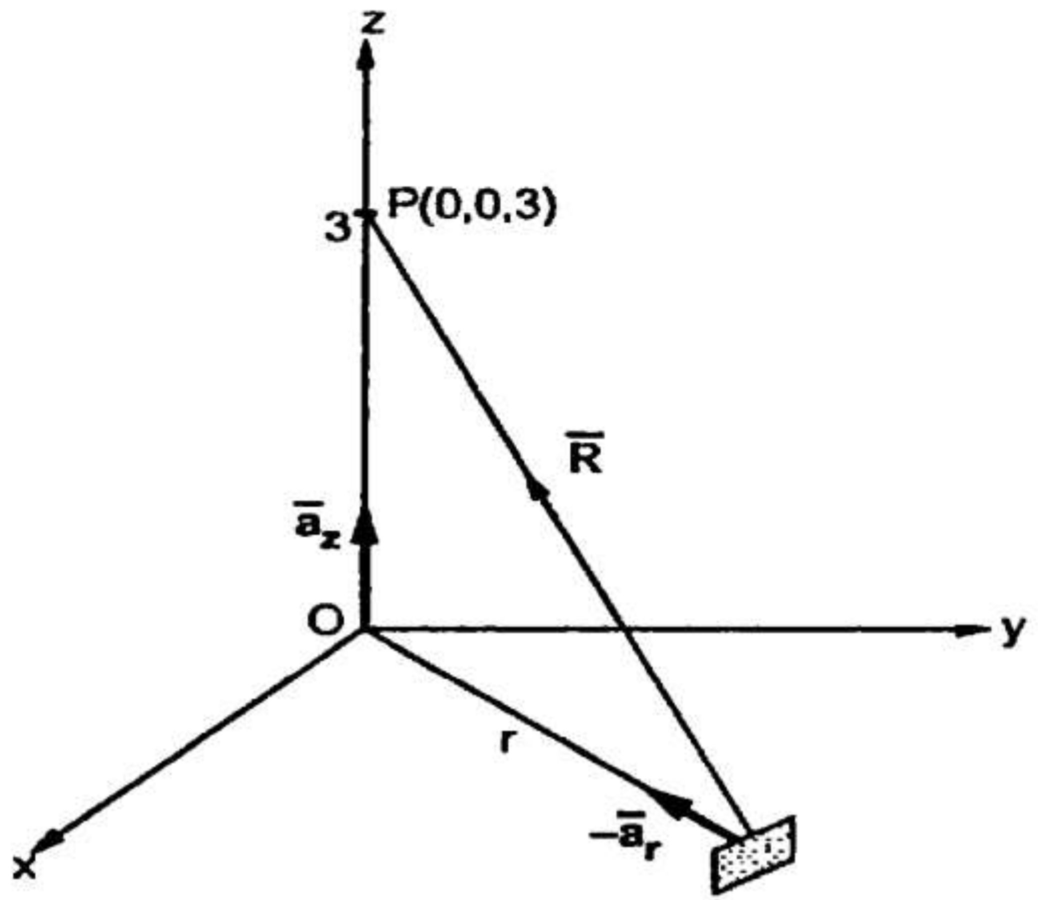
1. The component along $-\bar{a}_r$ having radius r i.e. $-r\bar{a}_r$.
2. The component z = 3 along $\bar{a}_z$ i.e. $3\bar{a}_z$.

$$\therefore \qquad \bar{R} = -r\,\bar{a}_r + 3\bar{a}_z$$

$$|\bar{R}| = \sqrt{(-r)^2 + (3)^2} = \sqrt{r^2+9}$$

$$\therefore \qquad \bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{-r\bar{a}_r + 3\bar{a}_z}{\sqrt{r^2+9}}$$

$$\therefore \qquad d\bar{E} = \frac{10^{-4}\, dr\, d\phi}{4\pi\varepsilon_0 \left(\sqrt{r^2+9}\right)^2}\left[\frac{-r\bar{a}_r + 3\bar{a}_z}{\sqrt{r^2+9}}\right]$$



It can be seen that due to symmetry about z axis, all radial components will cancell each other. Hence there will not be any component of $\bar{E}$ along $\bar{a}_r$. So in integration $\bar{a}_r$ need not be considered.

$$\therefore \quad \bar{E} = \int_{\phi=0}^{2\pi}\int_{r=0}^{4} \frac{10^{-4}\, dr\, d\phi}{4\pi\varepsilon_0 (r^2+9)^{3/2}} (3\bar{a}_z)$$

As there is no r dr in the numerator, use

$$r = 3\tan\theta, \quad dr = 3\sec^2\theta\, d\theta \quad \Big\} \quad \text{... Change of limits}$$

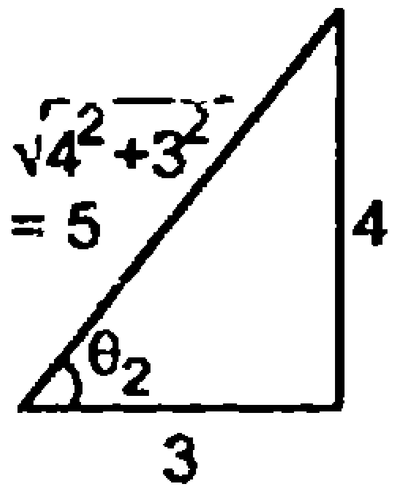
For $r = 0, \quad \theta_1 = 0$

For $r = 4, \quad \theta_2 = \tan^{-1} 4/3$

$$\bar{E} = \int_{\phi=0}^{2\pi}\int_{\theta_1=0}^{\theta_2} \frac{10^{-4}\, 3\sec^2\theta\, d\theta\, d\phi}{4\pi\varepsilon_0 [9\tan^2\theta+9]^{3/2}} (3\bar{a}_z) = \int_{\phi=0}^{2\pi}\int_{\theta_1=0^\circ}^{\theta_2} \frac{299.5914\times10^3 \sec^2\theta\, d\theta\, d\phi}{[1+\tan^2\theta]^{3/2}} \bar{a}_z$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta_1=0^\circ}^{\theta_2} \frac{299.5914\times10^3}{\sec\theta}\, d\theta\, d\phi\, \bar{a}_z = \int_{\phi=0}^{2\pi}\int_{\theta_1=0}^{\theta_2} 299.5914\times10^3\, d\theta\, d\phi [\cos\theta]\bar{a}_z$$

$$= 299.5914\times10^3 [\phi]_0^{2\pi} [\sin\theta]_{\theta_1=0^\circ}^{\theta_2} \bar{a}_z = 1.8823\times10^6 \sin\theta_2\, \bar{a}_z$$



$$\theta_2 = \tan^{-1}\frac{4}{3} \text{ i.e. } \tan\theta_2 = \frac{4}{3} \qquad \sin\theta_2 = \frac{4}{5} = 0.8$$

$$\bar{E} = 1.8823\times10^6\times0.8\,\bar{a}_z = 1.5059\times10^6\,\bar{a}_z \text{ V/m} = \mathbf{1.5059\,\bar{a}_z \text{ MV/m}}$$

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