JAIPUR ENGINEERING COLLEGE AND RESEARCH CENTRE

Year & Sem - B.Tech II year, sem- III

Subject - Electromagnetic Fields

Unit - I

Presented by - Ms. Ritu Soni

Designation - Assistant Professor

Department- Electrical Engineering

Vision and Mission of Institute

Vision of institute

To become a renowned centre of outcome based learning, and work towards, professional, cultural and social enrichment of the lives of individuals and communities.

Mission of institute

- **M1**.Focus on evaluation of learning outcomes and motivate students to inculcate research aptitude by project based learning.
- **M2**.Identify ,based on informed perception of Indian, regional and global needs, the areas of focus and provide platform to gain knowledge and solutions.
- **M3**.Offer opportunities for interaction between academia and industry.
- **M4**.Develop human potential to its fullest extent so that intellectually capable and imaginatively gifted leaders may emerge in a range of professions

Vision and Mission of Electrical Engineering Department

Vision of department

The Electrical Engineering department strives to recognized globally for outcome based technical knowledge and produce quality human being who can manage the advance technologies and contribute to society.

Mission Of department

M1. To impart quality technical knowledge to the learners to make them globally competitive Electrical Engineers.

M2. To provide the learners ethical guidelines along with excellent academic environment for a long productive career.

M3. To promote industry- institute relationship.

Syllabus of Electromagnetic fields

unit 1- Review of Vector Calculus

Vector algebra- addition, subtraction, components of vectors, scalar and vector multiplications, triple products, three orthogonal coordinate systems (rectangular, cylindrical and spherical). Vector calculus differentiation, partial differentiation, integration, vector operator del, gradient, divergence and curl; integral theorems of vectors. Conversion of a vector from one coordinate system to another.

Unit 2- Static Electric Field

Coulomb's law, Electric field intensity, Electrical field due to point charges. Line, Surface and Volume charge distributions. Gauss law and its applications. Absolute Electric potential, Potential difference, Calculation of potential differences for different configurations. Electric dipole, Electrostatic Energy and Energy density.

Unit 3- Conductors, Dielectrics and Capacitance

Current and current density, Ohms Law in Point form, Continuity of current, Boundary conditions of perfect dielectric materials. Permittivity of dielectric materials, Capacitance, Capacitance of a two wire line, Poisson's equation, Laplace's equation, Solution of Laplace and Poisson's equation, Application of Laplace's and Poisson's equations.

unit 4- Static Magnetic Fields

Biot-Savart Law, Ampere Law, Magnetic flux and magnetic flux density, Scalar and Vector Magnetic potentials. Steady magnetic fields produced by current carrying conductors.

Unit5- Magnetic Forces, Materials and Inductance

Force on a moving charge, Force on a differential current element, Force between differential current elements, Nature of magnetic materials, Magnetization and permeability, Magnetic boundary conditions, Magnetic circuits, inductances and mutual inductances.

Unit 6- Time Varying Fields and Maxwell's Equations

Faraday's law for Electromagnetic induction, Displacement current, Point form of Maxwell's equation, Integral form of Maxwell's equations, Motional Electromotive forces. Boundary Conditions.

Unit 7- Electromagnetic Waves

Derivation of Wave Equation, Uniform Plane Waves, Maxwell's equation in Phasor form, Wave equation in Phasor form, Plane waves in free space and in a homogenous material. Wave equation for a conducting medium, Plane waves in lossy dielectrics, Propagation in good conductors, Skin effect. Poynting theorem.

Course outcomes for Electromagnetic fields

CO1-Acquire basic understanding of vectors , their representation and conversion in different coordinate systems.

CO2-Able to compute the force, fields & energy of the electrostatic & magneto static fields. Able to analyze the materials, conductors, dielectrics, inductances and capacitances.

CO3-Understand the concept of time varying field and able to solve electromagnetic relation using Maxwell equations. Also able to analyze the electromagnetic waves.

Unit I - **Review of Vector Calculus**

Scalars and vectors

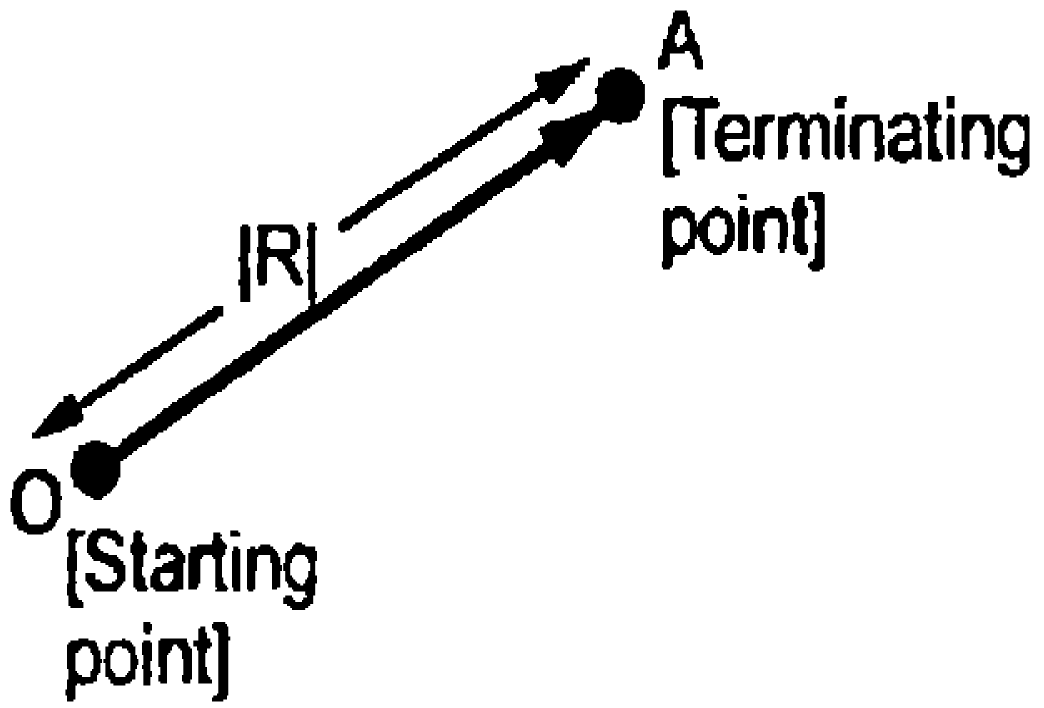
- **Scalars** - A scalar is a quantity which is wholly characterized by it's magnitude .Ex- temperature, mass, volume, density, speed, electric Charge etc..

- **Vector**- A vector is a quantity which is characterized by both magnitude and direction .Ex- Force, velocity, displacement, electric field intensity, acceleration etc.

- **Scalar field**- A field is a region in which a particular physical function has a value at each and every point in that region. Ex- Temperature of atmosphere.

- **Vector Field**- If a quantity which is specified in a region to define a field , is a vector then the corresponding field is called vector field Ex- wind velocity of atmosphere, displacement of flying bird in a space.

Representation of a vector



A vector can be represented by a straight line with an arrow in a plane

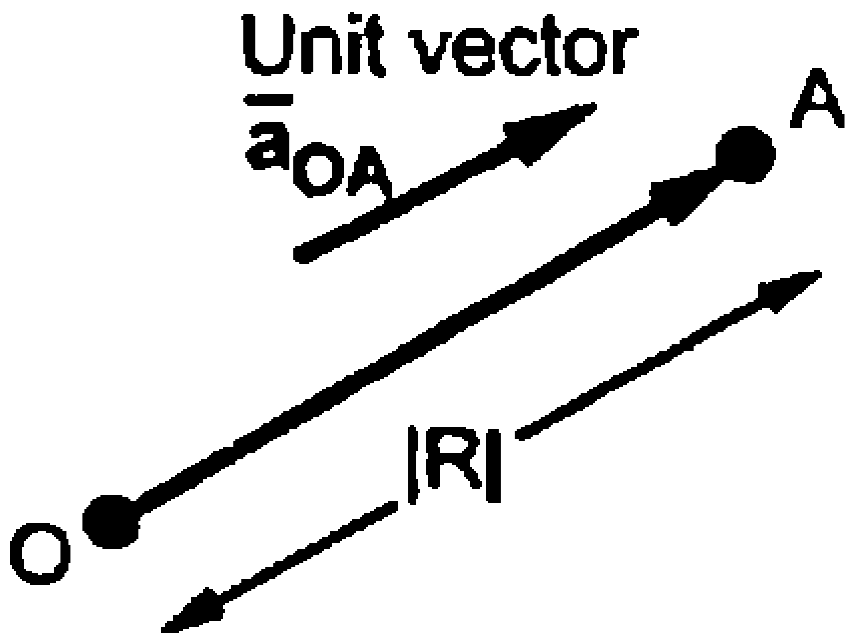
The length of the segment is magnitude while the arrow indicates the direction of the vector.

The vector showing is symbolically denoted as $\overline{OA}$.

$|\overline{OA}| = R$

Unit Vector

A unit vector has a function to indicate the direction, it's magnitude is always unity.



- Consider a unit vector $\bar{a}_{OA}$ in the direction of $\overline{OA}$ as shown.
- This vector indicates the direction of $\overline{OA}$ but it's magnitude is unity.

$$\overline{OA} = |\overline{OA}|\,\bar{a}_{OA} = R\,\bar{a}_{OA}$$

- If a vector is known then the unit vector along that vector can be obtain by dividing the vector by it's magnitude.

$$\text{Unit vector } \bar{a}_{OA} = \frac{\overline{OA}}{|\overline{OA}|}$$

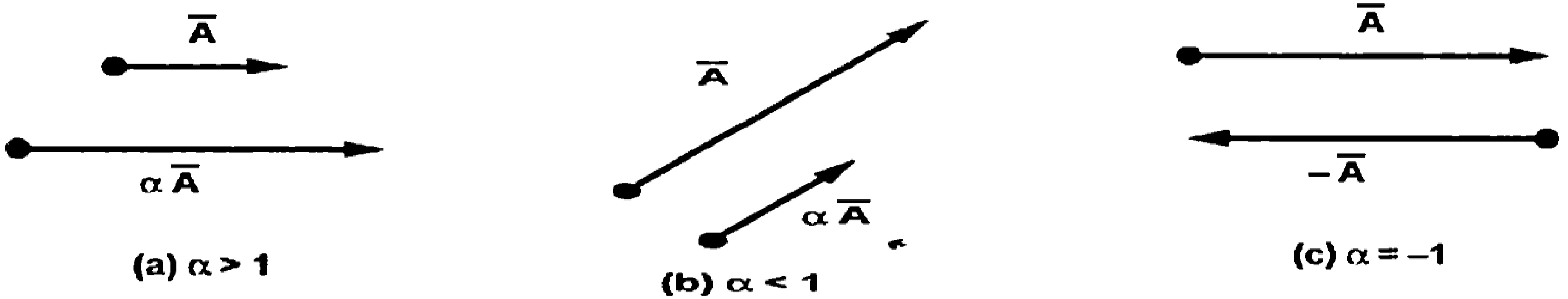
Vector algebra

The various mathematical operations can be performed with vectors such as:-

1) Scaling 2) Addition 3) Subtraction

1) Scaling of a vector

This is nothing but the multiplication by a scalar to vector. Such a multiplication changes the magnitude but not it's direction , when the scalar is positive.



2) Addition of vectors

consider two coplanar vectors as shown

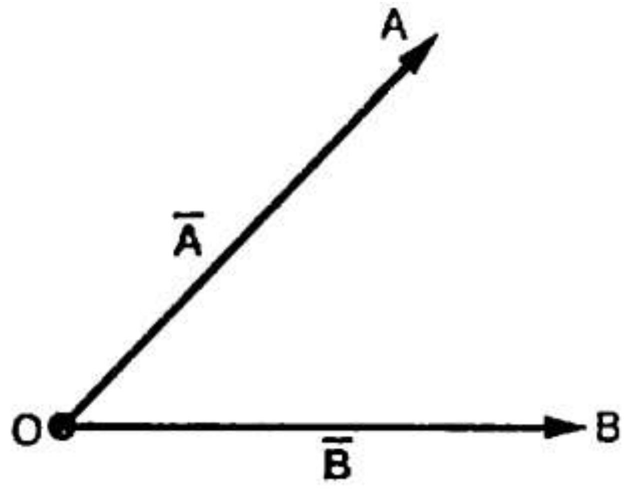


Fig (a) coplanar vector

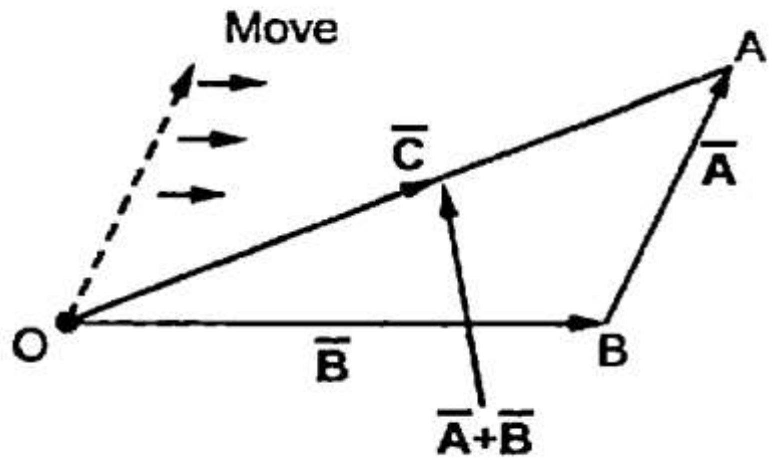


Fig.(b) Addition of vectors

- The procedure is to move one of the two vectors parallel to itself at the tip of the other vector.
- By the addition of $\bar{A}$ and $\bar{B}$ the resultant $\bar{C}$ is obtained.

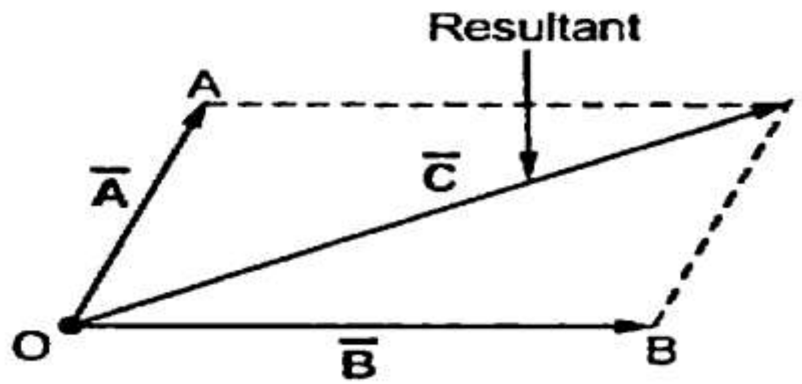
$$\bar{C} = \bar{A} + \bar{B}$$

- The direction of $\bar{C}$ is from origin O to the tip of the vector moved
- If $\bar{B}$ is moved parallel to itself at the tip of $\bar{A}$, we get same resultant $\bar{C}$. Thus

$$\bar{A} + \bar{B} = \bar{B} + \bar{A}.$$

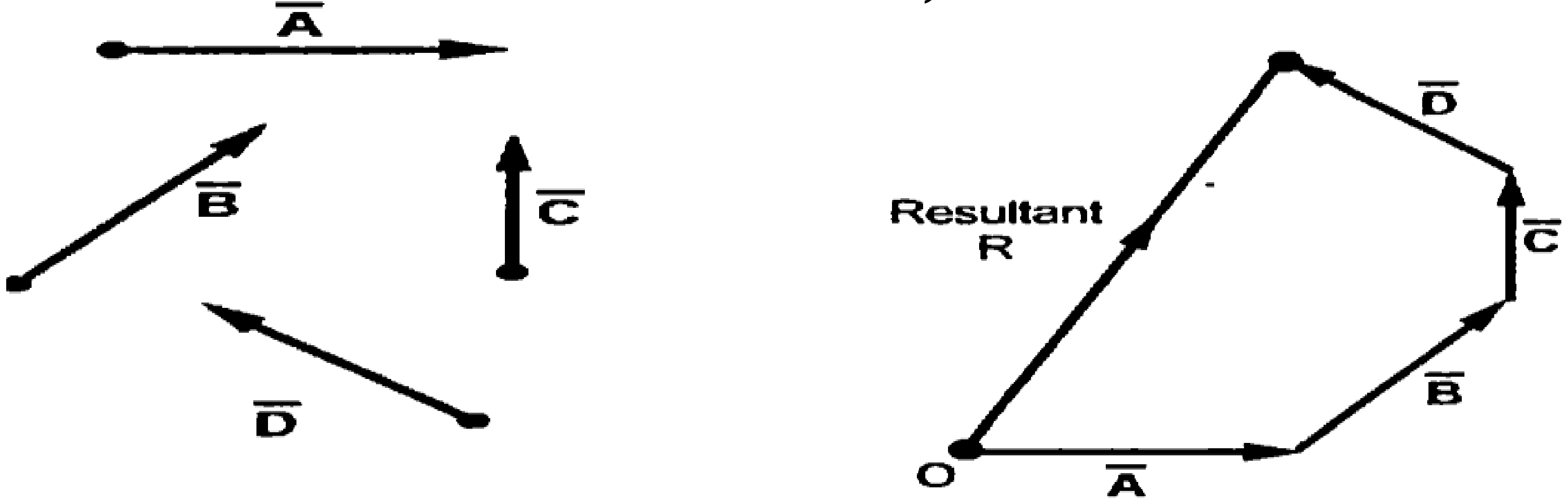
Another method of performing the addition of vectors is parallelogram rule

- complete the parallelogram as shown . Then diagonal of the parallelogram represents the addition of the two vectors.



Parallelogram rule for addition

•By using any of these two methods , no. of vectors can be added to obtain resultant



(a) Four vectors

(b) Sum of the four vectors

$$\bar{R} = \bar{A} + \bar{B} + \bar{C} + \bar{D}$$

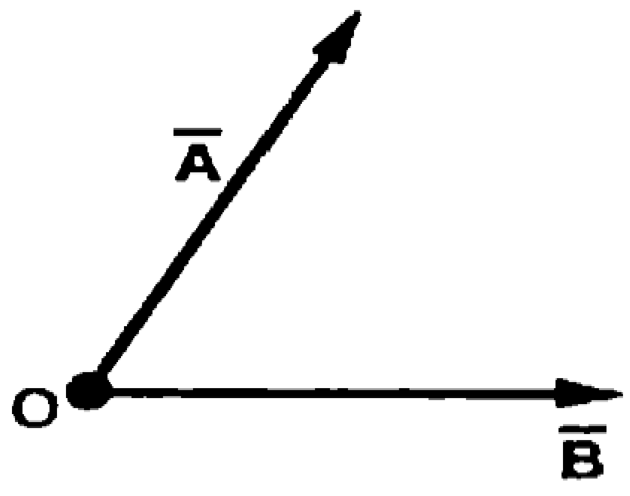
- The following basic laws of algebra are obeyed by the vectors

Law	Addition	Multiplication by scalar
Commutative	$\bar{A} + \bar{B} = \bar{B} + \bar{A}$	$\alpha\bar{A} = \bar{A}\alpha$
Associative	$\bar{A} + (\bar{B} + \bar{C}) = (\bar{A} + \bar{B}) + \bar{C}$	$\beta(\alpha\bar{A}) = (\beta\alpha)\bar{A}$
Distributive	$\alpha(\bar{A} + \bar{B}) = \alpha\bar{A} + \alpha\bar{B}$	$(\alpha + \beta)\bar{A} = \alpha\bar{A} + \beta\bar{A}$

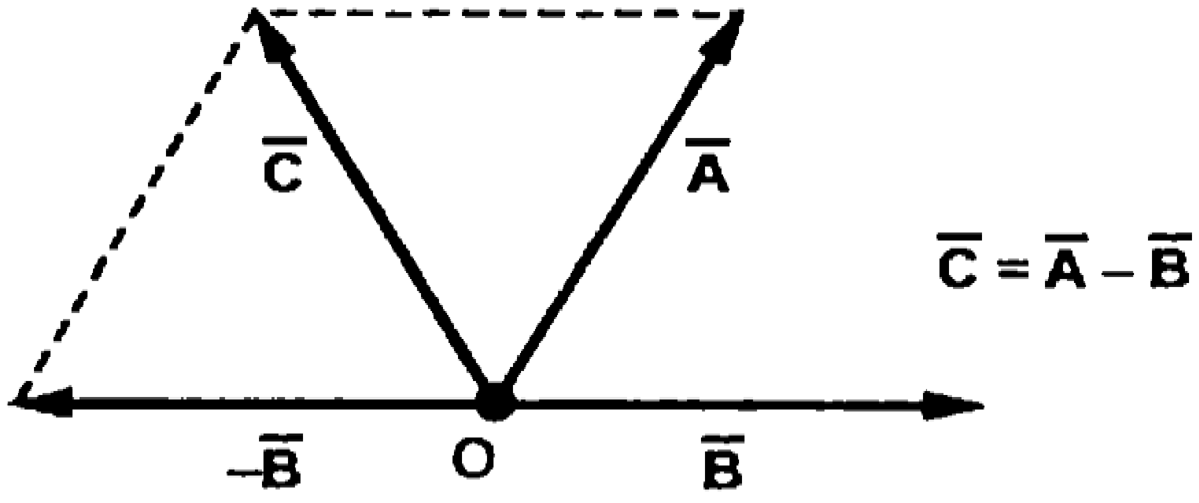
α and β are scalars

3) Subtraction of vectors

The subtraction of vectors can be obtained from the rules of addition. If B is to be subtracted from A then based on addition it can be represented as $\bar{C} = \bar{A} + (-\bar{B})$



(a) Vectors



(b) Subtraction of vectors

Identical vector

Two vectors are said to be identical if there difference is zero

The co-ordinate system

To describe a vector accurately and to express a vector in terms of its components it is necessary to have some reference directions. Such directions are represented in terms of various co-ordinate systems. These are-

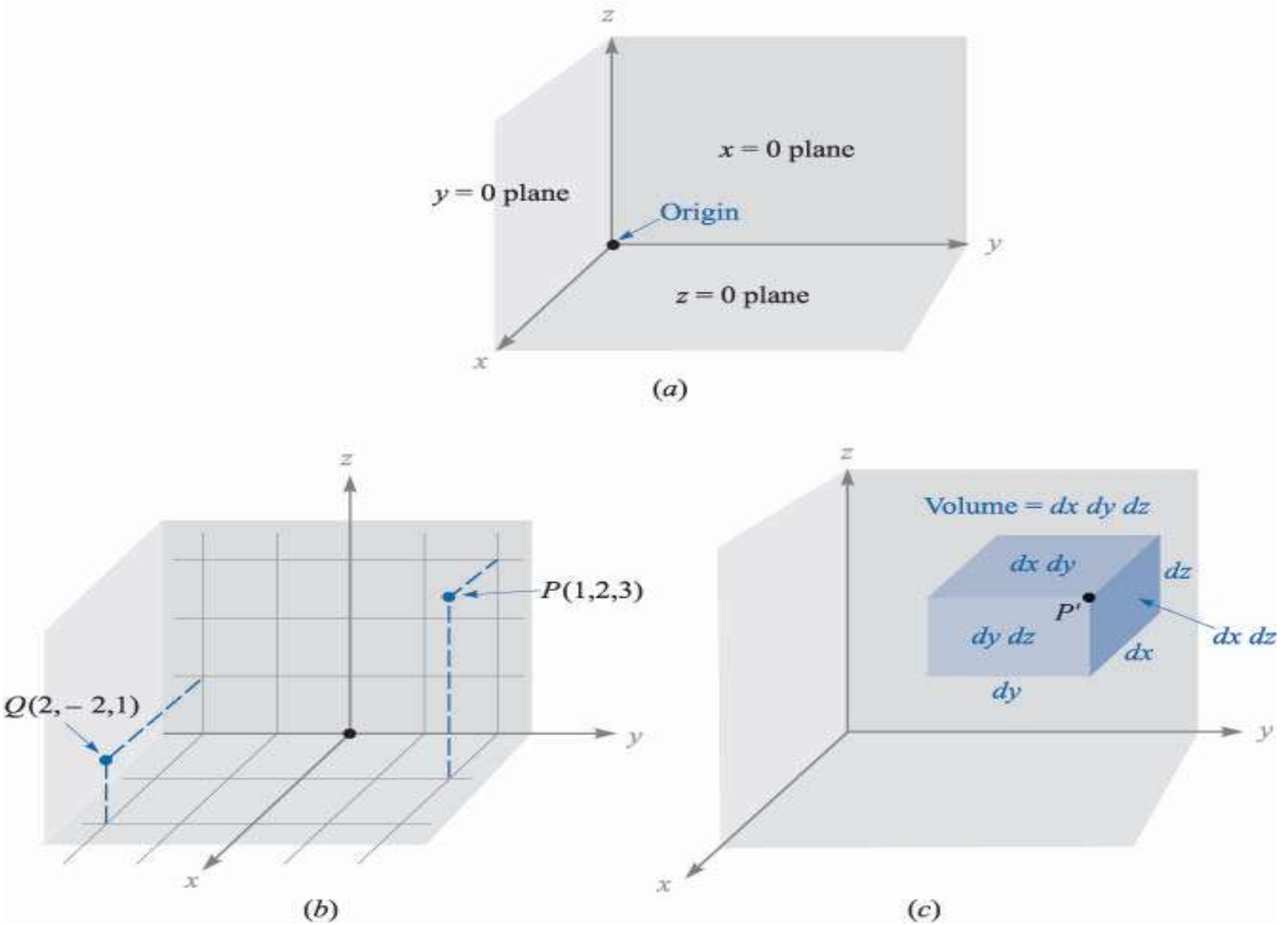
1) Cartesian or rectangular co-ordinate system
2) cylindrical co-ordinate system
3) Spherical co-ordinate system

1) Cartesian or rectangular co-ordinate system

This system has three co-ordinate axes represented as x , y and z which are mutually perpendicular to each other. These three axes intersects at a common point called origin of the system.

There are two types of such systems called

a) Right handed system and b) Left handed system.



•Right handed system means if x axes is rotated towards y axes through a smaller angle, then this rotation causes the upward movement of right handed screw in the z axes. If the right hand is used then the thumb , forefinger and middle finger may be identified , respectively as x, y and z axes.

(a) Right handed Cartesian co-ordinate system , (b) The location of point P (1,2,3) and Q(2,-2,1), (c) The differential volume element, dx, dy and dz are in general independent differentials.

- A point is located by giving its x, y and z co-ordinates.These are respectively the distances from the origin to the intersection of a perpendicular dropped from the point to the x,y and z axes.
- The alternative method is- to consider a point as being at the common intersection of three surfaces, the planes x=constant, y= constant and z = constant. The constant being the co-ordinate values of the point
- If we visualize three planes intersecting at a general point P (x,y,z),we may increase each co-ordinate value by a differential amount and obtain three slightly displaced planes intersecting at P' (x+dx, y+dy, z+dz)
- The six planes define a rectangular parallelepiped whose volume is **dv= dx dy dz.**
- The surfaces has different areas **ds** of **dx dy , dy dz** and **dz dx.**
- Finally the distance dl from P to P' is the diagonal of the parallelepiped and has the length of $|\bar{dl}| = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$

Point P' is indicated , but point P is located the only invisible corner.

- **Unit vectors-** three unit vectors are $\bar{\mathbf{a}}_x$, $\bar{\mathbf{a}}_y$ **and** $\bar{\mathbf{a}}_z$.
- **position or distance vectors** are – consider a point P (x_1,y_1,z_1) . The distance of point P from origin is represented by position vector or radius vector.

$$\bar{r}_{OP} = x_1 \bar{a}_x + y_1 \bar{a}_y + z_1 \bar{a}_z$$

The magnitude of this vector is given by $|\bar{r}_{OP}| = \sqrt{(x_1)^2 + (y_1)^2 + (z_1)^2}$

- Now if $\bar{P} = x_1 \bar{a}_x + y_1 \bar{a}_y + z_1 \bar{a}_z$

$$\bar{Q} = x_2 \bar{a}_x + y_2 \bar{a}_y + z_2 \bar{a}_z$$

Then the displacement from P to Q is represented by $\overline{PQ} = \bar{Q} - \bar{P} = [x_2 - x_1]\bar{a}_x + [y_2 - y_1]\bar{a}_y + [z_2 - z_1]\bar{a}_z$

$$|\overline{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

This is the length of vector PQ

We can find a unit vector along the direction PQ as $\bar{a}_{PQ} = \text{Unit vector along PQ} = \dfrac{\overline{PQ}}{|\overline{PQ}|}$

Differential elements in Cartesian co-ordinate system

dx= differential length in x direction

dy= differential length in y direction

dz = differential length in z direction

differential vector length is $\overline{dl} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z$

distance from P to P' $\left|\overline{dl}\right| = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$

Differential Volume $dv = dx\,dy\,dz$

Differential surface area $d\bar{S} = dS\,\bar{a}_n$

$\bar{a}_n$ = Unit vector normal to the surface dS

$d\bar{S}_x$ = Differential vector surface area normal to x direction

$= dydz\,\bar{a}_x$

$d\bar{S}_y$ = Differential vector surface area normal to y direction

$= dxdz\,\bar{a}_y$

$d\bar{S}_z$ = Differential vector surface area normal to z direction

$= dxdy\,\bar{a}_z$

- **Example 1-** Obtain the unit vector in the direction from the origin towards the point P(3,-3,-2).

Solution : The origin O (0, 0, 0) while P (3, – 3, – 2) hence the distance vector $\overline{OP}$ is,

$$\overline{OP} = (3-0)\bar{a}_x + (-3-0)\bar{a}_y + (-2-0)\bar{a}_z = 3\bar{a}_x - 3\bar{a}_y - 2\bar{a}_z$$

$$\therefore \qquad |\overline{OP}| = \sqrt{(3)^2 + (-3)^2 + (-2)^2} = 4.6904$$

Hence the unit vector along the direction OP is,

$$\bar{a}_{OP} = \frac{\overline{OP}}{|\overline{OP}|} = \frac{3\bar{a}_x - 3\bar{a}_y - 2\bar{a}_z}{4.6904}$$

$$= 0.6396\,\bar{a}_x - 0.6396\,\bar{a}_y - 0.4264\,\bar{a}_z$$

- **Example-2** Two points A(2,2,1) and B(3,-4,2)are given in Cartesian system. Obtain the vector from A to B and a unit vector directed from A to B.

Solution : The starting point is A and terminating point is B.

Now $\bar{A} = 2\,\bar{a}_x + 2\bar{a}_y + \bar{a}_z$ and $\bar{B} = 3\bar{a}_x - 4\bar{a}_y + 2\bar{a}_z$

$\therefore \quad \overline{AB} = \bar{B} - \bar{A} = (3-2)\bar{a}_x + (-4-2)\bar{a}_y + (2-1)\bar{a}_z$

$\therefore \quad \overline{AB} = \bar{a}_x - 6\,\bar{a}_y + \bar{a}_z$

This is the vector directed from A to B.

Now $|\overline{AB}| = \sqrt{(1)^2 + (-6)^2 + (1)^2} = 6.1644$

Thus unit vector directed from A to B is,

$$\bar{a}_{AB} = \frac{\overline{AB}}{|\overline{AB}|} = \frac{\bar{a}_x - 6\bar{a}_y + \bar{a}_z}{6.1644}$$

$$= 0.1622\,\bar{a}_x - 0.9733\,\bar{a}_y + 0.1622\,\bar{a}_z$$

It can be cross checked that magnitude of this unit vector is unity i.e.

$\sqrt{(0.1622)^2 + (-0.9733)^2 + (0.1622)^2} = 1.$

- **Example-3** Given A(3,-2,1), B(-3,-3,5) , C(2,6,-4) Find-

(i) Vector from A to C, (ii) unit vector from B to A, (iii) the distance from B to C and (iv) The vector from A to midpoint of the straight line joining B to C.

Solution : The position vectors for the given points are,

$$\bar{A} = 3\bar{a}_x - 2\bar{a}_y + \bar{a}_z, \quad \bar{B} = -3\bar{a}_x - 3\bar{a}_y + 5\bar{a}_z, \quad \bar{C} = 2\bar{a}_x + 6\bar{a}_y - 4\bar{a}_z$$

i) The vector from A to C is,

$$\overline{AC} = \bar{C} - \bar{A} = [2-3]\bar{a}_x + [6-(-2)]\bar{a}_y + [-4-1]\bar{a}_z$$

$$= -\bar{a}_x + 8\bar{a}_y - 5\bar{a}_z$$

ii) For unit vector from B to A, obtain distance vector $\overline{BA}$ first.

$$\therefore \quad \overline{BA} = \bar{A} - \bar{B} \qquad \text{... as starting is B and terminating is A}$$

$$= [3-(-3)]a_x + [(-2)-(-3)]\bar{a}_y + [1-5]\bar{a}_z$$

$$= 6\bar{a}_x + \bar{a}_y - 4\bar{a}_z$$

$$\therefore \quad |\overline{BA}| = \sqrt{(6)^2 + (1)^2 + (-4)^2} = 7.2801$$

$$\therefore \bar{a}_{BA} = \frac{\overline{BA}}{|\overline{BA}|} = \frac{6\bar{a}_x + \bar{a}_y - 4\bar{a}_z}{7.2801} = 0.8241\,\bar{a}_x + 0.1373\,\bar{a}_y - 0.5494\,\bar{a}_z$$

iii) For distance between B and C, obtain $\overline{BC}$

$$\overline{BC} = \overline{C} - \overline{B} = [2-(-3)]\,\bar{a}_x + [6-(-3)]\bar{a}_y + [(-4)-(5)]\,\bar{a}_z = 5\,\bar{a}_x + 9\,\bar{a}_y - 9\,\bar{a}_z$$

$$\therefore \quad \text{Distance BC} = \sqrt{(5)^2 + (9)^2 + (-9)^2} = 13.6747$$

iv) Let $B(x_1, y_1, z_1)$ and $C(x_2, y_2, z_2)$ then the co-ordinates of midpoint of BC are $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}, \dfrac{z_1+z_2}{2}\right)$.

$$\therefore \text{ Midpoint of BC} = \left(\frac{-3+2}{2}, \frac{-3+6}{2}, \frac{5-4}{2}\right) = (-0.5, 1.5, 0.5)$$

Hence the vector from A to this midpoint is

$$= [-0.5-3]\bar{a}_x + [1.5-(-2)]\bar{a}_y + [0.5-1]\,\bar{a}_z = -3.5\,\bar{a}_x + 3.5\,\bar{a}_y - 0.5\,\bar{a}_z$$

Vector Multiplication

- Scalar or Dot Product
- Vector or Cross Product

Scalar or Dot Product – is defined as a product of magnitude of **A** and **B** ,and the cosine of the smaller angle b/w them.

$$\bar{A} \cdot \bar{B} = |\bar{A}|\,|\bar{B}| \cos \theta_{AB}$$

The result of such a Dot product is Scalar hence it is also called as Scalar product.

Properties of Dot Product-

(i) If the two vectors are parallel to each other ie. $\theta = 0^{o}$, then $\cos\theta = 1$ thus $\bar{A} \cdot \bar{B} = |\bar{A}|\ |\bar{B}|$ **for parallel vectors**

(ii) If two vectors are perpendicular to each other ie. $\theta = 90^{o}$, then $\cos\theta = 0$ thus $\bar{A} \cdot \bar{B} = 0$ **for perpendicular vectors**

(iii) The Dot product obeys commutative Law- $\bar{A} \cdot \bar{B} = \bar{B} \cdot \bar{A}$

(iv) The Dot product obeys Distributive Law- $\bar{A} \cdot (\bar{B} + \bar{C}) = \bar{A} \cdot \bar{B} + \bar{A} \cdot \bar{C}$

(v) If the Dot product of vector with itself is performed then $\bar{A} \cdot \bar{A} = |\bar{A}|\ |\bar{A}| \cos 0^\circ = |\bar{A}|^2$

(vi) Consider the unit vectors $\bar{a}_x$, $\bar{a}_y$ and $\bar{a}_z$ in Cartesian co-ordinate system. All these vectors are mutually perpendicular to each other, hence there Dot product is zero.

$$\bar{a}_x \cdot \bar{a}_y = \bar{a}_y \cdot \bar{a}_x = \bar{a}_x \cdot \bar{a}_z = \bar{a}_z \cdot \bar{a}_x = \bar{a}_y \cdot \bar{a}_z = \bar{a}_z \cdot \bar{a}_y = 0$$

(vii) Any unit vectors dotted with itself is unity, $\bar{a}_x \cdot \bar{a}_x = \bar{a}_y \cdot \bar{a}_y = \bar{a}_z \cdot \bar{a}_z = 1$

(viii) Consider two vectors in Cartesian co-ordinate system,

$$\bar{A} = A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z \quad \text{and} \quad \bar{B} = B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_z$$

$$\text{Now } \bar{A} \cdot \bar{B} = (A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z) \cdot (B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_z)$$

This product has nine scalar terms as dot product obeys distributive law, but six terms out of nine will be zero involving the Dot product of different unit vectors. While the remaining three terms involve the unit vector doted with itself, the result of which is unity.

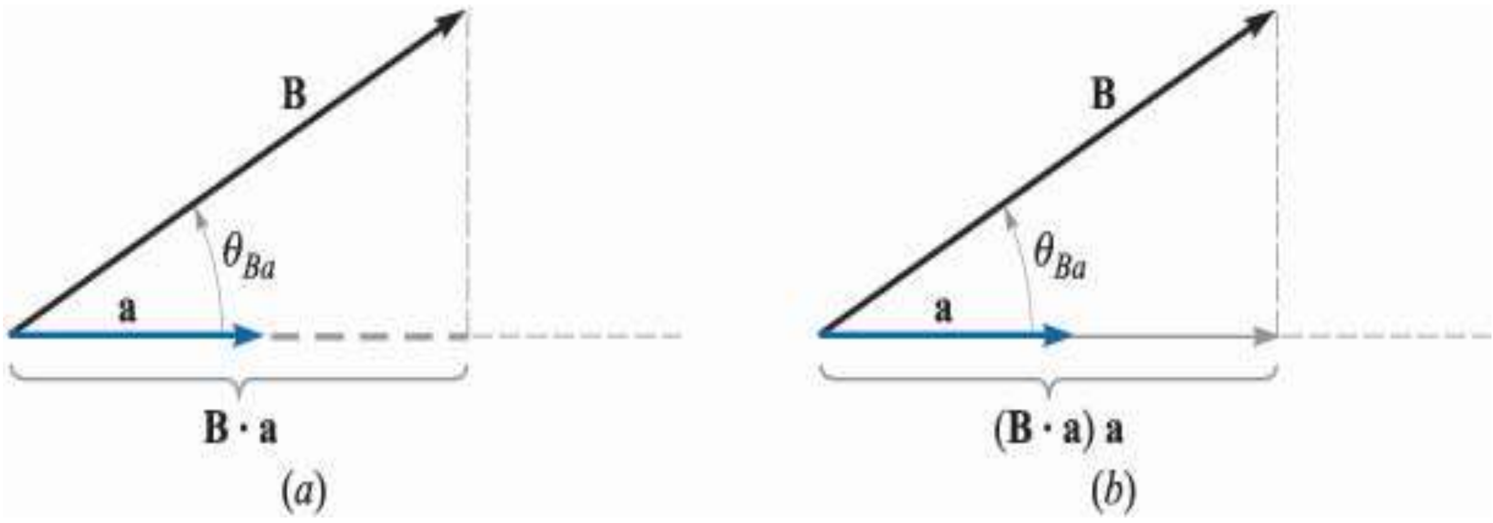
$$\bar{A} \cdot \bar{B} = A_x B_x (\bar{a}_x \cdot \bar{a}_x) + A_y B_y (\bar{a}_y \cdot \bar{a}_y) + A_z B_z (\bar{a}_z \cdot \bar{a}_z)$$

$$\bar{A} \cdot \bar{B} = A_x B_x + A_y B_y + A_z B_z$$

Application of Dot product-

(i) To determine angle b/w two vectors as- $\theta = \cos^{-1}\left\{\frac{\bar{A} \cdot \bar{B}}{|\bar{A}|\ |\bar{B}|}\right\}$

(ii) To find a component of a vector in the given direction-



From fig. (a) we can obtain the component (scalar) of **B** in the direction specified by the unit vector **a** as- $\mathbf{B} \cdot \mathbf{a} = |\mathbf{B}|\ |\mathbf{a}| \cos\theta_{Ba} = |\mathbf{B}| \cos\theta_{Ba}$

This sign of component is positive if $0 \le \theta < 90°$ and negative whenever $90° < \theta \le 180°$.

In order to obtain component vector of **B** in the direction of **a** , we simply multiply the component (scalar) by **a** as illustrated by fig(b).

For example- the component of **B** in the direction of $\mathbf{a_x}$ is $\mathbf{B} \cdot \mathbf{a_x} = B_x$ and the component vector is $B_x \mathbf{a_x}$ or $(\mathbf{B} \cdot \mathbf{a_x})\,\mathbf{a_x}$

The Geometrical term projection is also used with the dot product. Thus $\mathbf{B} \cdot \mathbf{a}$ is the projection of **B** in the direction of **a** .

Example 5- given two vectors- $\bar{A} = 2\bar{a}_x - 5\bar{a}_y - 4\bar{a}_z$ *and* $\bar{B} = 3\bar{a}_x + 5\bar{a}_y + 2\bar{a}_z$

Find the dot product and angle between the two vectors.

Solution- The dot product is-

$$\bar{A} \cdot \bar{B} = A_x B_x + A_y B_y + A_z B_z$$

$$= (2 \times 3) + (-5)(5) + (-4)(2) = 6 - 25 - 8$$

$$= -27$$

$$|\bar{A}| = \sqrt{(2)^2 + (-5)^2 + (-4)^2}$$

$$= \sqrt{45}$$

$$|\bar{B}| = \sqrt{(3)^2 + (5)^2 + (2)^2}$$

$$= \sqrt{38}$$

$$\theta = \cos^{-1}\left\{\frac{\bar{A} \cdot \bar{B}}{|\bar{A}|\ |\bar{B}|}\right\}$$

$$= \cos^{-1}\left\{\frac{-27}{\sqrt{45}\sqrt{38}}\right\}$$

$$= 130.762^\circ$$

- **Example 6-** Given vector field $\bar{G} = (y-1)\bar{a}_x + 2x\bar{a}_y$. Find this vector field at P(2,3,1) and it's projection on $\bar{B} = 5\bar{a}_x - \bar{a}_y + 2\bar{a}_z$.

Solution- The vector field $\bar{G}$ at P

$\bar{G}$ at P $= 2\bar{a}_x + 4\bar{a}_y$ substituting co-ordinates of P in **G**

To find its projection on $\bar{B}$, first find $\bar{a}_B$, the unit vector in the direction of $\bar{B}$.

$$\therefore \quad \bar{a}_B = \frac{\bar{B}}{|\bar{B}|} = \frac{5\bar{a}_x - \bar{a}_y + 2\bar{a}_z}{\sqrt{(5)^2 + (-1)^2 + (2)^2}}$$

$$= 0.9128\,\bar{a}_x - 0.1825\,\bar{a}_y + 0.3651\,\bar{a}_z$$

Hence projection of $\bar{G}$ at P on the vector $\bar{B}$ is,

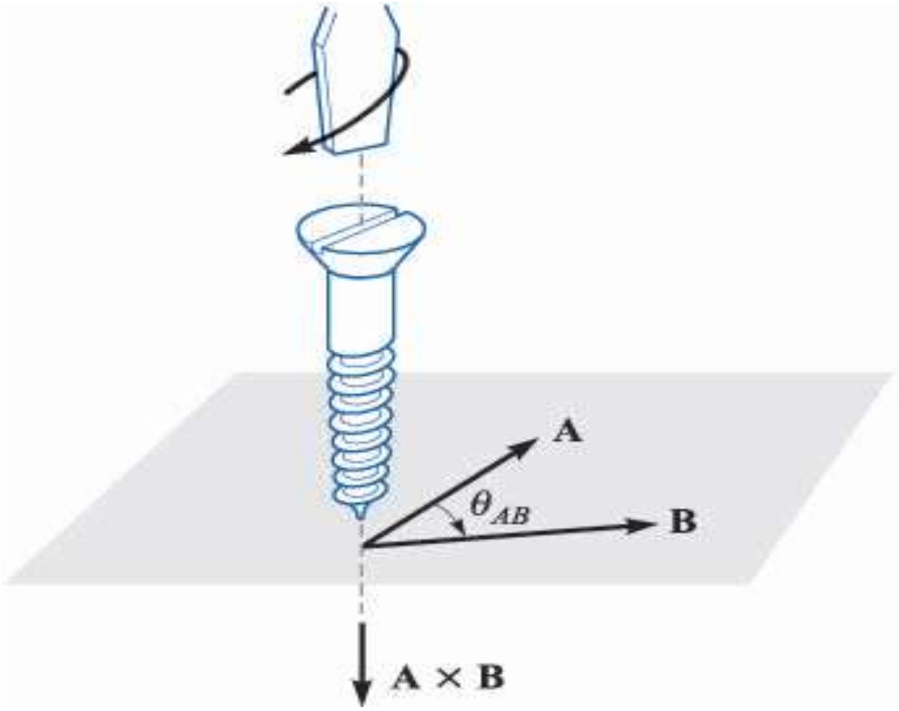
$$= (\bar{G} \text{ at } P) \cdot \bar{a}_B$$

$$= (2\times 0.9128) + (4\times -0.1825) + (0\times 0.3651) = 1.0956$$

Vector Product or Cross Product- is defined as a product of magnitude of **A** and **B** ,and the sine of the smaller angle b/w them. It is denoted as **A** × **B**. This product is the vector quantity hence the direction of **A** × **B** is perpendicular to the plane containing **A** and **B** , and is along that one of the two perpendicular which is in the direction of advance of a right- handed screw as **A** is turned into **B**.

$$\bar{A} \times \bar{B} = |\bar{A}| \; |\bar{B}| \sin\theta_{AB} \, \bar{a}_N$$

Where $\bar{a}_N$ = unit vector perpendicular to the plane of A and B in the direction decided by right handed screw rule.



Properties of Cross Product-

(i) The cross product is not commutative-

Reversing the order of vectors **A** and **B** results in a unit vector in opposite direction.

$$\bar{B} \times \bar{A} = -[\bar{A} \times \bar{B}]$$

(ii)The cross product is not associative, thus – $\bar{A}\times(\bar{B}\times\bar{C}) \neq (\bar{A}\times\bar{B})\times\bar{C}$

(iii) With respect to addition the cross product is distributive. Thus, $\bar{A}\times(\bar{B}+\bar{C}) = \bar{A}\times\bar{B}+\bar{A}\times\bar{C}$

(iv) If the two vectors are parallel to each other ie. $\theta = 0^{o}$, then $\sin\theta = 0$ thus cross product of such two vectors is zero.

(v) If the Cross product of a vector with itself is performed then $\bar{A}\times\bar{A} = 0$

(vi) Consider the unit vectors $\bar{a}_x$, $\bar{a}_y$ and $\bar{a}_z$ in Cartesian co-ordinate system. All these vectors are mutually perpendicular to each other, then

$$\bar{a}_x \times \bar{a}_y = |\bar{a}_x|\,|\bar{a}_y|\,\sin(90^{\circ})\bar{a}_N$$ In this case, $\bar{a}_N = \bar{a}_z$

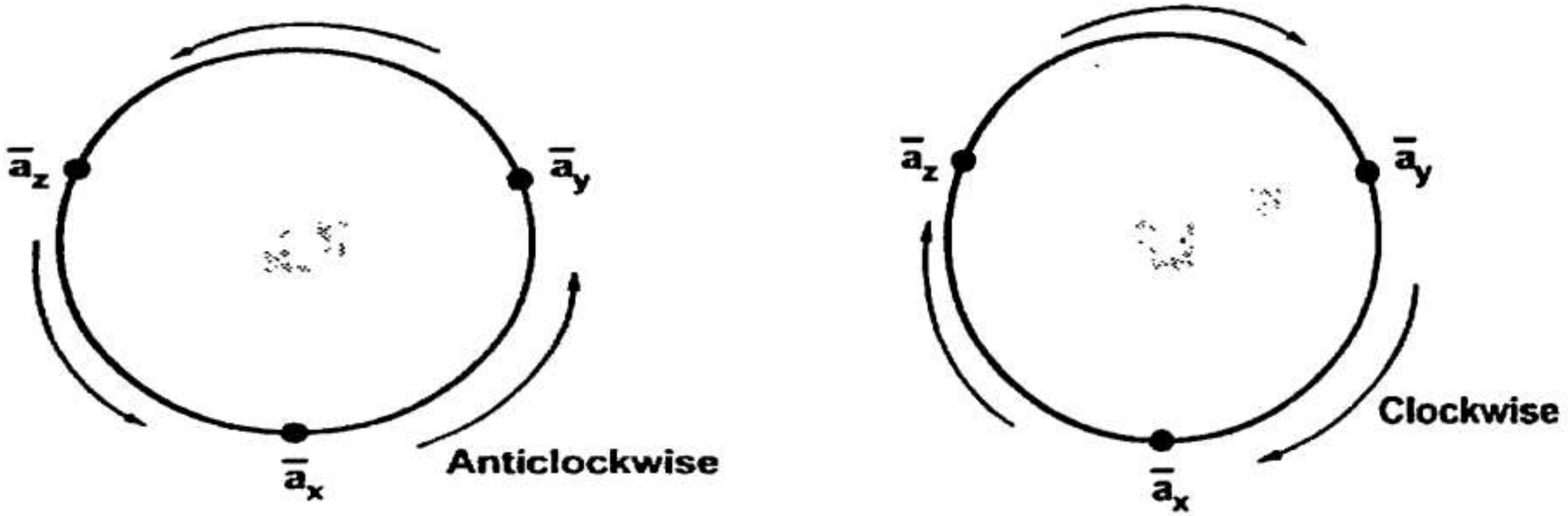
Hence,

$$\bar{a}_x \times \bar{a}_y = \bar{a}_z$$
$$\bar{a}_y \times \bar{a}_z = \bar{a}_x$$
$$\bar{a}_z \times \bar{a}_x = \bar{a}_y$$

But if the order of unit vectors is reversed, the result is negative of the remaining third unit vector. Thus -

$$\bar{a}_y \times \bar{a}_x = -\bar{a}_z, \quad \bar{a}_z \times \bar{a}_y = -\bar{a}_x, \quad \bar{a}_x \times \bar{a}_z = -\bar{a}_y$$

This can be remembered by a circle indicating cyclic permutations of cross product of unit vectors shown in fig



(a) Positive result (b) Negative result

(vii) Cross product in determinant form-

Consider two vectors in Cartesian co-ordinate system,

$$\bar{A} = A_x \bar{a}_x + A_y \bar{a}_y + A_z \bar{a}_z \quad \text{and} \quad \bar{B} = B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_z$$

$$\begin{aligned}\bar{A} \times \bar{B} \; = \; & A_x B_x (\bar{a}_x \times \bar{a}_x) + A_x B_y (\bar{a}_x \times \bar{a}_y) + A_x B_z (\bar{a}_x \times \bar{a}_z) \\ & + A_y B_x (\bar{a}_y \times \bar{a}_x) + A_y B_y (\bar{a}_y \times \bar{a}_y) + A_x B_z (\bar{a}_y \times \bar{a}_z) \\ & + A_z B_x (\bar{a}_z \times \bar{a}_x) + A_z B_y (\bar{a}_z \times \bar{a}_y) + A_z B_z (\bar{a}_z \times \bar{a}_z)\end{aligned}$$

$$= \left(A_y B_z - A_z B_y\right)\bar{a}_x + \left(A_z B_x - A_x B_z\right)\bar{a}_y + \left(A_x B_v - A_y B_x\right)\bar{a}_z$$

This result can be expressed in determinant form

$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

- **Example 7 -** Given two coplanar vector $\bar{\mathbf{A}} = 3\bar{\mathbf{a}}_x + 4\bar{\mathbf{a}}_y - 5\bar{\mathbf{a}}_z$ *and* $\bar{\mathbf{B}} = -6\bar{\mathbf{a}}_x + 2\bar{\mathbf{a}}_y + 4\bar{\mathbf{a}}_z$ obtain the unit vector normal to the plane containing the vectors **A** and **B** .

Solution-

$$\bar{\mathbf{A}} \times \bar{\mathbf{B}} = \begin{vmatrix} \bar{\mathbf{a}}_x & \bar{\mathbf{a}}_y & \bar{\mathbf{a}}_z \\ 3 & 4 & -5 \\ -6 & 2 & 4 \end{vmatrix}$$

$$= \bar{\mathbf{a}}_x \begin{vmatrix} 4 & -5 \\ 2 & 4 \end{vmatrix} - \bar{\mathbf{a}}_y \begin{vmatrix} 3 & -5 \\ -6 & 4 \end{vmatrix} + \bar{\mathbf{a}}_z \begin{vmatrix} 3 & 4 \\ -6 & 2 \end{vmatrix}$$

$$= 26\bar{\mathbf{a}}_x + 18\bar{\mathbf{a}}_y + 30\bar{\mathbf{a}}_z$$

$$\bar{\mathbf{a}}_N = \frac{\bar{\mathbf{A}} \times \bar{\mathbf{B}}}{|\bar{\mathbf{A}} \times \bar{\mathbf{B}}|} = \frac{26\bar{\mathbf{a}}_x - 18\bar{\mathbf{a}}_y + 30\bar{\mathbf{a}}_z}{\sqrt{(26)^2 + (18)^2 + (30)^2}}$$

$$= 0.5964\,\bar{\mathbf{a}}_x + 0.4129\,\bar{\mathbf{a}}_y + 0.6882\,\bar{\mathbf{a}}_z$$

This is unit vector normal to the plane containing the vectors **A** and **B** .

Products of three vectors

- Scalar triple product
- Vector triple product

Scalar triple product- The scalar triple product of three vectors **A**, **B** and **C** is mathematically defined as- $\bar{A}\cdot(\bar{B}\times\bar{C}) = \bar{B}\cdot(\bar{C}\times\bar{A}) = \bar{C}\cdot(\bar{A}\times\bar{B})$

Thus if,

$$\bar{A} = A_x\,\bar{a}_x + A_y\,\bar{a}_y + A_z\,\bar{a}_z$$

$$\bar{B} = B_x\,\bar{a}_x + B_y\,\bar{a}_y + B_z\,\bar{a}_z$$

$$\bar{C} = C_x\,\bar{a}_x + C_y\,\bar{a}_y + C_z\,\bar{a}_z$$

Then the scalar triple product is obtained by the determinant

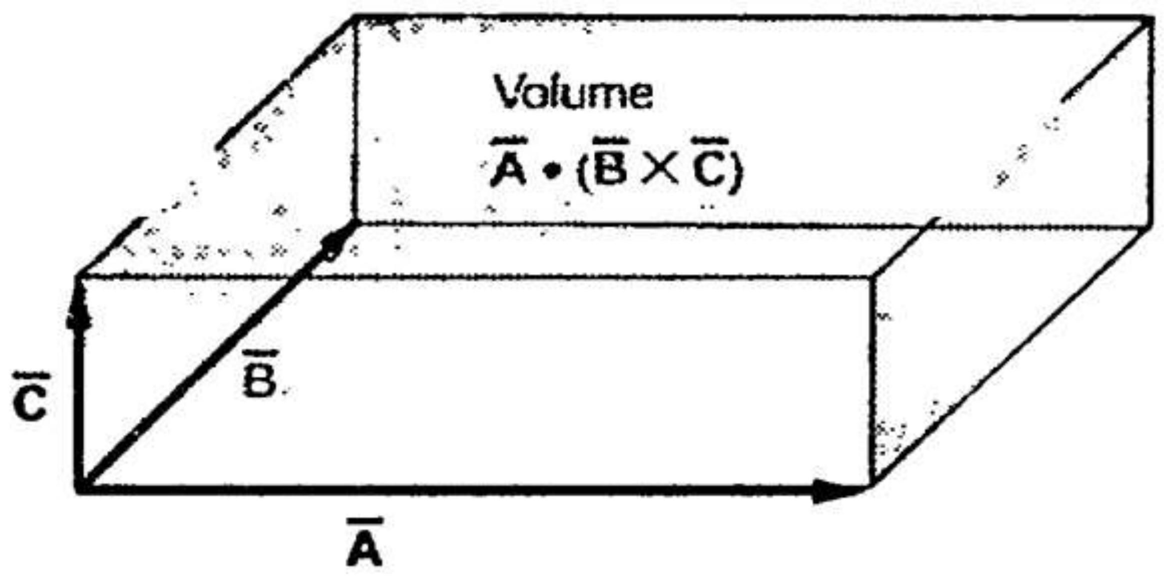
$$\bar{A}\cdot(\bar{B}\times\bar{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

The result of this product is scalar hence the product is called scalar triple product.

The cyclic order a, b, c is important.

Characteristics of scalar triple product-

- This product represents the volume of a parallelepiped with edges **A**, **B** and **C** drawn from the same origins as shown in fig.



- This product depends only on the cyclic order of ‘ a b c’ and not on the position of $\cdot$ and $\times$ in the product. If cyclic order is broken by permuting two of the vectors, the sign is reversed. $\bar{A} \cdot (\bar{B} \times \bar{C}) = -\bar{B} \cdot (\bar{A} \times \bar{C})$
- If two of the three vectors are equal then the result of the scalar triple product is zero.

$$\bar{A} \cdot (\bar{A} \times \bar{C}) = 0$$

- The scalar triple product is distributive.

Vector triple product - The vector triple product of three vectors **A**, **B** and **C** is mathematically defined as- $\bar{A}\times(\bar{B}\times\bar{C}) = \bar{B}(\bar{A}\cdot\bar{C}) - \bar{C}(\bar{A}\cdot\bar{B})$

The rule can be remembered as 'bac-cab' rule. The above rule can easily be proved by writing the Cartesian components of each term in equation. The position of the brackets is very important.

Characteristics of vector triple product-

- It must be noted that in the vector triple product- $(\bar{A}\cdot\bar{B})\bar{C} \neq \bar{A}(\bar{B}\cdot\bar{C})$

  but $(\bar{A}\cdot\bar{B})\bar{C} = \bar{C}(\bar{A}\cdot\bar{B})$

  This is because $\bar{A}\cdot\bar{B}$ is a scalar and multiplication by scalar to a vector is commutative.

- From the basic definition we can write- $\bar{B}\times(\bar{C}\times\bar{A}) = \bar{C}(\bar{B}\cdot\bar{A}) - \bar{A}(\bar{B}\cdot\bar{C})$

  $\bar{C}\times(\bar{A}\times\bar{B}) = \bar{A}(\bar{C}\cdot\bar{B}) - \bar{B}(\bar{C}\cdot\bar{A})$

- But dot product is commutative hence $\bar{C}\cdot\bar{A} = \bar{A}\cdot\bar{C}$ and so on. Hence

$$\bar{A}\times(\bar{B}\times\bar{C}) + \bar{B}\times(\bar{C}\times\bar{A}) + \bar{C}\times(\bar{A}\times\bar{B}) = 0$$

the result of the vector triple product is a vector.

Example-8- The three fields are given by - **A**= 2 $\mathbf{a}_x$ - 3 $\mathbf{a}_z$, **B**= 2 $\mathbf{a}_x$ - $\mathbf{a}_y$ + 2 $\mathbf{a}_z$ and **C**= 2 $\mathbf{a}_x$ -3 $\mathbf{a}_y$ + $\mathbf{a}_z$. Find scalar and vector triple product.

Solution- The scalar triple product is,

$$\bar{A} \cdot (\bar{B} \times \bar{C}) = \begin{vmatrix} 2 & 0 & -1 \\ 2 & -1 & 2 \\ 2 & -3 & 1 \end{vmatrix} = 14$$

The vector triple product is,

$$\bar{A} \times (\bar{B} \times \bar{C}) = \bar{B}(\bar{A} \cdot \bar{C}) - \bar{C}(\bar{A} \cdot \bar{B})$$

$$\bar{A} \cdot \bar{C} = (2)(2) + (0)(-3) + (-1)(1) = 3$$

$$\bar{A} \cdot \bar{B} = (2)(2) + (0)(-1) + (-1)(2) = 2$$

$$\therefore \quad \bar{A} \times (\bar{B} \times \bar{C}) = 3\bar{B} - 2\bar{C} = 3[2\bar{a}_x - \bar{a}_y + 2\bar{a}_z] - 2[2\bar{a}_x - 3\bar{a}_y + \bar{a}_z]$$

$$= 2\bar{a}_x + 3\bar{a}_y + 4\bar{a}_z$$

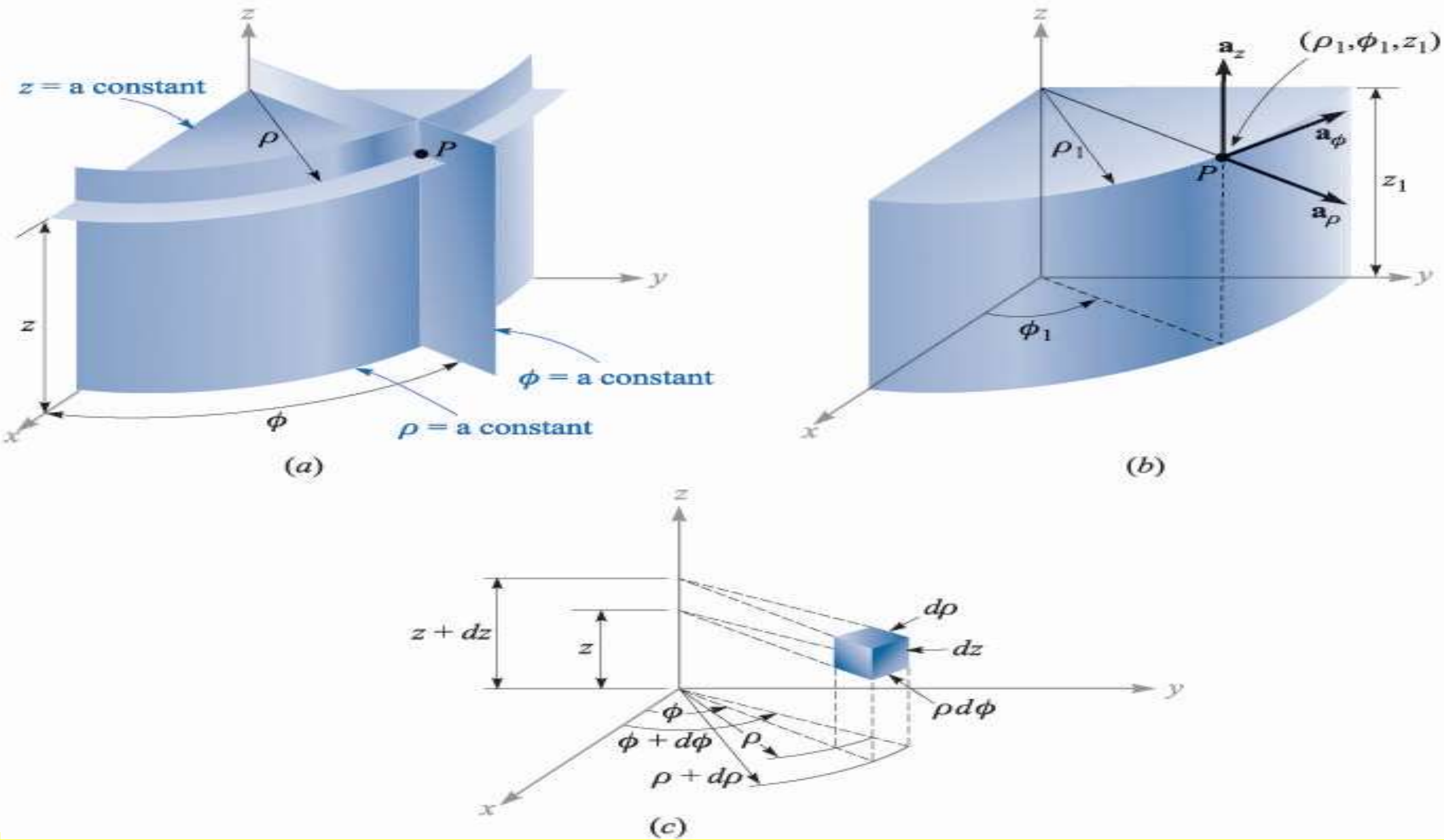
2) cylindrical co-ordinate system

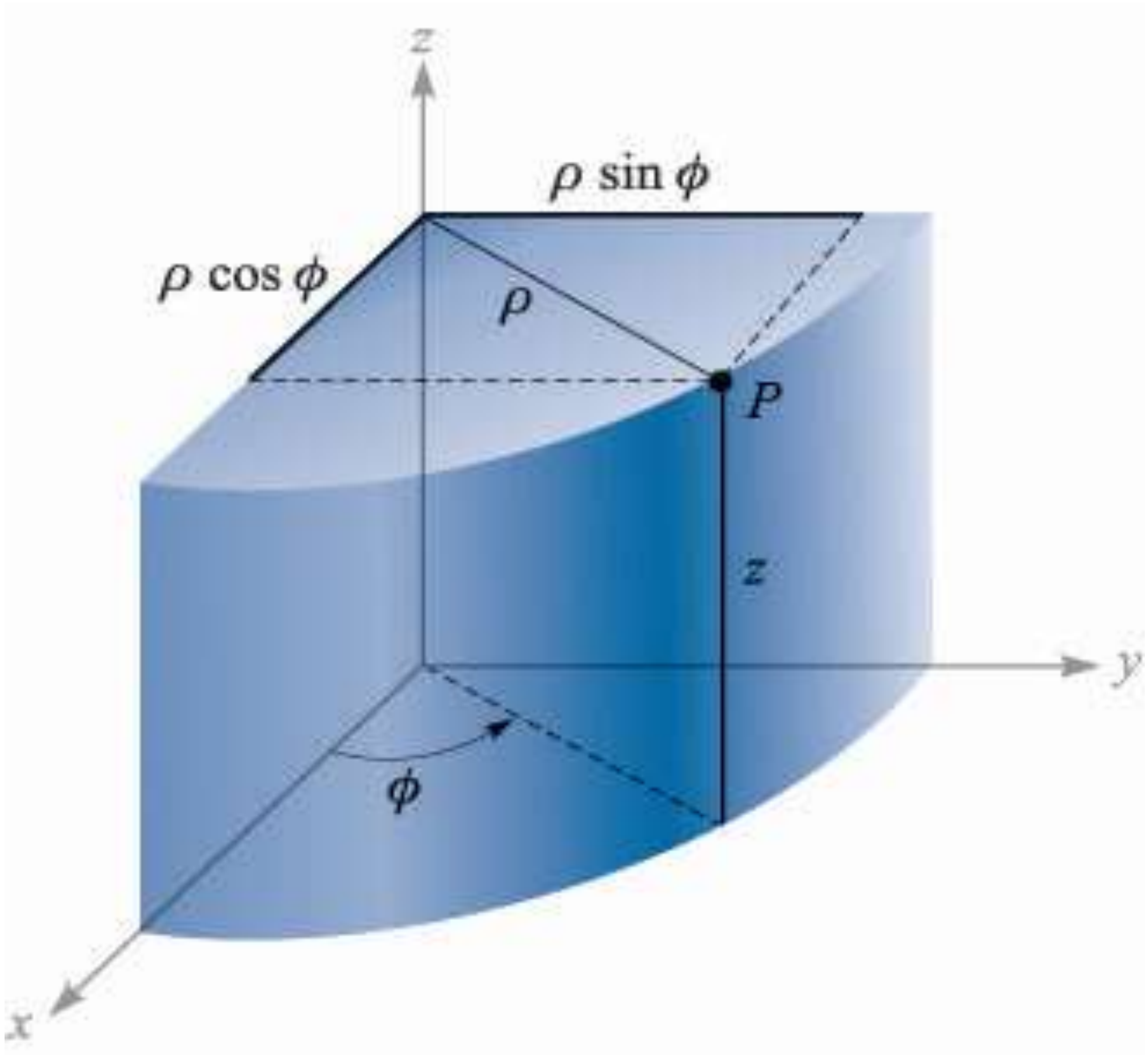
- consider any point as the intersection of three mutually perpendicular surfaces.
- These surfaces are a circular cylinder (ρ =constant), a plane (ϕ=constant), and

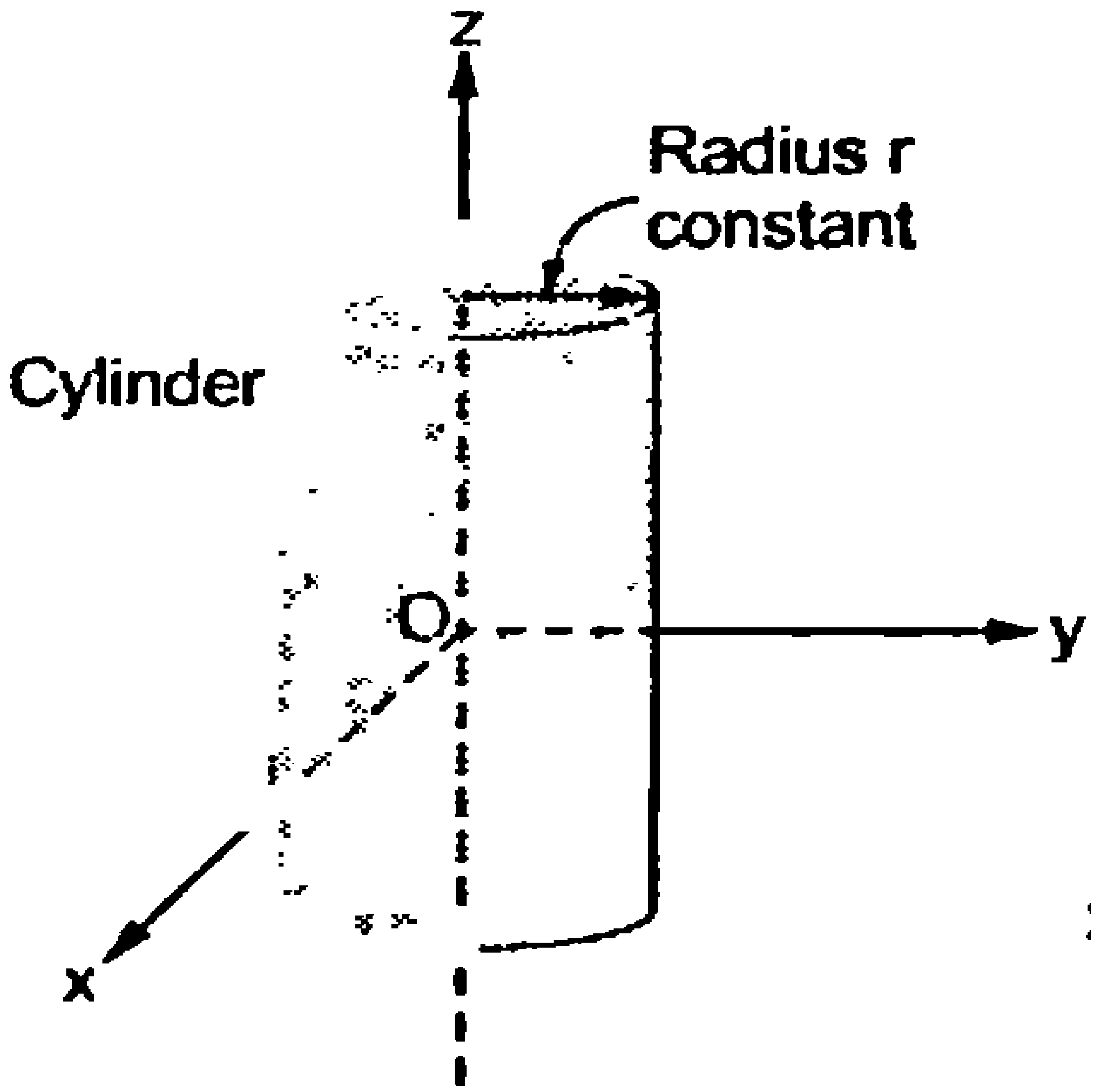
  another plane (z= constant).
- Three unit vectors must also be defined-

  they are directed toward increasing coordinate values and are perpendicular to the surface on which that coordinate value is constant (i.e., the unit vector $\mathbf{a_x}$ is normal to the plane x =constant and points toward larger values of x).

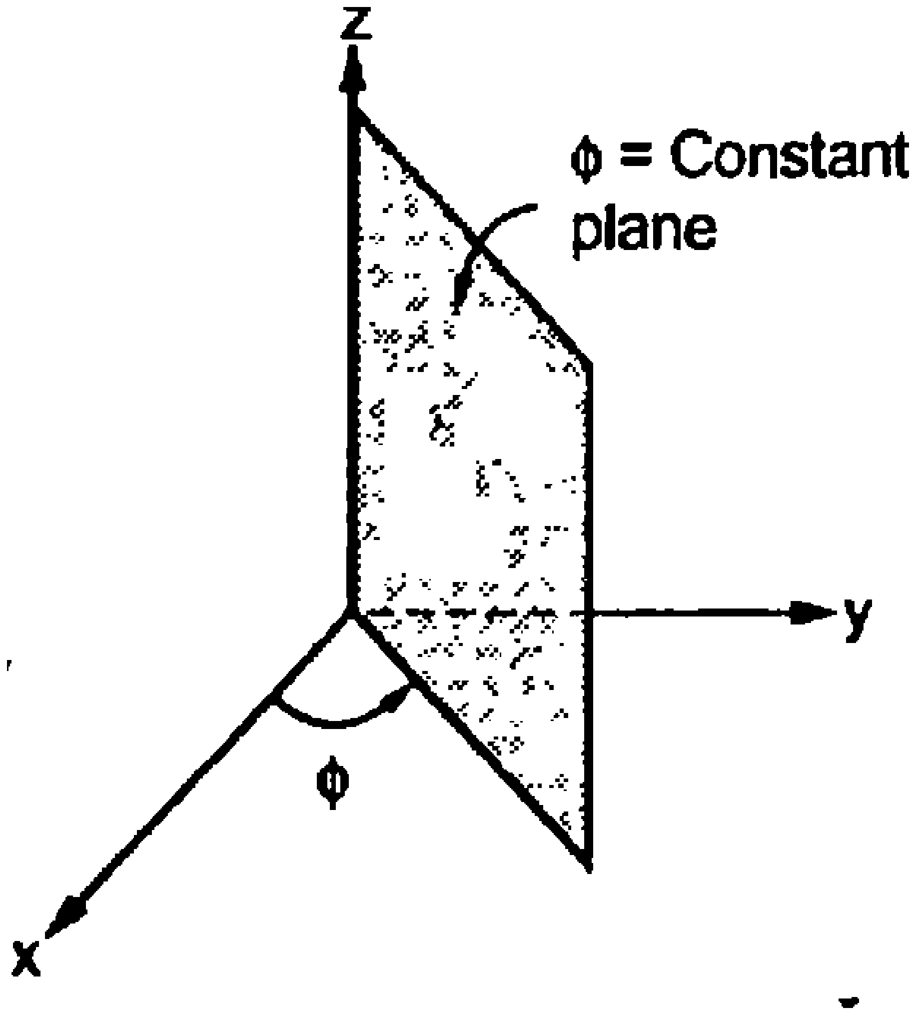
  three unit vectors in cylindrical coordinates , are- $\mathbf{a}_\rho$, $\mathbf{a}_\phi$ and $\mathbf{a}_z$
- The unit vector $\mathbf{a}_\rho$ at a point $P(\rho_1,\phi_1,z_1)$ is directed radially outward, normal to the cylindrical surface $\rho= \rho_1$. It lies in the planes $\phi=\phi_1$ and $z=z_1$.
- The unit vector $\mathbf{a}_\phi$ is normal to the plane $\phi= \phi_1$, points in the direction of increasing ϕ, lies in the plane $z = z_1$, and is tangent to the cylindrical surface $\rho= \rho_1$.
- The unit vector $\mathbf{a}_z$ is the same as the unit vector $\mathbf{a}_z$ of the cartesian coordinate system.


z = a constant
P
ρ
z
φ
φ = a constant
ρ = a constant
(a)
(ρ_1, ϕ_1, z_1)
$\mathbf{a}_z$
$\mathbf{a}_\phi$
$\mathbf{a}_\rho$
ρ_1
P
z_1
ϕ_1
(b)
$d\rho$
dz
$z + dz$
z
$\rho\, d\phi$
φ
$\phi + d\phi$
ρ
$\rho + d\rho$
(c)

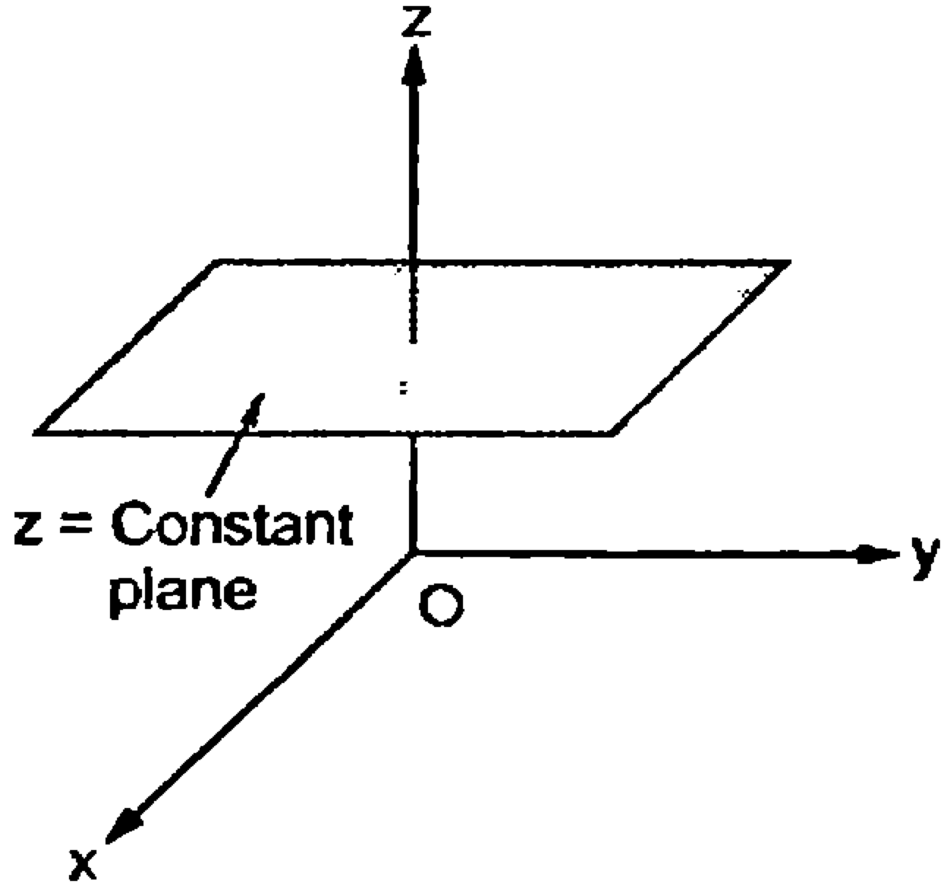

z
ρ sin φ
ρ cos φ
ρ
P
z
y
φ
x



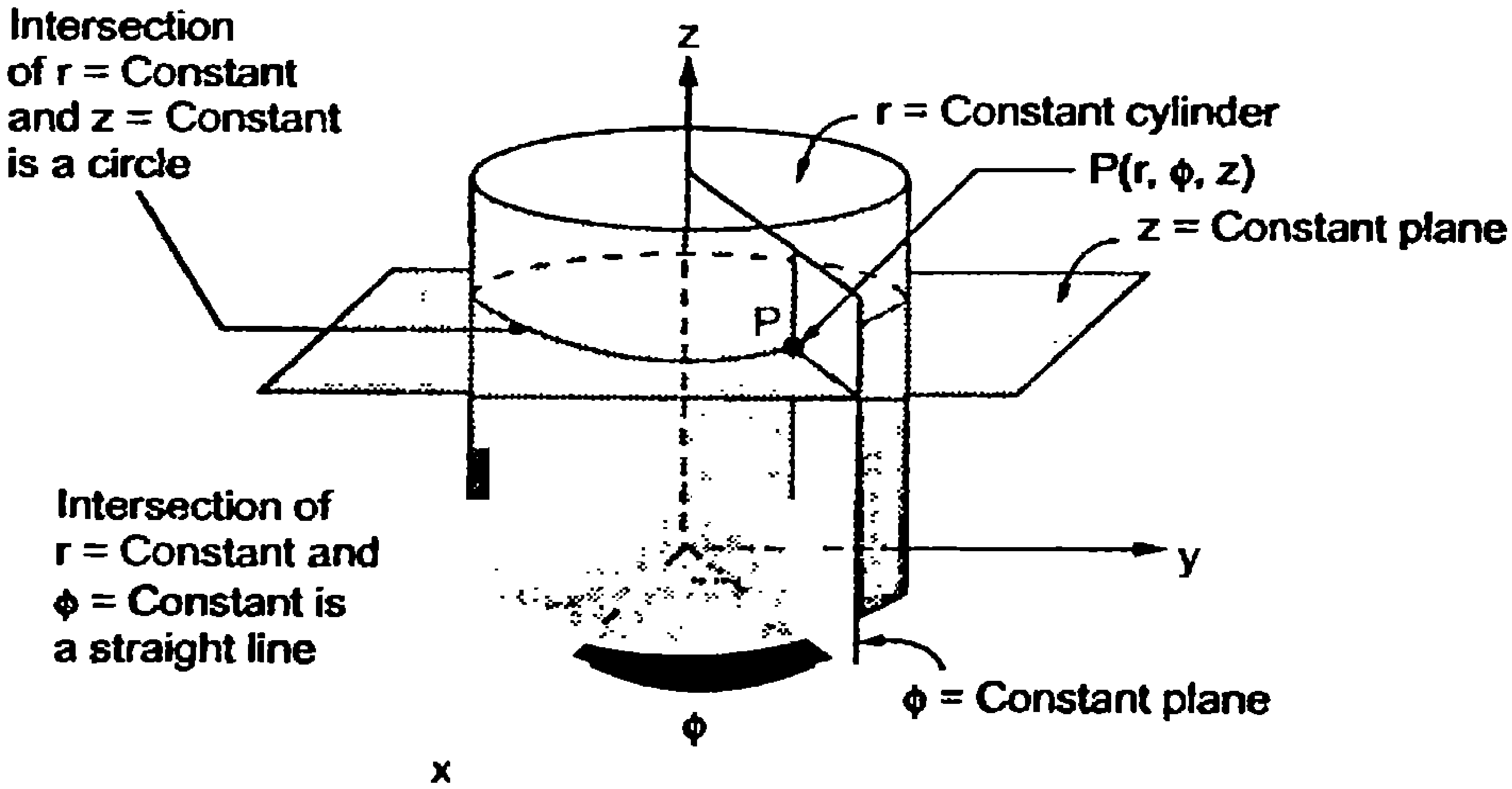
(a) r = Constant

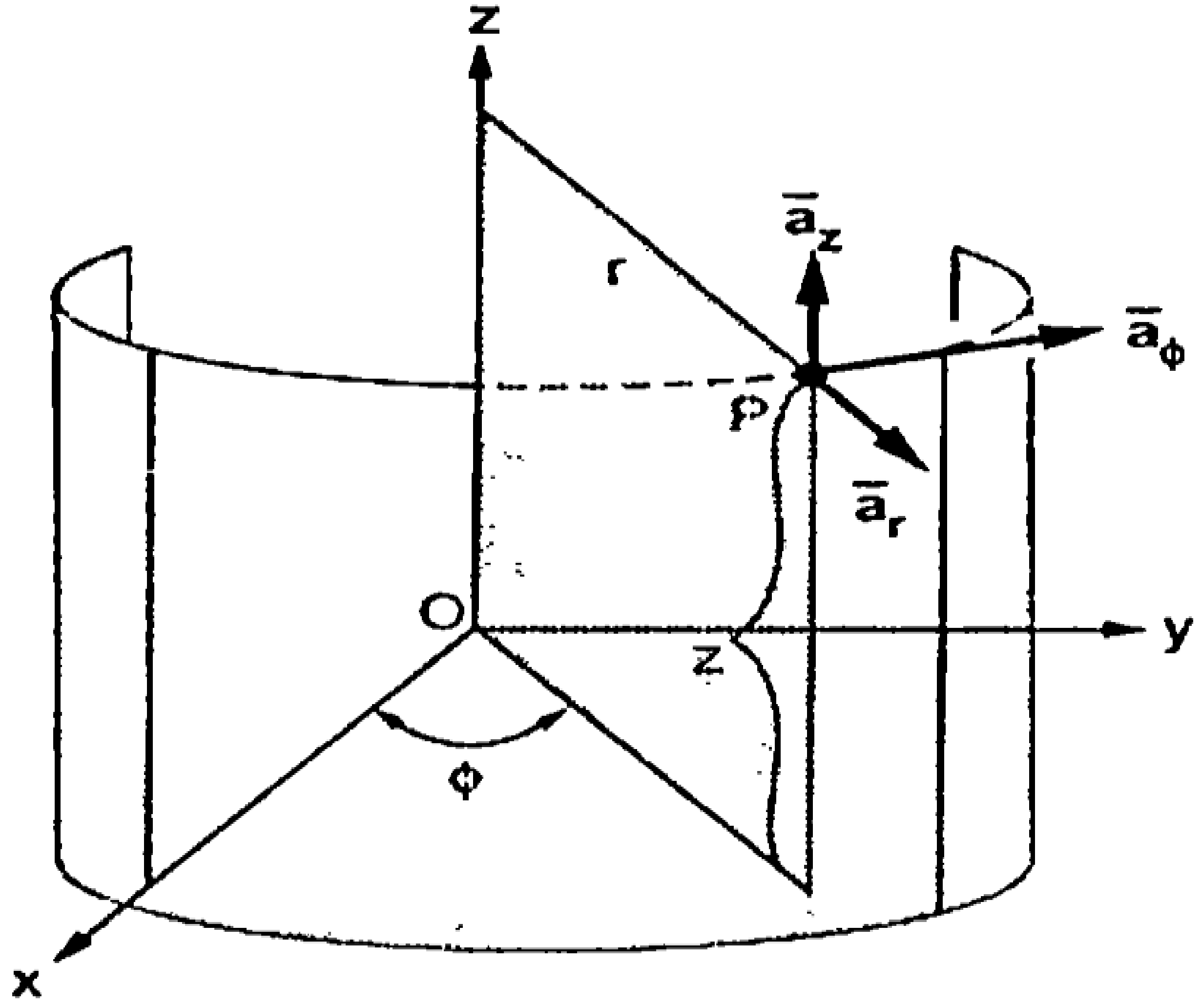


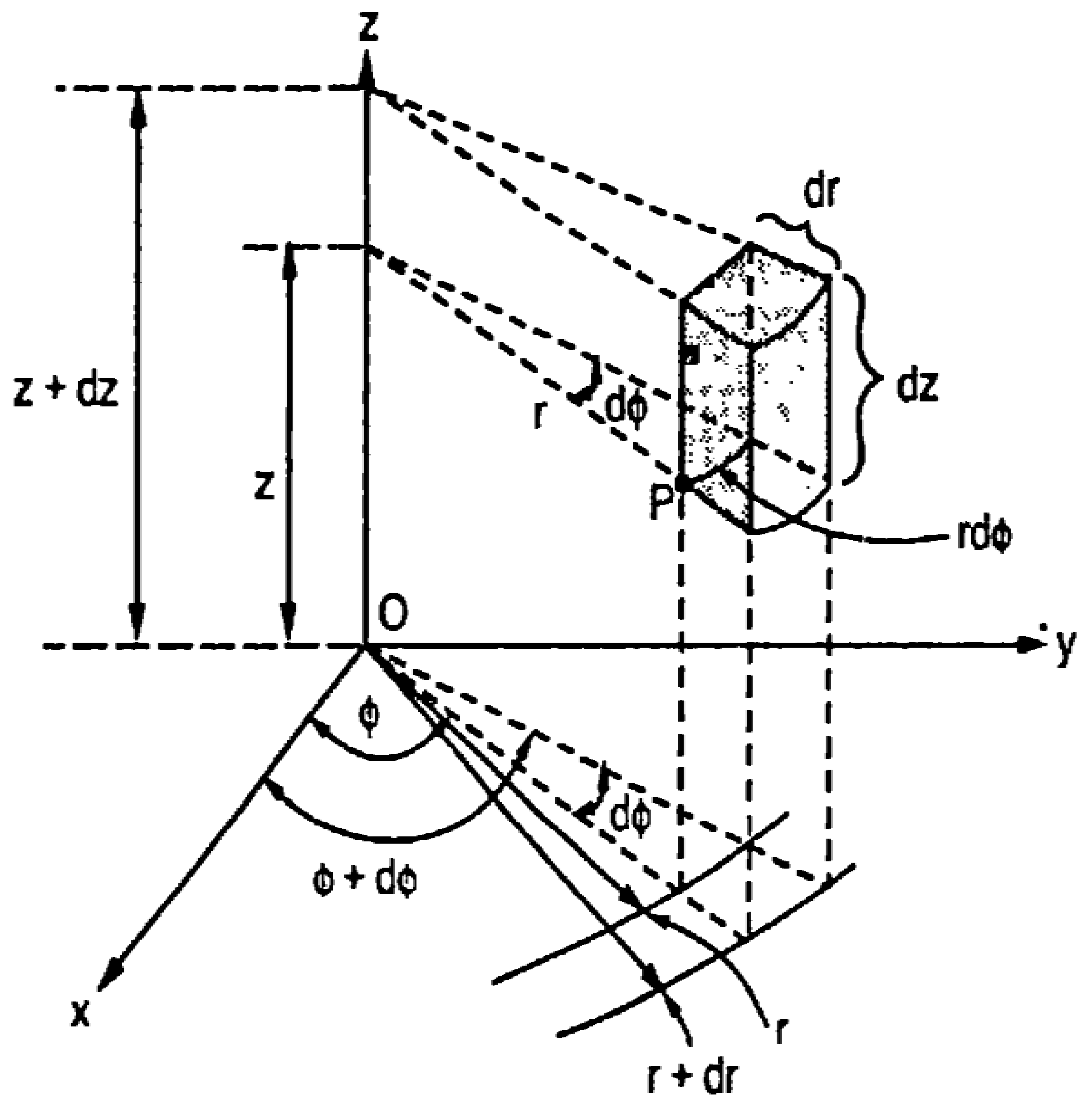
(b) ϕ = Constant



(c) z = Constant


Intersection
of r = Constant
and z = Constant
is a circle
z
r = Constant cylinder
P(r, φ, z)
z = Constant plane
P
Intersection of
r = Constant and
φ = Constant is
a straight line
y
φ = Constant plane
φ
x


z
r
$\bar{a}_z$
$\bar{a}_\phi$
ρ
$\bar{a}_r$
O
y
z
ϕ
x


z
dr
z + dz
dz
r
dϕ
z
P
rdϕ
O
y
ϕ
dϕ
ϕ + dϕ
x
r
r + dr

- A differential volume element in cylindrical coordinates may be obtained by increasing ρ, ϕ and z by the differential increments $d\rho$, $d\phi$ and dz.

The two cylinders of radius ρ and $\rho + d\rho$, the two radial planes at angles ϕ and $\phi + d\phi$, and the two ``horizontal" planes at ``elevations" z and dz. Now enclose a small volume, as shown in Fig. having the shape of a truncated wedge.

As the volume element becomes very small, its shape approaches that of a rectangular parallelepiped having sides of length $d\rho$; $d\phi$ and dz.

Note that $d\rho$ and dz are dimensionally lengths, but $d\phi$ is not; $\rho d\phi$ is the length.

- The surfaces have areas of $\rho\, d\rho\, d\phi$, $d\rho\, dz$, and $\rho\, d\phi\, dz$.
- and the volume becomes $\rho\, d\rho\, d\phi\, dz$:
- The variables of the rectangular and cylindrical coordinate systems are easily related to each other - $x = \rho \cos\phi$

$$y = \rho \sin\phi$$

$$z = z$$

- we may express the cylindrical variables in terms of x, y, and z:

$$\rho = \sqrt{x^2 + y^2} \qquad \emptyset = tan^{-1}\frac{y}{x} \quad \text{and} \quad z = z$$

- Using the above equations, scalar functions given in one coordinate system are easily transformed into the other system.
- A vector function in one coordinate system, requires two steps to transform it to another coordinate system,

In cartesian coordinate system, Vector $\mathbf{A} = A_x \mathbf{a_x} + A_y \mathbf{a_y} + A_z \mathbf{a_z}$

in cylindrical coordinates coordinates - $\mathbf{A} = A_\rho \mathbf{a_\rho} + A_\phi \mathbf{a_\phi} + A_z \mathbf{a_z}$

To find any desired component of a vector, from the definition of the dot product –

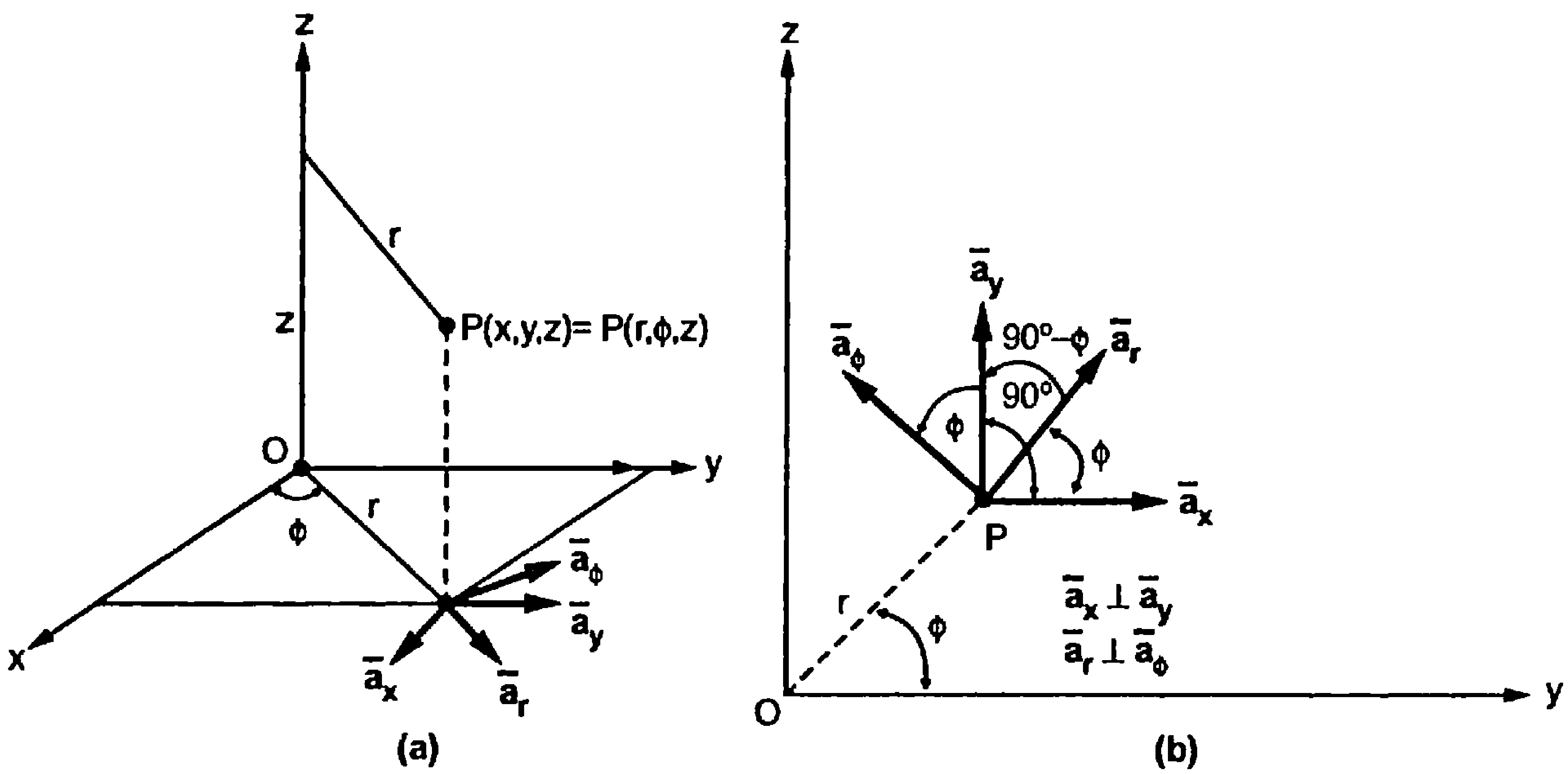
$A_\rho = \mathbf{A} \cdot \mathbf{a_\rho}$ and $A_\phi = \mathbf{A} \cdot \mathbf{a_\phi}$

Expending these products-

$A_\rho = (A_x \mathbf{a_x} + A_y \mathbf{a_y} + A_z \mathbf{a_z}) \cdot \mathbf{a_\rho} = A_x \mathbf{a_x} \cdot \mathbf{a_\rho} + A_y \mathbf{a_y} \cdot \mathbf{a_\rho}$

$A_\phi = (A_x \mathbf{a_x} + A_y \mathbf{a_y} + A_z \mathbf{a_z}) \cdot \mathbf{a_\phi} = A_x \mathbf{a_x} \cdot \mathbf{a_\phi} + A_y \mathbf{a_y} \cdot \mathbf{a_\phi}$

$A_z = (A_x \mathbf{a_x} + A_y \mathbf{a_y} + A_z \mathbf{a_z}) \cdot \mathbf{a_z} = A_z \mathbf{a_z} \cdot \mathbf{a_z} = A_z$


z
r
z
P(x,y,z)= P(r,φ,z)
O
y
r
φ
x
$\bar{a}_\phi$
$\bar{a}_y$
$\bar{a}_x$
$\bar{a}_r$
(a)
z
$\bar{a}_y$
$\bar{a}_\phi$
90°−φ
$\bar{a}_r$
90°
φ
φ
$\bar{a}_x$
P
$\bar{a}_x \perp \bar{a}_y$
$\bar{a}_r \perp \bar{a}_\phi$
r
φ
O
y
(b)

- Dot products of unit vectors - $\mathbf{a}_x \cdot \mathbf{a}_\rho$, $\mathbf{a}_y \cdot \mathbf{a}_\rho$, $\mathbf{a}_x \cdot \mathbf{a}_\phi$, $\mathbf{a}_y \cdot \mathbf{a}_\phi$ Can be determined by the following table-

$\bullet$	$\mathbf{a}_\rho$	$\mathbf{a}_\phi$	$\mathbf{a}_z$
$\mathbf{a}_x$	$\cos\phi$	$-\sin\phi$	0
$\mathbf{a}_y$	$\sin\phi$	$\cos\phi$	0
$\mathbf{a}_z$	0	0	1

Q. – (a) Give the Cartesian coordinates of the point C ($\rho = 4.4$, $\phi = -115^\circ$, $z = 2$).

(b) Give the Cylindrical coordinates of the point D ($x = -3.1$, $y = 2.6$, $z = -3$).

(c) specify the distance from C to D.

Ans- C ($x = -1.860$, $y = 3.99$, $z = 2$) ; D ($\rho = 4.05$, $\phi = 140.0^\circ$, $z = -3$); 8.36

Example-9-Transform the vector field $\overline{W} = 10\bar{a}_x - 8\bar{a}_y + 6\bar{a}_z$ to cylindrical co-ordinate system at point P(10,-8,6)

Solution- from the given field **W**,

$W_x = 10$, $W_y = -8$ and $W_z = 6$

Now W_ρ = $\mathbf{W} \bullet \mathbf{a}_\rho$ = $[10a_x - 8a_y + 6a_z] \bullet a_\rho$ = $10(a_x \bullet a_\rho) - 8(a_y \bullet a_\rho) + 6(a_z \bullet a_\rho)$

$= 10\cos\emptyset - 8\sin\emptyset + 0$

For point P, x= 10, and y= -8 $\emptyset = \tan^{-1}\left[\frac{y}{x}\right] = \tan^{-1}\left(\frac{-8}{10}\right) = -38.6598°$

$\cos\phi = 0.7808$ and $\sin\phi = -0.6246$

So, $W_\rho = 10\times(0.7808) - 8\times(-0.6246) = 12.804$

Now $W_\phi = \overline{W} \bullet \bar{a}_\phi = 10\bar{a}_x \bullet \bar{a}_\phi - 8\bar{a}_y \bullet \bar{a}_\phi + 6\bar{a}_z \bullet \bar{a}_\phi$

$= 10(-\sin\phi) - 8\cos\phi + 0 = 0$

$W_z = \overline{W} \bullet \bar{a}_z = 10\bar{a}_x \bullet \bar{a}_z - 8\bar{a}_y \bullet \bar{a}_z + 6\bar{a}_z \bullet \bar{a}_z$

$= 10\times0 - 8\times0 + 6\times1 = 6$

Hence, **W** = 12.804 $\mathbf{a}_\rho$ + 6 $\mathbf{a}_z$ in cylindrical system

Example-10- Give the Cartesian coordinates of the vector $\boldsymbol{H} = 20\boldsymbol{a}_\rho - 10\boldsymbol{a}_\emptyset + 3\boldsymbol{a}_z$

At point P(x=5, y= 2, z= -1)

Solution- the given vector is in cylindrical system

$$\boldsymbol{H}_x = \boldsymbol{H} \bullet \boldsymbol{a}_x = (20\boldsymbol{a}_\rho - 10\boldsymbol{a}_\emptyset + 3\boldsymbol{a}_z) \bullet \boldsymbol{a}_x$$

$$\boldsymbol{H}_x = 20(\boldsymbol{a}_\rho \bullet \boldsymbol{a}_x) - \mathbf{10}(\boldsymbol{a}_\emptyset \bullet \boldsymbol{a}_x) + \mathbf{3}(\boldsymbol{a}_z \bullet \boldsymbol{a}_x)$$

$$H_x = 20\cos\emptyset - 10(-\sin\emptyset) + 3(0)$$

At point P, $x = 5$, $y = 2$ and $z = -1$

Now $\phi = \tan^{-1}\dfrac{y}{x} = \tan^{-1}\dfrac{2}{5} = 21.8014°$

$\therefore \cos\phi = 0.9284$ and $\sin\phi = 0.3714$

$\therefore H_x = 20\times(0.9284)+10\times 0.3714 = 22.282$

$$H_y = \boldsymbol{H} \bullet \boldsymbol{a}_y = 20\boldsymbol{a}_\rho \bullet \boldsymbol{a}_y - 10\boldsymbol{a}_\emptyset \bullet \boldsymbol{a}_y + 3\boldsymbol{a}_z \bullet \boldsymbol{a}_y = 20\sin\phi - 10\cos\phi + 0$$

$$= 20\times(0.3714) - 10\times(0.9284) = -1.856$$

$$H_z = 3$$

$\overline{\mathrm{H}} = 22.282\,\bar{a}_x - 1.856\,\bar{a}_y + 3\,\bar{a}_z$ **in cartesian system.**

3) Spherical co-ordinate system

- The surfaces which are used to define spherical co-ordinate system on the three Cartesian axis are –

  Sphere of radius r, origin as the centre of sphere

  A right circular cone, with it's apex at the origin and it's axis as Z axis. It's half angle is θ. It rotates about Z axis and θ varies from 0 to 180º.

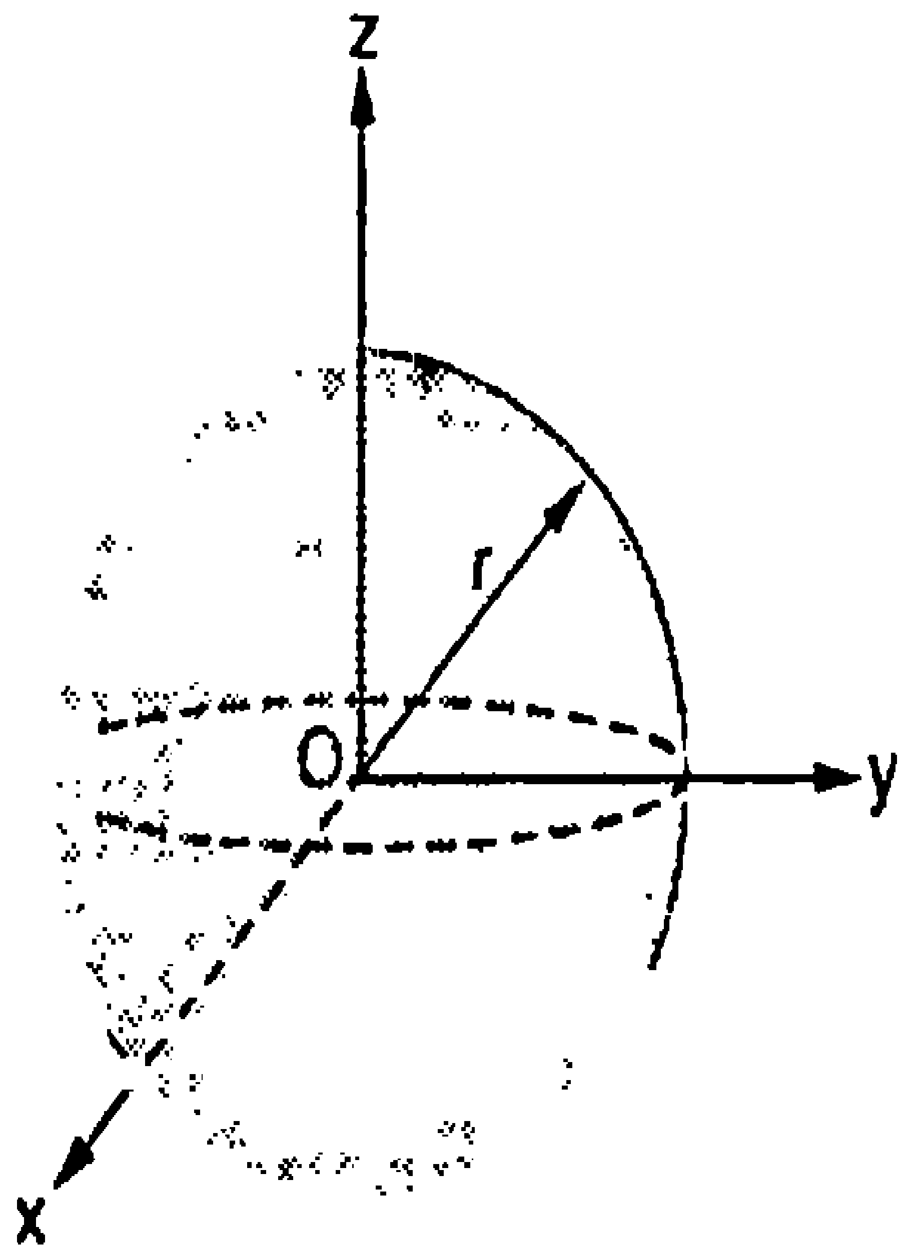
  A half plane perpendicular to xy plane containing Z axis, making an angle ϕ with the xy plane.

- The three co-ordinates of point P are (r, θ, ϕ)
- The ranges of the variables are-

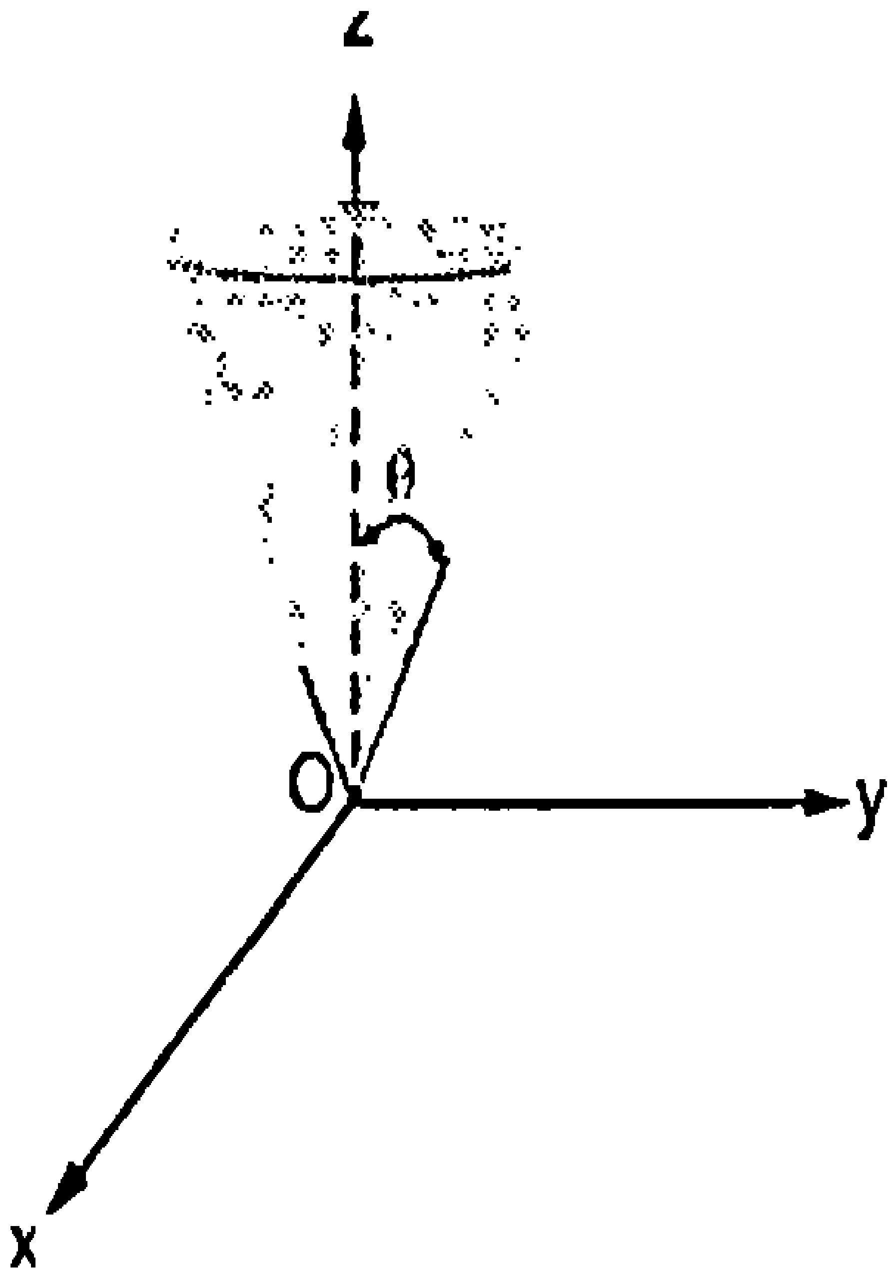
$$0 \leq r < \infty$$

$$0 \leq \phi \leq 2\pi$$

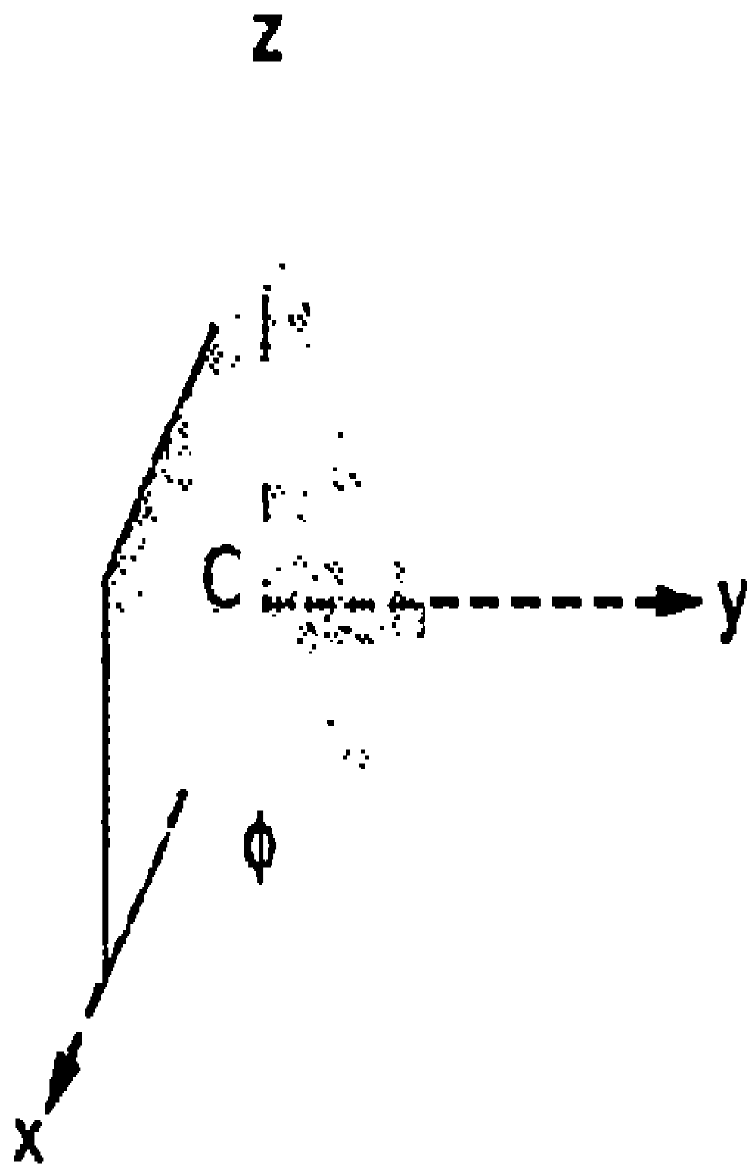
$$0 \leq \theta \leq \pi \text{ as half angle}$$



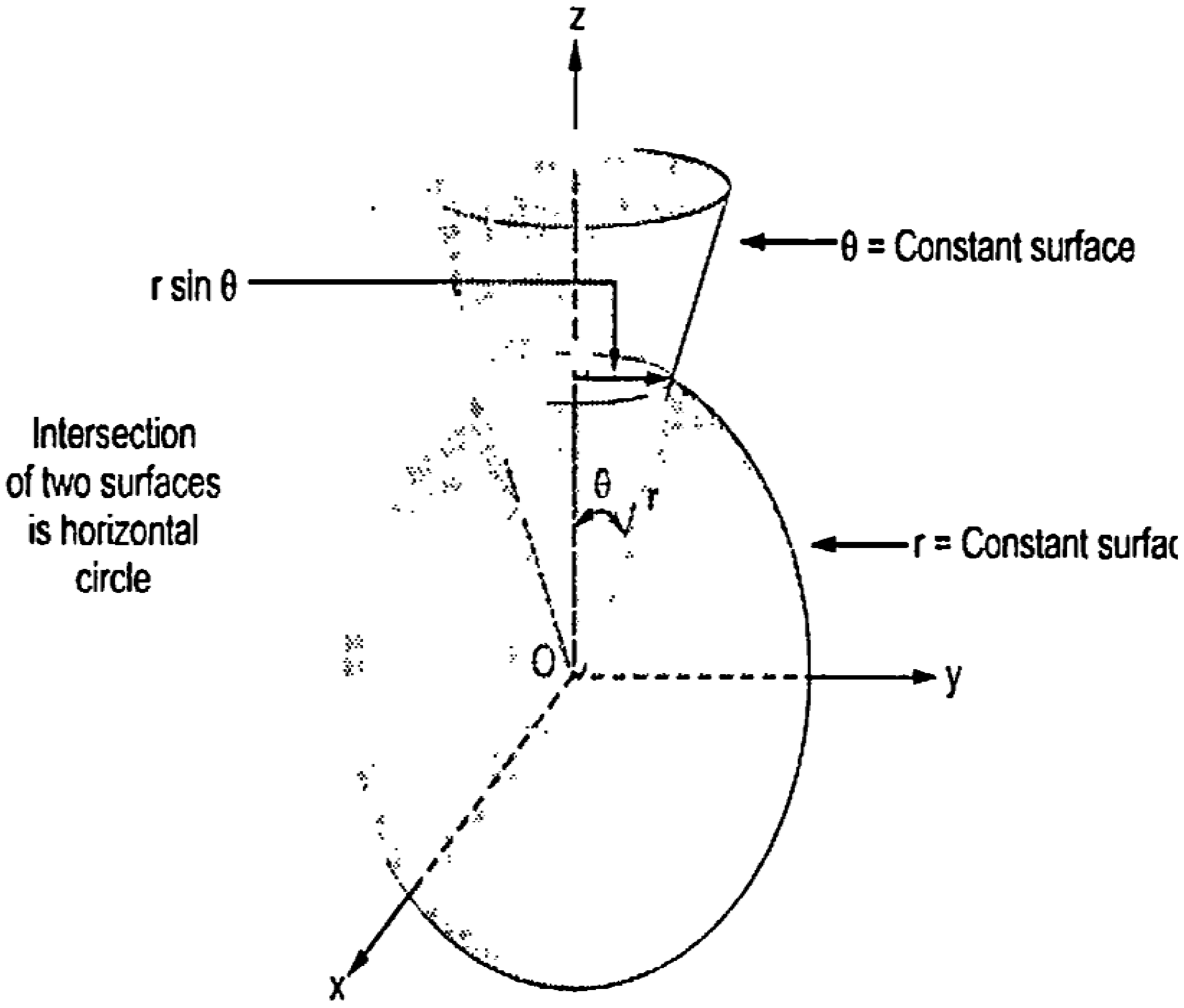
(a) Sphere of radius r with centre as origin

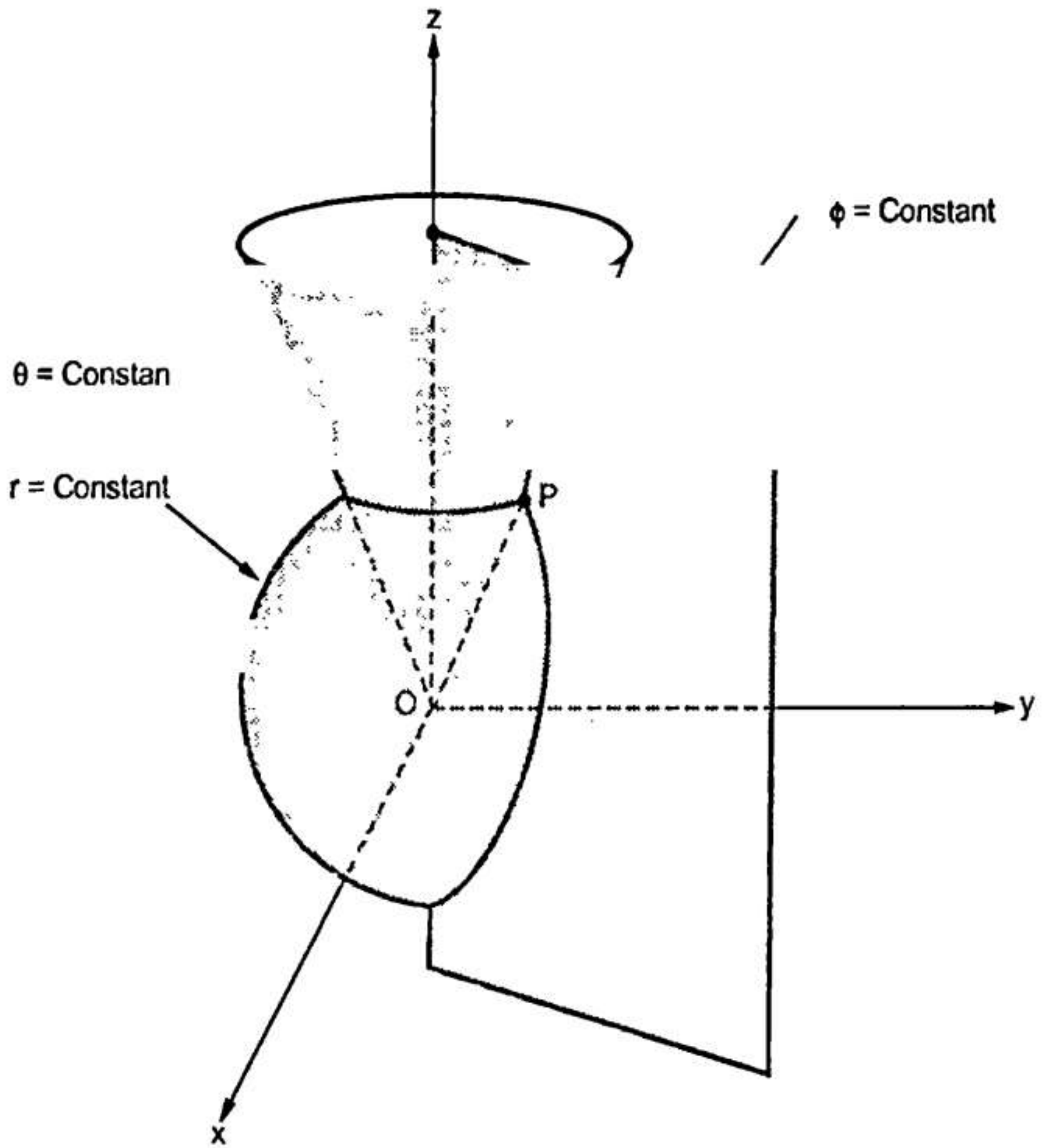


(b) Right circular cone with apex at origin

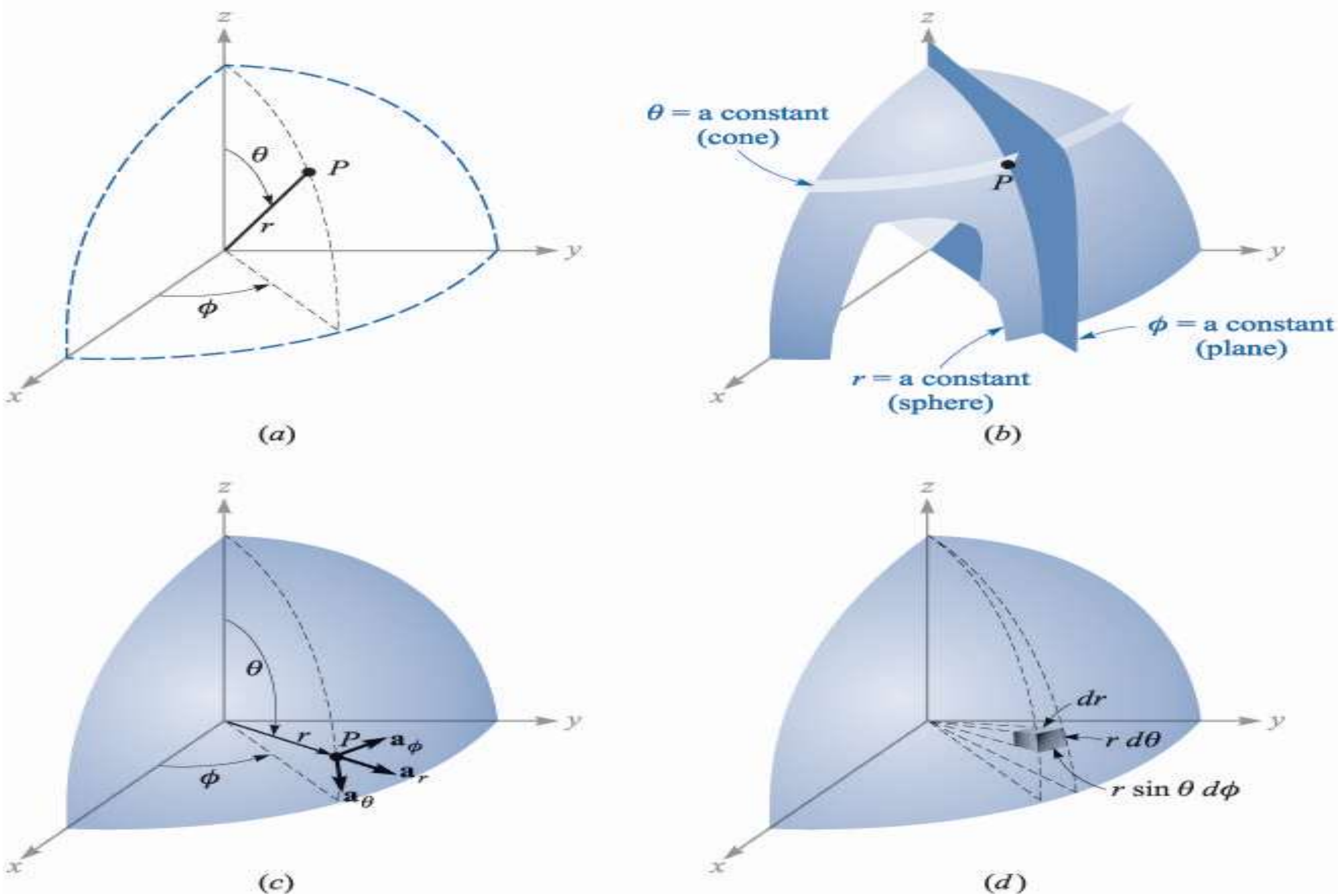


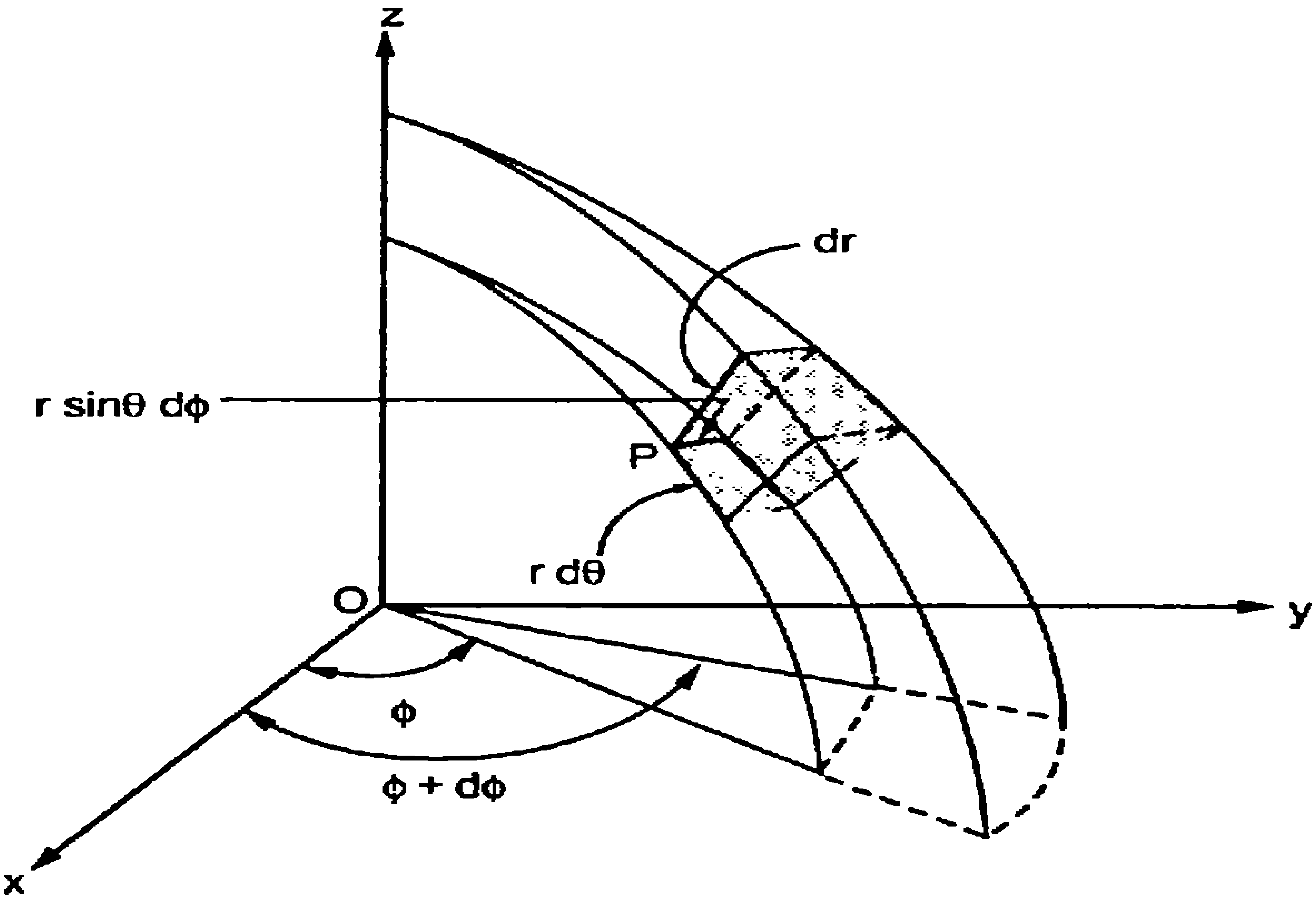
(c) Half plane perpendicular to xy plane


z
θ = Constant surface
r sin θ
Intersection of two surfaces is horizontal circle
θ
r
r = Constant surfac
O
y
x


z
ϕ = Constant
θ = Constan
r = Constant
P
O
y
x

- The point P is defined as the intersection of three surfaces- r = a constant, θ = a constant and ϕ = a constant.
- The intersection of a sphere r = constant and cone – θ = constant is a horizontal circle whose radius is r sinθ.
- Now consider the intersection of ϕ = constant plane with r = constant and θ = constant planes as shown in fig., this defines a point P.
- Three unit vectors may be defined at any point. Each unit vector is perpendicular to one of the three mutually perpendicular surfaces and oriented in that direction in which the coordinate increases.
- The unit vector is $\mathbf{a}_r$ directed radially outward, normal to the sphere r = constant, and lies in the cone θ = constant and the plane ϕ = constant.
- The unit vector $\mathbf{a}_\theta$ is normal to the conical surface, lies in the plane, and is tangent to the sphere and oriented in the direction of increasing θ.
- The third unit vector $\mathbf{a}_\phi$ is the same as in cylindrical coordinates, being normal to the plane and tangent to both the cone and sphere. It is oriented in the direction of increasing ϕ


z
θ
P
r
y
φ
x
(a)
z
θ = a constant
(cone)
P
y
φ = a constant
(plane)
r = a constant
(sphere)
x
(b)
z
θ
r
P
$\mathbf{a}_\phi$
$\mathbf{a}_r$
$\mathbf{a}_\theta$
φ
y
x
(c)
z
dr
r dθ
r sin θ dφ
y
x
(d)


z
dr
r sinθ dφ
P
r dθ
O
y
φ
φ + dφ
x

- A differential volume element may be constructed in spherical coordinates by increasing r, θ , and ϕ by dr, dθ , and dϕ as shown in Fig.
- The distance between the two spherical surfaces of radius r and r + dr is dr; the distance between the two cones having generating angles of θ and dθ is rdθ ; and the distance between the two radial planes at angles ϕ and dϕ is found to be r sinθ dϕ.
- The surfaces have areas of r dr dθ , r sinθ dr dϕ , and r^2 sinθ dθ dϕ.
- The volume is r^2 sinθ dr dθ dϕ.
- The transformation of scalars from the cartesian to the spherical coordinate system is easily made by using the relation:

  $x = r \sin\theta \cos\phi$

  $y = r \sin\theta \sin\phi$ and $z = r \cos\theta$
- The transformation in the reverse direction is achieved with the help of

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \cos^{-1}\left[\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right] \text{ and } \phi = \tan^{-1}\frac{y}{x}$$

- **Differential elements in Spherical co-ordinate system**

dr = Differential length in r direction

$r\,d\theta$ = Differential length in θ direction

$r\sin\theta\,d\phi$ = Differential length in ϕ direction

differential vector length is $\overline{dl} = dr\,\bar{a}_r + r\,d\theta\,\bar{a}_\theta + r\sin\theta\,d\phi\,\bar{a}_\phi$

and $|\overline{dl}| = \sqrt{(dr)^2 + (r\,d\theta)^2 + (r\sin\theta\,d\phi)^2}$

Differential Volume $dv = r^2 \sin\theta\,dr\,d\theta\,d\phi$

Differential surface areas are

$d\bar{S}_r$ = Differential vector surface area normal to r direction

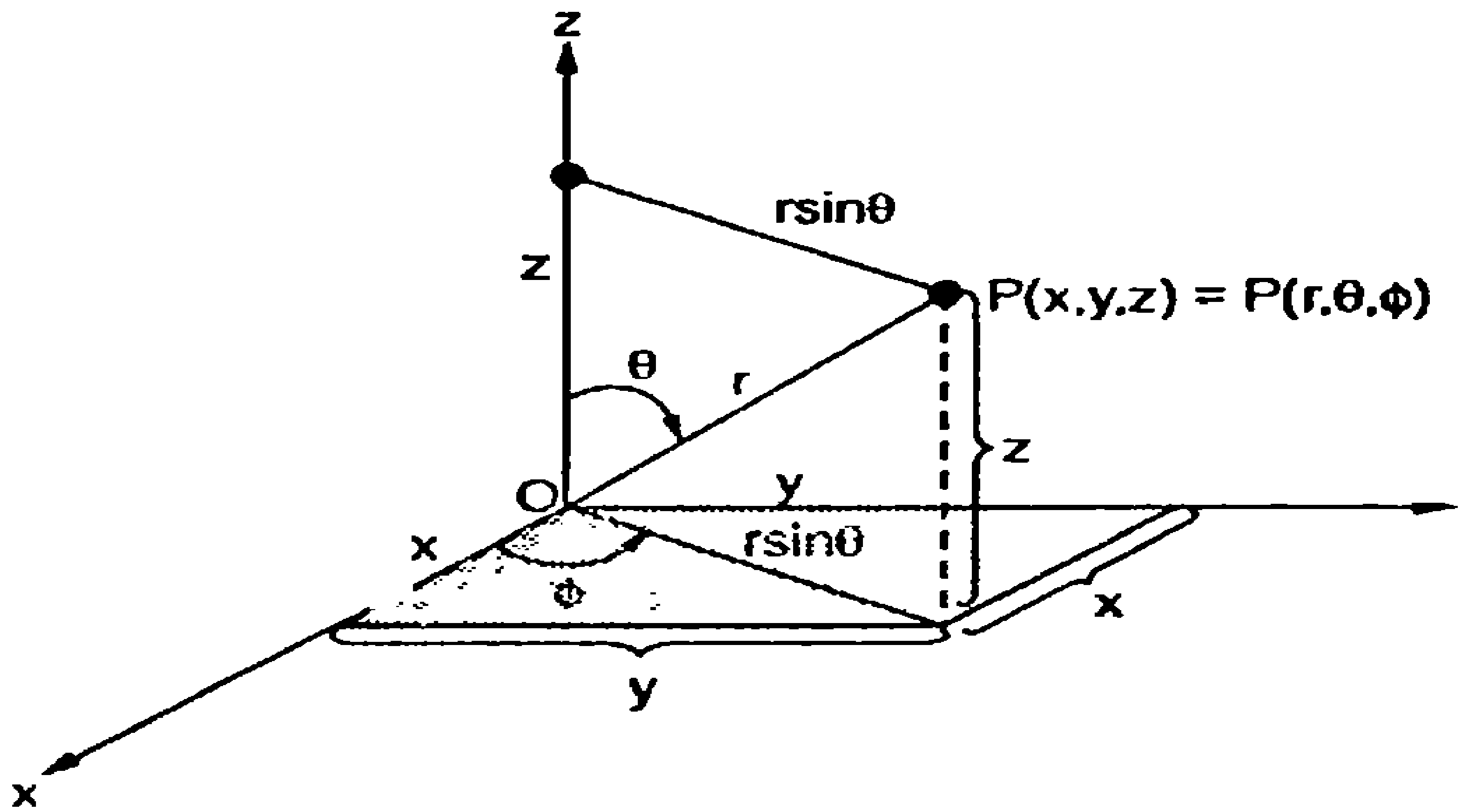
$= r^2 \sin\theta\,d\theta\,d\phi$

$d\bar{S}_\theta$ = Differential vector surface area normal to θ direction

$= r\sin\theta\,dr\,d\phi$

$d\bar{S}_\phi$ = Differential vector surface area normal to ϕ direction

$= r\,dr\,d\theta$



Relationship between Cartesian and spherical co-ordinate system

- Dot products of unit vectors in spherical and Cartesian coordinate systems

Dot operator •	$\bar{a}_r$	$\bar{a}_\theta$	$\bar{a}_\phi$
$\bar{a}_x$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\bar{a}_y$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\bar{a}_z$	$\cos\theta$	$-\sin\theta$	0

While calculating ϕ make sure the signs of x and y. If both are positive, ϕ is positive in the first quadrant. If x is negative and y is positive then the point is in the second quadrant hence ϕ must be within $+90°$ and $+180°$ i.e. within $-180°$ and $-270°$. Thus for $x=-2$ and $y=1$ we get $\phi = \tan^{-1}\left[\frac{1}{-2}\right] = -26.56°$ but it should be taken as $-26.56°+180°$ i.e. $154.43°$. Hence when x is negative, it is necessary to add $180°$ to the ϕ calculated using $\tan^{-1}$ function, to obtain accurate ϕ corresponding to the point.

When y is negative and x is positive then ϕ is in fourth quadrant i.e. within $0°$ and $-90°$ i.e. $270°$ and $360°$. Similarly when x is negative and y is also negative the point is in third quadrant and accordingly ϕ must be between $-90°$ to $-180°$ i.e. $+180°$ and $+270°$. So $180°$ must be subtracted from the ϕ calculated by $\tan^{-1}$ function, to get accurate ϕ when both x and y are negative. Thus if $x=y=-3$ then $\phi = \tan^{-1}\left[\frac{-3}{-3}\right] = +45°$ but actually it is $45°-180° = -135°$ i.e. $-135°+360° = +225°$.

Example-11- *Obtain the spherical coordinates of 10 $\bar{a}_x$ at the point $P(x=-3, y=2, z=4)$.*

Slution- Given vector is in cartesian system say $\bar{F} = 10\bar{a}_x$.

Then $$F_r = \bar{F}\bullet\bar{a}_r = 10\bar{a}_x \bullet \bar{a}_r$$

$$= 10 \sin\theta \cos\phi$$

At point P, x = – 3, y = 2, z = 4

Using the relationship between cartesian and spherical,

$$x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$$

$$\therefore \quad \phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2}{-3} = -33.69°$$

But x is negative and y is positive hence ϕ must be between +90° and +180°. So add 180° to the ϕ to get correct ϕ.

$$\therefore \quad \phi = -33.69° + 180° = +146.31°$$

$$\therefore \quad \cos\phi = -0.832 \quad \text{and} \quad \sin\phi = 0.5547$$

- -

And $$\theta = \cos^{-1}\frac{z}{r} = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$= \cos^{-1}\frac{4}{\sqrt{(-3)^2+(2)^2+(4)^2}} = 42.0311°$$

$$\cos\theta = 0.7428 \quad \text{and} \quad \sin\theta = 0.6695$$

$$F_r = 10\times 0.6695\times(-0.832) = -5.5702$$

$$F_\theta = \bar{F}\bullet\bar{a}_\theta = 10\,\bar{a}_x\bullet\bar{a}_\theta = 10\cos\theta\cos\phi$$

$$= 10\times 0.7428\times(-0.832) = -6.18$$

$$F_\phi = \bar{F}\bullet\bar{a}_\phi = 10\,\bar{a}_x\bullet\bar{a}_\phi = 10(-\sin\phi)$$

$$= 10\times(-0.5547) = -5.547$$

$$\bar{F} = -5.5702\,\bar{a}_r - 6.18\,\bar{a}_\theta - 5.547\,\bar{a}_\phi \quad \text{in spherical system.}$$

Example-12- *Express* $\bar{B} = r^2\,\bar{a}_r + \sin\theta\,\bar{a}_\phi$ *in the cartesian co-ordinates. Hence obtain* $\bar{B}$ *at* $P(1,2,3)$.

Solution-

$$B_x = \bar{B}\bullet\bar{a}_x = r^2\,\bar{a}_r\bullet\bar{a}_x + \sin\theta\,\bar{a}_\phi\bullet\bar{a}_x$$

$$= r^2\sin\theta\cos\phi + \sin\theta(-\sin\phi)$$

$$B_y = \bar{B}\bullet\bar{a}_y = r^2\,\bar{a}_r\bullet\bar{a}_y + \sin\theta\,\bar{a}_\phi\bullet\bar{a}_y$$

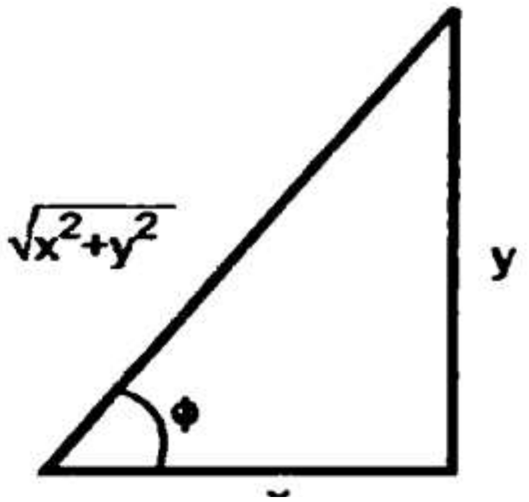
$$= r^2\sin\theta\sin\phi + \sin\theta\cos\phi$$

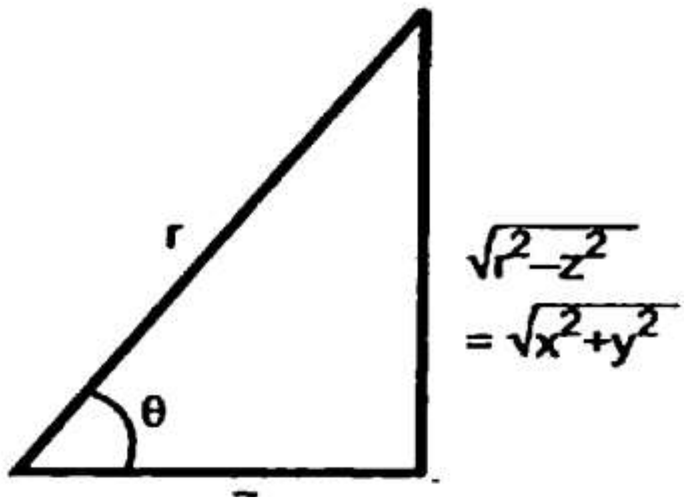
$$B_z = \bar{B}\bullet\bar{a}_z = r^2\,\bar{a}_r\bullet\bar{a}_z + \sin\theta\,\bar{a}_\phi\bullet\bar{a}_z$$

$$= r^2\cos\theta + \sin\theta(0) = r^2\cos\theta$$

Now it is known that,

$$r = \sqrt{x^2+y^2+z^2}, \quad \phi = \tan^{-1}\frac{y}{x} \quad \text{and} \quad \theta = \cos^{-1}\frac{z}{r}$$





$$\sin\phi = \frac{y}{\sqrt{x^2+y^2}}, \; \cos\phi = \frac{x}{\sqrt{x^2+y^2}}$$

$$\sin\theta = \frac{\sqrt{x^2+y^2}}{r} \quad \text{and} \quad \cos\theta = \frac{z}{r}$$

$$B_x = r^2\frac{\sqrt{x^2+y^2}}{r}\frac{x}{\sqrt{x^2+y^2}} + \frac{\sqrt{x^2+y^2}}{r}\left(-\frac{y}{\sqrt{x^2+y^2}}\right)$$

$$= (r\,x) - \frac{y}{r} = \sqrt{x^2+y^2+z^2}\,(x) - \frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$B_y = r^2\frac{\sqrt{x^2+y^2}}{r}\frac{y}{\sqrt{x^2+y^2}} + \frac{\sqrt{x^2+y^2}}{r}\frac{x}{\sqrt{x^2+y^2}}$$

$$= (r\,y) + \frac{x}{r} = \sqrt{x^2+y^2+z^2}\,(y) + \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$B_z = r^2 \times \frac{z}{r} = (r\,z) = \sqrt{x^2+y^2+z^2}\,(z)$$

$$\bar{B} = B_x\,\bar{a}_x + B_y\,\bar{a}_y + B_z\,\bar{a}_z$$

Thus $\bar{B}$ at P (1, 2, 3) is, $\bar{B} = 3.207\,\bar{a}_x + 7.7504\,\bar{a}_y + 11.2248\,\bar{a}_z$

- **Distance in all co-ordinate systems-**

Consider two points A and B with the position vectors as

$$\bar{A} = x_1\,\bar{a}_x + y_1\,\bar{a}_y + z_1\,\bar{a}_z \quad \text{and} \quad \bar{B} = x_2\,\bar{a}_x + y_2\,\bar{a}_y + z_2\,\bar{a}_z$$

Then the distance d between two points in all the three co-ordinate systems are given by-

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \qquad \text{... Cartesian}$$

$$d = \sqrt{{\rho_2}^2 + {\rho_1}^2 - 2\rho_1\rho_2\cos(\emptyset_2 - \emptyset_1) + (z_2 - z_1)^2} \qquad \text{... Cylindrical}$$

$$d = \sqrt{{r_2}^2 + {r_1}^2 - 2r_1r_2\cos\theta_2\cos\theta_1 - 2r_1r_2\sin\theta_2\sin\theta_1\cos(\emptyset_2 - \emptyset_1)} \qquad \text{... Spherical}$$

- **Transformation of vectors from spherical to cylindrical and from cylindrical to Spherical system**

Let the vector **A** in spherical system $\bar{\mathbf{A}} = A_r \bar{a}_r + A_\theta \bar{a}_\theta + A_\phi \bar{a}_\phi$

The components of vector in cylindrical system is given by-

$$A_\rho = A_r \bar{a}_r \cdot \bar{a}_\rho + A_\theta \bar{a}_\theta \cdot \bar{a}_\rho + A_\phi \bar{a}_\phi \cdot \bar{a}_\rho$$

$$A_\phi = A_r \bar{a}_r \cdot \bar{a}_\phi + A_\theta \bar{a}_\theta \cdot \bar{a}_\phi + A_\phi \bar{a}_\phi \cdot \bar{a}_\phi$$

$$A_z = A_r \bar{a}_r \cdot \bar{a}_z + A_\theta \bar{a}_\theta \cdot \bar{a}_z + A_\phi \bar{a}_\phi \cdot \bar{a}_z$$

Now $\bar{a}_r \cdot \bar{a}_\rho = \sin\theta, \quad \bar{a}_\theta \cdot \bar{a}_\rho = \cos\theta, \quad \bar{a}_\phi \cdot \bar{a}_\rho = 0$

$$\bar{a}_r \cdot \bar{a}_\phi = 0, \quad \bar{a}_\theta \cdot \bar{a}_\phi = 0, \quad \bar{a}_\phi \cdot \bar{a}_\phi = 1$$

$$\bar{a}_r \cdot \bar{a}_z = \cos\theta, \quad \bar{a}_\theta \cdot \bar{a}_z = -\sin\theta, \quad \bar{a}_\phi \cdot \bar{a}_z = 0$$

- Dot products of unit vectors - $\mathbf{a_\rho \cdot a_r}$, $\mathbf{a_\rho \cdot a_\theta}$, $\mathbf{a_\rho \cdot a_\phi}$, $\mathbf{a_\phi \cdot a_\theta}$, , $\mathbf{a_z \cdot a_r}$, , $\mathbf{a_z \cdot a_\theta}$, and , $\mathbf{a_z \cdot a_\phi}$ Can be determined by the following table-

•	$\mathbf{a_r}$	$\mathbf{a_\theta}$	$\mathbf{a_\phi}$
a_ρ	$\sin\theta$	$\cos\theta$	0
a_ϕ	0	0	1
a_z	$\cos\theta$	$-\sin\theta$	0

- Similarly a vector in cylindrical can easily be transferred into the Spherical system

Types of integral related to Electromagnetic Theory

A charge can exist in point , line , surface and volume form. Hence while dealing with such charge distribution, the following types if integrals are required-

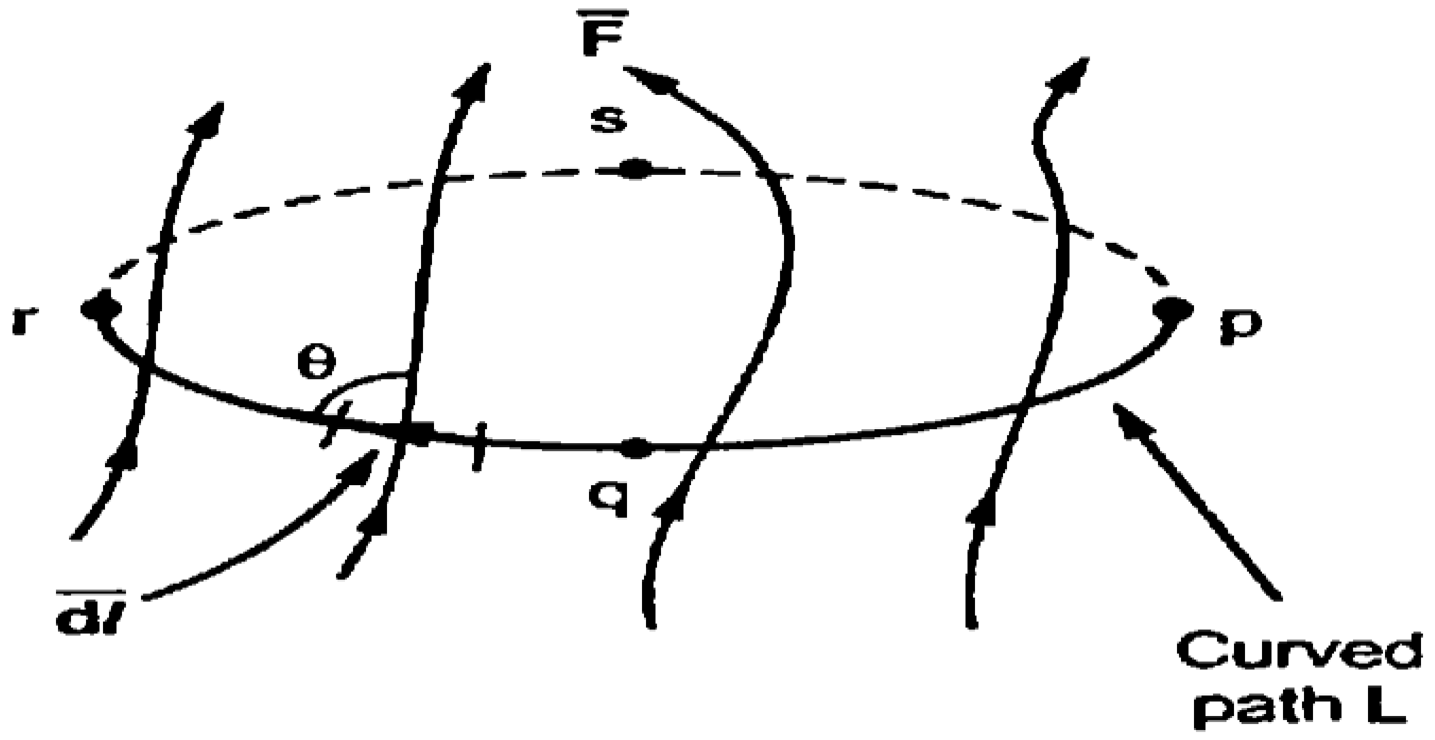
- Line integral
- Surface integral
- Volume integral

1) Line integral- a line can exist as a straight line or it can be a distance travelled along a curve, thus in general from mathematical point of view, a line is a curved path in space.

Consider a vector field **F** as shown in fig., the curved path shown in the field is p-r . This is called the path of integration and corresponding integral can be defined as-

$$\int_L \overline{F} \cdot \overline{dl} = \int_p^r |F|\, dl \cos\theta$$

Where **dl** = elementary length

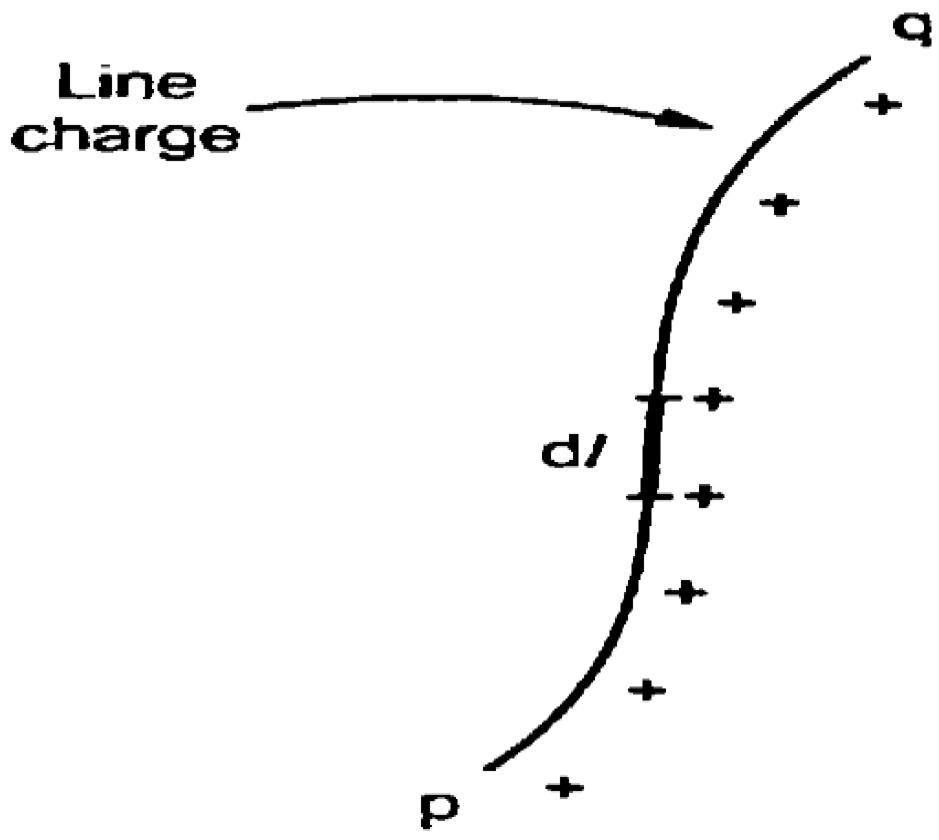


- This is called the line integral of F around the curved path L. It represents the tangential component of F along the path L
- The curved path can be of two types-

(i) open path as p-r

(ii) closed path as p-q-r-s-p

- The closed path is also called Contour. The corresponding integral is called contour integral, closed integral or circular integral. Mathematically- $\oint_L \bar{F} \cdot \overline{dl}$ = Circular integral



Line charge

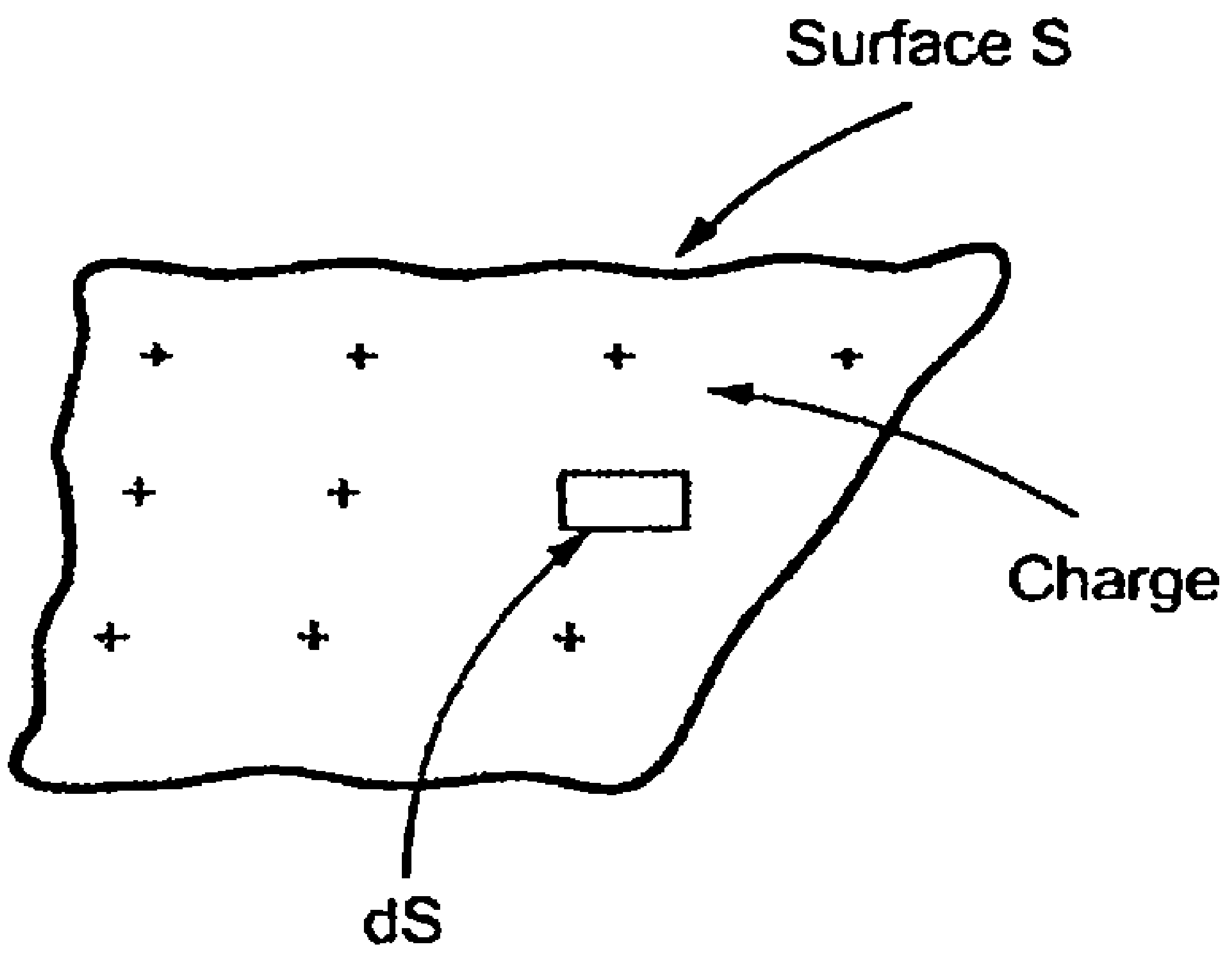
if there exist a charge along a line as shown , then the total charge is obtained by calculating a line integral.

$$Q = \int_L \rho_L \, dl$$

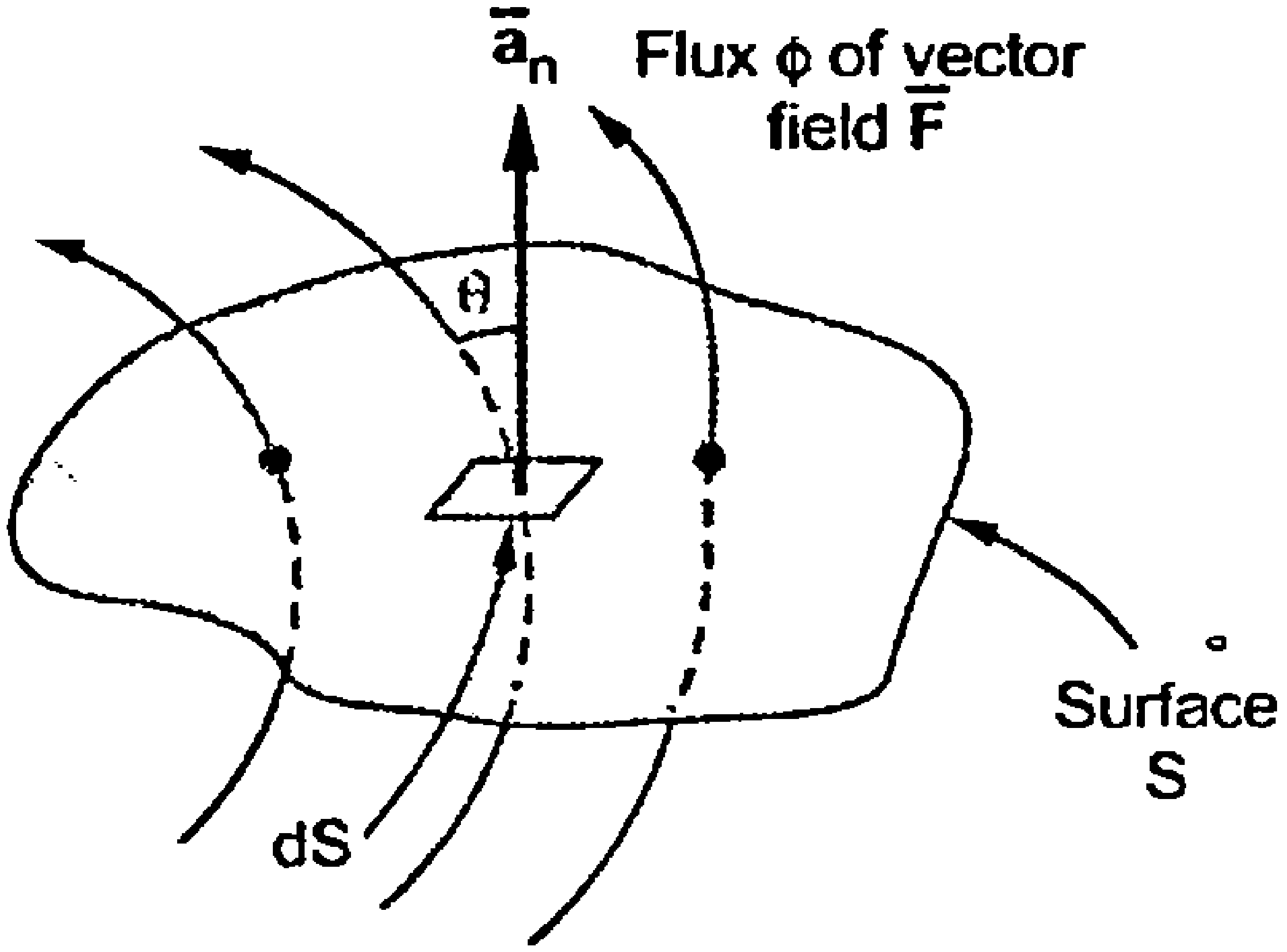
where ρ_L = Line charge density (charge per unit length (c/m))

2) **Surface Integral** – in electromagnetic theory, a charge may exist in a distributed form . It may be spreaded over a surface as shown in fig(a). similarly a flux ф may pass through a surface as shown in fig (b).

While doing analysis of such cases an integral is required is called surface integral, to be carried out over a surface related to a vector field.



(a) Surface charge



b) Flux crossing a surface

- For charge distribution shown in fig (a), we can write for the total charge existing on the surface as

$$Q = \int_S \rho_S \, dS$$

- Where ρ_S = surface charge density in C/m^2 ; dS= elementary surface
- Similarly for the fig.(b), the total flux crossing the surface S can be expressed as,

$$\phi = \int \bar{F} \cdot d\bar{S} = \int |\bar{F}| \, dS \cos\theta = \int \bar{F} \cdot \bar{a}_n \, dS$$

where $\bar{a}_n$ = **Unit vector normal to the surface S** ;

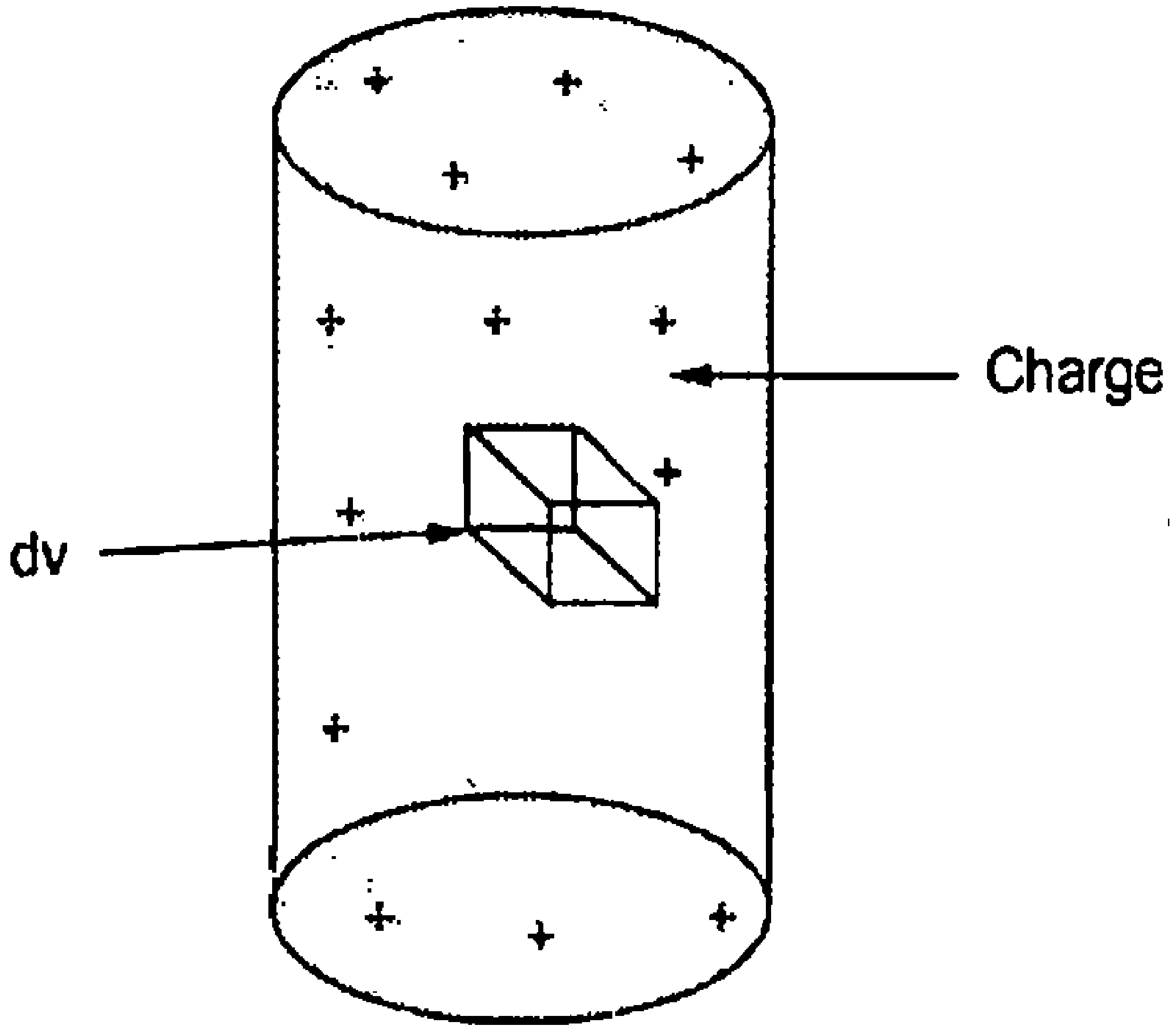
- Both the above equations represents the surface integrals and mathematically it becomes a double integration while solving the problems.
- If the surface is closed, then it defines as a volume and corresponding surface integration is given by, $$\phi = \oint_S \bar{F} \cdot d\bar{S}$$
- This represents the net outward flux of vector field **F** from surface S.

3) Volume integral – if the charge distribution exists in a three dimensional volume form as shown in fig. then a volume integral is required to calculate the total charge.

thus if ρ_v is the volume charge density over a volume V then the volume integral is defined as-

$$Q = \int_v \rho_v \, dv$$

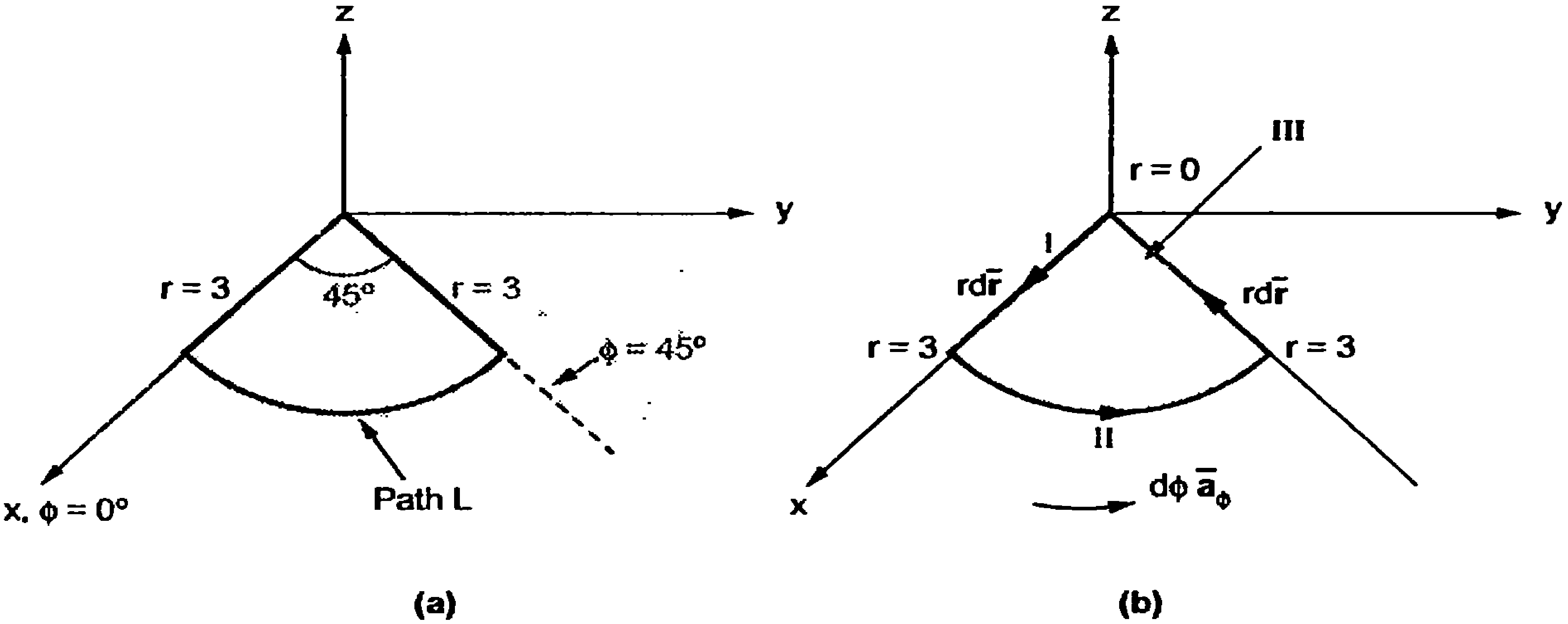
Where dv= elementary volume



Volume charge

Example -13 calculate the circulation of vector field, $\bar{F} = r^2 \cos\phi \, \bar{a}_r + z \sin\phi \, \bar{a}_z$

Around the path L defined by- $0 \le r \le 3$, $0 \le \phi \le 45^o$ and $Z = 0$ as shown in fig.(a)



Solution- Divide the given path L into three sections.

Section I : r varies from 0 to 3, $\phi = 0°$ and $z = 0$

$\therefore \quad d\bar{l} = dr\,\bar{a}_r$ ... Along radial direction

$$\therefore \quad \int_{I} \bar{F} \cdot d\bar{l} = \int_{r=0}^{3} (r^2 \cos\phi\ \bar{a}_r + z \sin\phi\,\bar{a}_z) \cdot dr\,\bar{a}_r$$

$$= \int_{r=0}^{3} r^2 \cos\phi\ dr \qquad \ldots\ \bar{a}_r \cdot \bar{a}_r = 1,\ \ \bar{a}_z \cdot \bar{a}_r = 0$$

$$= \left[\frac{r^3}{3}\right]_0^3 \cos 0° = \left[\frac{27}{3}\right][1] = 9$$

on II : r is constant 3, ϕ varies from 0 to 45°, $z = 0$

$\therefore \quad d\bar{l} = r\,d\phi\,\bar{a}_\phi$ ... Along ϕ direction

$$\therefore \quad \int_{II} \bar{F} \cdot d\bar{l} = \int_{\phi=0}^{45°} (r^2 \cos\phi\ \bar{a}_r + z \sin\phi\,\bar{a}_z) \cdot r\,d\phi\,\bar{a}_\phi$$

$$= 0 \qquad \ldots\ \bar{a}_r \cdot \bar{a}_\phi = \bar{a}_z \cdot \bar{a}_\phi = 0$$

Harmonic field- A scalar field is said to be harmonic in a given region, if it's laplacian vanishes in the region.

Mathematically for a scalar field V to be harmonic,

$$\nabla^2 V = 0$$

This equation is called Laplace's equation.

Section III : r varies from 3 to 0, $\phi = 45°$ and $z = 0$

$$d\bar{l} = dr\,\bar{a}_r$$

Note that $d\bar{l}$ is always positive, limits of integration from r = 3 to 0 taking care of direction.

$$\therefore \qquad \int_{III} \bar{F} \cdot d\bar{l} = \int_{r=3}^{0} (r^2 \cos\phi\,\bar{a}_r + z \sin\phi\,\bar{a}_z) \cdot dr\,\bar{a}_r$$

$$= \int_{r=3}^{0} r^2 \cos\phi\,dr \qquad \ldots \bar{a}_r \cdot \bar{a}_r = 1,\ \bar{a}_z \cdot \bar{a}_r = 0$$

$$= \cos 45° \left[\frac{r^3}{3}\right]_3^0 = 0.7071 \left[\frac{-27}{3}\right] = -6.3639$$

$$\therefore \qquad \oint_L \bar{F} \cdot d\bar{l} = 9 + 0 - 6.3639 = 2.636$$

Del operator- The Del operator , written ∇ is the vector differential operator.

In Cartesian coordinates-

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{a}_x + \frac{\partial}{\partial y}\boldsymbol{a}_y + \frac{\partial}{\partial z}\boldsymbol{a}_z$$

The operator is useful in defining-

1) The gradient of scalar V, written as ∇ V.

2) The divergence of a vector **A**, written as $\nabla \cdot \mathbf{A}$

3)The curl of a Vector **A**, written as, $\nabla \times \mathbf{A}$

4) The Laplacian of a scalar V, written as $\nabla^2 V$

Del operator in cylindrical coordinates- $\nabla = \frac{\partial}{\partial \rho}\boldsymbol{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \emptyset}\boldsymbol{a}_\emptyset + \frac{\partial}{\partial z}\boldsymbol{a}_z$

Del operator in spherical coordinates- $\nabla = \mathbf{a}_r\frac{\partial}{\partial r} + \mathbf{a}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{a}_\emptyset\frac{1}{r\sin\theta}\frac{\partial}{\partial \emptyset}$

- **Divergence-** it is seen that $\oint_S \bar{F} \cdot d\bar{S}$ gives the flux flowing across the surface S. then mathematically **Divergence** is defined as the net outward flow of the flux per unit volume over a close incremental surface. It is denoted as $\text{div}\ \bar{F}$ and is given by -

$$\text{div}\ \bar{F} = \lim_{\Delta v \to 0} \frac{\oint_S \bar{F} \cdot d\bar{S}}{\Delta v} = \text{Divergence of } \bar{F}$$

where Δv = Differential volume element

Symbolically it is denoted as,

$$\nabla \cdot \bar{F} = \text{Divergence of } \bar{F}$$

where ∇ = Vector operator $= \frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z$

But $\bar{F} = F_x\bar{a}_x + F_y\bar{a}_y + F_z\bar{a}_z$

$$\therefore \quad \boxed{\nabla \cdot \bar{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \text{div}\ \bar{F}}$$

This is divergence of $\bar{F}$ in Cartesian system.

Similarly divergence in other co-ordinate systems are,

$$\nabla \cdot \bar{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z} \quad \text{Cylindrical}$$

$$\nabla \cdot \bar{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi} \quad \text{Spherical}$$

- Physically the divergence at a point indicate how much that vector field diverges from that point.

- **Divergence Theorem**- From the definition of divergence, we know that-

$$\nabla \cdot \bar{F} = \lim_{\Delta v \to 0} \frac{\oint_S \bar{F} \cdot d\bar{S}}{\Delta v}$$

From this definition it can be written as-

$$\oint_S \bar{F} \cdot d\bar{S} = \int_v (\nabla \cdot \bar{F})\, dv \qquad \textbf{……(i)}$$

This equation (i) is known as the divergence theorem.

It states that- **“The integral of the normal component of any vector field over a closed surface is equal the integral of the divergence of this vector field throughout the volume enclosed by that closed surface”**.

The divergence theorem converts the surface integral into a volume integral, provided that the closed surface encloses certain volume.

$$\therefore \quad \oint_v (\nabla \cdot \overline{F})\, dv = \int_{z=0}^{1} \int_{\phi=0}^{2\pi} \int_{r=0}^{4} \left(3r\cos^2\phi + \frac{z\cos\phi}{r} \right) r\, dr\, d\phi\, dz$$

$$= \int_{z=0}^{1} \int_{\phi=0}^{2\pi} \left[\frac{3r^3}{3} \cos^2\phi + z\cos\phi\, r \right]_0^4 d\phi\, dz$$

$$= \int_{z=0}^{1} \int_{\phi=0}^{2\pi} \left\{ 4^3 \left[\frac{1+\cos 2\phi}{2} \right] + 4z\cos\phi \right\} d\phi\, dz$$

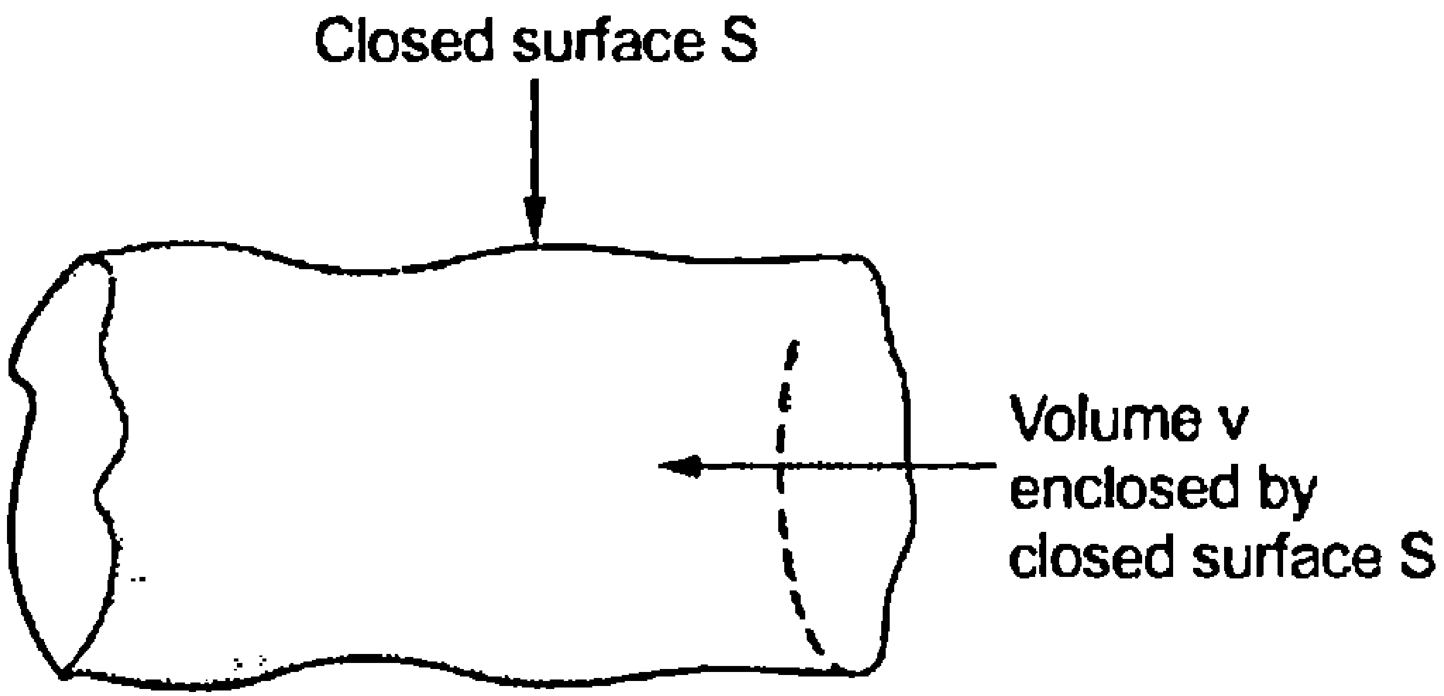
$$= \int_{z=0}^{1} \left\{ 32\left[\phi + \frac{\sin 2\phi}{2} \right]_0^{2\pi} + 4z[\sin\phi]_0^{2\pi} \right\} dz$$

$$= \int_{z=0}^{1} \{32 \times [2\pi + 0] + 4z[0]\}\, dz = \int_{z=0}^{1} 64\pi\, dz$$

$$= 64\pi[z]_0^1 = 64\pi$$

Thus $\int_S \overline{F} \cdot d\overline{S} = \oint_v (\nabla \cdot \overline{F})\, dv$ and divergence theorem is verified.

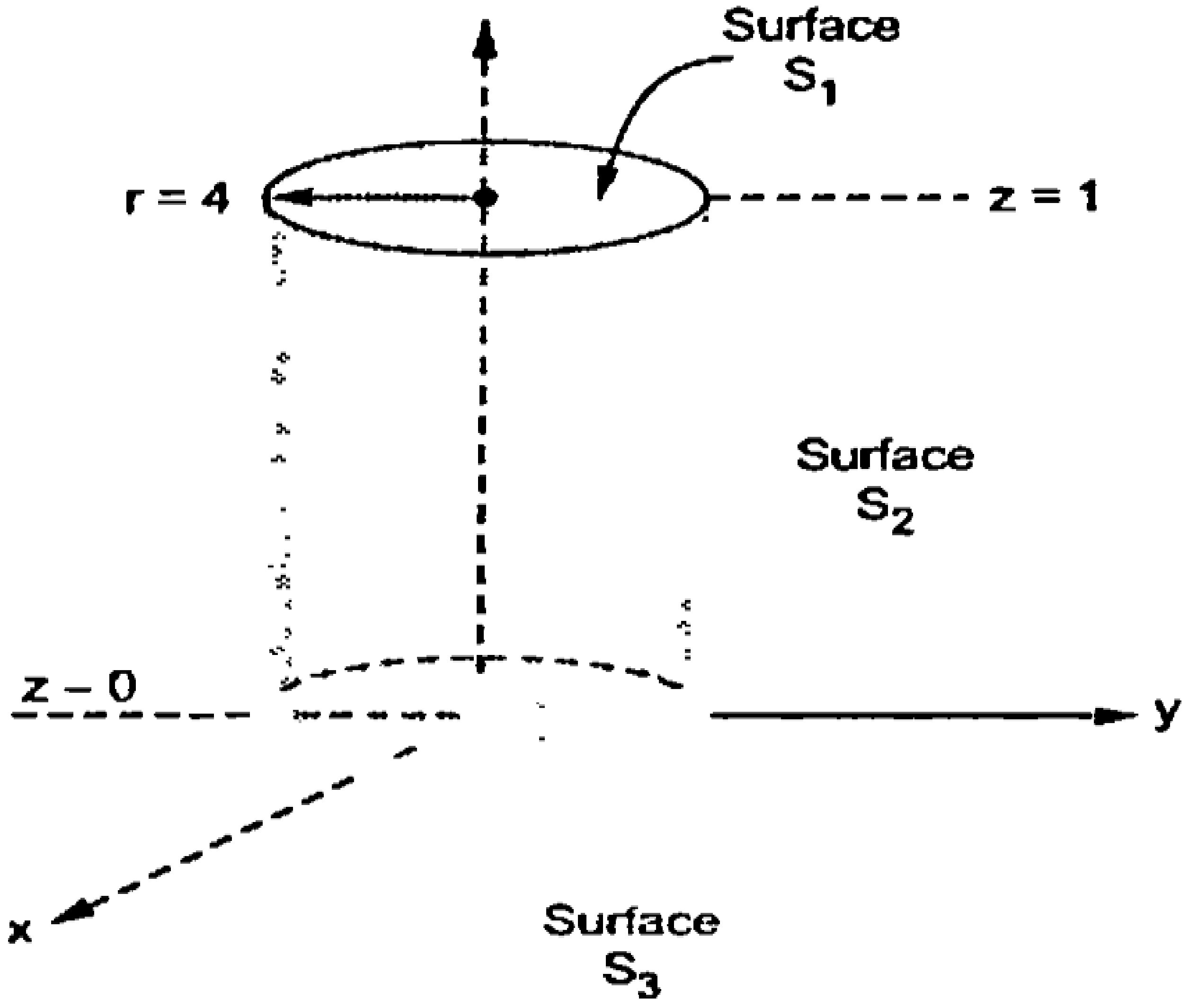
- The theorem can be applied to any vector field but partial derivatives of that vector field must exist. The divergence theorem as applied to flux density, both sides of the divergence theorem give the net charge enclose by the closed surface(i.e. net flux crossing the closed surface).



- Figure shows how closed surface S encloses a volume v for which divergence theorem is applicable.

Example- 14- A particular vector field $\bar{F} = r^2 \cos^2 \phi \, \bar{a}_r + z \sin \phi \, \bar{a}_\phi$ is in cylindrical system. Find the flux emanating due to this field from the closed surface of the cylinder $0 \leq z \leq 1, r = 4$. verify divergence theorem.

Solution- the outward flux is given by, $\phi = \oint_S \bar{F} \cdot d\bar{S}$ over a closed surface S



The surface made up of-

1) Top surface S_1 for which Z=1, r varies from 0 4 and ϕ varies from 0 to 2π.
2) Lateral surface for which Z varies from 0 to 1 , ϕ from 0 to 2π and r = 4.
3) Bottom surface S_3 for which Z=0, r varies from 0 to 4 and ϕ varies from 0 to 2π

For S_1, $\quad d\bar{S} = r\,dr\,d\phi\,\bar{a}_z$

For S_2, $\quad d\bar{S} = r\,dz\,d\phi\,\bar{a}_r$

For S_3, $\quad d\bar{S} = r\,dr\,d\phi\,(-\bar{a}_z)$

$$\oint_{S_1} \bar{F} \cdot d\bar{S} = \oint_{S_1} (r^2 \cos^2 \phi \bar{a}_r + z \sin \phi \bar{a}_\phi) \cdot (r\,dr\,d\phi \bar{a}_z)$$

$$= 0 \qquad \ldots \bar{a}_r \cdot \bar{a}_z = \bar{a}_\phi \cdot \bar{a}_z = 0$$

$$\oint_{S_3} \bar{F} \cdot d\bar{S} = \oint_{S_3} (r^2 \cos^2 \phi \bar{a}_r + z \sin \phi \bar{a}_\phi) \cdot [r\,dr\,d\phi\,(-\bar{a}_z)]$$

$$= 0 \qquad \ldots \bar{a}_r \cdot \bar{a}_z = \bar{a}_\phi \cdot \bar{a}_z = 0$$

$$\oint_{S_2} \bar{F} \cdot d\bar{S} = \oint_{S_2} (r^2 \cos^2 \phi \bar{a}_r + z \sin \phi \bar{a}_\phi) \cdot (r\,dz\,d\phi \bar{a}_r)$$

$$= \int_{z=0}^{1} \int_{\phi=0}^{2\pi} r^2 \cos^2 \phi\, r\,dz\,d\phi \qquad \ldots \bar{a}_r \cdot \bar{a}_z = 1,\ \bar{a}_\phi \cdot \bar{a}_r = 0\ r = 4$$

$$= (4)^3 \int_{z=0}^{1} \int_{\phi=0}^{2\pi} dz \cos^2 \phi \ \ d\phi = \ 64 \int_0^1 dz \int_{\phi=0}^{2\pi} \frac{1+\cos 2\phi}{2} d\phi$$

$$= 64 \times [z]_0^1 \times \frac{1}{2} \times \left\{ [\phi]_0^{2\pi} + \left[+ \frac{\sin 2\phi}{2} \right]_0^{2\pi} \right\}$$

$$= 64 \times 1 \times \frac{1}{2} \times 2\pi = 64\pi$$

$$\oint_S \overline{F} \cdot d\overline{S} = 0 + 64\pi + 0 = 64\pi$$

Let us verify divergence theorem which states that,

$$\oint_S \overline{F} \cdot d\overline{S} = \oint_v (\nabla \cdot \overline{F})\, dv$$

Now $\qquad dv = r\, dr\, d\phi\, dz$

$$\nabla \cdot \overline{F} = \frac{1}{r} \frac{\partial}{\partial r}(r\, F_r) + \frac{1}{r} \frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$

$$= \frac{1}{r} \frac{\partial}{\partial r}(r \times r^2 \cos^2 \phi) + \frac{1}{r} \frac{\partial}{\partial \phi}(z \sin \phi) + 0$$

$$= \frac{\cos^2 \phi}{r} \times 3r^2 + \frac{z}{r}(+\cos \phi) = 3\, r \cos^2 \phi + \frac{z \cos \phi}{r}$$

Gradient of a scalar- consider that in space let W be the unique function of x,y and z co-ordinates in Cartesian system. This is the scalar function and denoted as W(x,y,z) . Consider the vector operator in Cartesian system denoted as ∇ called del. It is defined as,

$$\nabla \text{ (del)} = \frac{\partial}{\partial x} \bar{a}_x + \frac{\partial}{\partial y} \bar{a}_y + \frac{\partial}{\partial z} \bar{a}_z$$

- The operation of the vector operator del (∇) on a scalar function is called gradient of scalar.

$$\text{Grad } W = \nabla W = \left(\frac{\partial}{\partial x} \bar{a}_x + \frac{\partial}{\partial y} \bar{a}_y + \frac{\partial}{\partial z} \bar{a}_z \right) W$$

$$\text{Grad } W = \frac{\partial W}{\partial x} \bar{a}_x + \frac{\partial W}{\partial y} \bar{a}_y + \frac{\partial W}{\partial z} \bar{a}_z$$

- Gradient of a scalar is a vector.

- The gradient of a scalar W in various co-ordinate systems are given by-

Sr. No	Co-ordinate system	Grad W = ∇ W
1.	Cartesian	$\nabla W = \frac{\partial W}{\partial x}\bar{a}_x + \frac{\partial W}{\partial y}\bar{a}_y + \frac{\partial W}{\partial z}\bar{a}_z$
2.	Cylindrical	$\nabla W = \frac{\partial W}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial W}{\partial \phi}\bar{a}_\phi + \frac{\partial W}{\partial z}\bar{a}_z$
3.	Spherical	$\nabla W = \frac{\partial W}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial W}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial W}{\partial \phi}\bar{a}_\phi$

- **Properties of gradient of scalar-**

1) The gradient ∇ W gives maximum rate of change of W per unit distance.

2) The ∇ W always indicates the direction of maximum rate of change of W.

3) The ∇ W at any point is perpendicular to the constant W surface, which passes through the point.

4) The directional derivative of W along the unit vector $\bar{a}$ is $\nabla W \cdot \vec{a}$ (dot product), which is the projection of ∇ W in the direction of $\bar{a}$.

If U is another scalar function then,

5) $\nabla(U+W) = \nabla U + \nabla W$

6) $\nabla(UW) = U\nabla W + W\nabla U$

7) $\nabla\left(\frac{U}{W}\right) = \frac{W\nabla U - U\nabla W}{W^2}$

Example 15- *A particular scalar field α is given by,* a) $\alpha = 20\, e^{-x} \sin\left(\frac{\pi y}{6}\right)$ …in Cartesian b) $\alpha = 25\, r \sin\phi$ ….. In cylindrical and c) $\alpha = \frac{40 \cos\theta}{r^2}$ ……in Spherical.

Find it's gradient at P(0,1,1) for Cartesian, $P\left(\sqrt{2}, \frac{\pi}{2}, 5\right)$ for cylindrical and P(3,60°,30°) for spherical.

Solution- a) $\alpha = 20\, e^{-x} \sin\left(\frac{\pi y}{6}\right)$ in cartesian

$$\nabla \alpha = \frac{\partial \alpha}{\partial x}\bar{a}_x + \frac{\partial \alpha}{\partial y}\bar{a}_y + \frac{\partial \alpha}{\partial z}\bar{a}_z$$

$$\frac{\partial \alpha}{\partial x} = \frac{\partial}{\partial x}\left[20\, e^{-x} \sin\left(\frac{\pi y}{6}\right)\right] = -20\, e^{-x} \sin\left(\frac{\pi y}{6}\right)$$

$$\frac{\partial \alpha}{\partial y} = \frac{\partial}{\partial y}\left[20\, e^{-x} \sin\left(\frac{\pi y}{6}\right)\right] = 20\, e^{-x} \cos\left(\frac{\pi y}{6}\right) \times \frac{\pi}{6}$$

$$\frac{\partial \alpha}{\partial z} = \frac{\partial}{\partial z}\left[20\, e^{-x} \sin\left(\frac{\pi y}{6}\right)\right] = 0$$

$$\nabla \alpha = -20\ e^{-x} \sin\left(\frac{\pi y}{6}\right)\bar{a}_x + 20\ e^{-x} \frac{\pi}{6} \cos\left(\frac{\pi y}{6}\right)\bar{a}_y$$

$\therefore$ At P (0, 1, 1) the $\nabla \alpha = -10\ \bar{a}_x + 9.0689\ \bar{a}_y$

b) $\alpha = 25\ r \sin\phi$ in cylindrical.

$$\therefore \quad \nabla \alpha = \frac{\partial \alpha}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial \alpha}{\partial \phi}\bar{a}_\phi + \frac{\partial \alpha}{\partial z}\bar{a}_z$$

$$\frac{\partial \alpha}{\partial r} = 25 \sin\phi \qquad \frac{\partial \alpha}{\partial \phi} = 25\, r \cos\phi, \qquad \frac{\partial \alpha}{\partial z} = 0$$

$$\therefore \quad \nabla \alpha = 25 \sin\phi\ \bar{a}_r + 25 \cos\phi\ \bar{a}_\phi$$

$\therefore$ At $P\left(\sqrt{2}, \frac{\pi}{2}, 5\right)$ the $\nabla \alpha = 25\ \bar{a}_r$

c) $\alpha = \dfrac{40\cos\theta}{r^2}$ in spherical.

$$\therefore \qquad \nabla\alpha = \frac{\partial\alpha}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial\alpha}{\partial\theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial\alpha}{\partial\phi}\bar{a}_\phi$$

$$\frac{\partial\alpha}{\partial r} = 40\cos\theta\left[-2r^{-3}\right] = -80\frac{\cos\theta}{r^3}$$

$$\frac{\partial\alpha}{\partial\theta} = -\frac{40}{r^2}\sin\theta\,, \qquad \frac{\partial\alpha}{\partial\phi} = 0$$

$$\therefore \qquad \nabla\alpha = -\frac{80\cos\theta}{r^3}\bar{a}_r - \frac{40}{r^3}\sin\theta\,\bar{a}_\theta$$

$\therefore$ At P $(3, 60°, 30°)$ the $\nabla\alpha = -1.4814\,\bar{a}_r - 0.9362\,\bar{a}_\theta$

Curl of a vector- The circulation of a vector field around a closed path is given by curl of a vector. Mathematically it is defined as-

$$\text{Curl of } \bar{F} = \lim_{\Delta S_N \to 0} \frac{\oint \bar{F} \cdot d\bar{l}}{\Delta S_N}$$

Where ΔS_N = area enclosed by the line integral in normal direction.

- Thus maximum circulation of $\bar{F}$ per unit area as area tends to zero whose direction is normal to the surface is called curl of $\bar{F}$.

Symbolically it is expressed as- $\nabla \times \bar{F} = \text{curl of } \bar{F}$

- Curl indicates the rotational property of vector field. If curl of vector is zero, the vector field is irrigational.
- In various coordinate systems, the curl of vector is given by-

$$\nabla \times \bar{F} = \left[\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right]\bar{a}_x + \left[\frac{\partial F_x}{\partial_z} - \frac{\partial F_z}{\partial x}\right]\bar{a}_y + \left[\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right]\bar{a}_z$$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \quad \text{Cartesian}$$

$$\nabla \times \bar{F} = \left[\frac{1}{r}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right]\bar{a}_r + \left[\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right]\bar{a}_\phi + \left[\frac{1}{r}\frac{\partial (r F_\phi)}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial \phi}\right]\bar{a}_z$$

$$\nabla \times \bar{F} = \frac{1}{r}\begin{vmatrix} \bar{a}_r & r\bar{a}_\phi & \bar{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_r & rF_\phi & F_z \end{vmatrix} \quad \text{Cylindrical}$$

$$\nabla \times \bar{F} = \frac{1}{r\sin\theta}\left[\frac{\partial F_\phi \sin\theta}{\partial \theta} - \frac{\partial F_\theta}{\partial \phi}\right]\bar{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial \phi} - \frac{\partial (r F_\phi)}{\partial r}\right]\bar{a}_\theta + \frac{1}{r}\left[\frac{\partial (r F_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta}\right]\bar{a}_\phi$$

$$\nabla \times \bar{F} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \bar{a}_r & r\bar{a}_\theta & r\sin\theta\,\bar{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta\, F_\phi \end{vmatrix} \quad \text{Spherical}$$

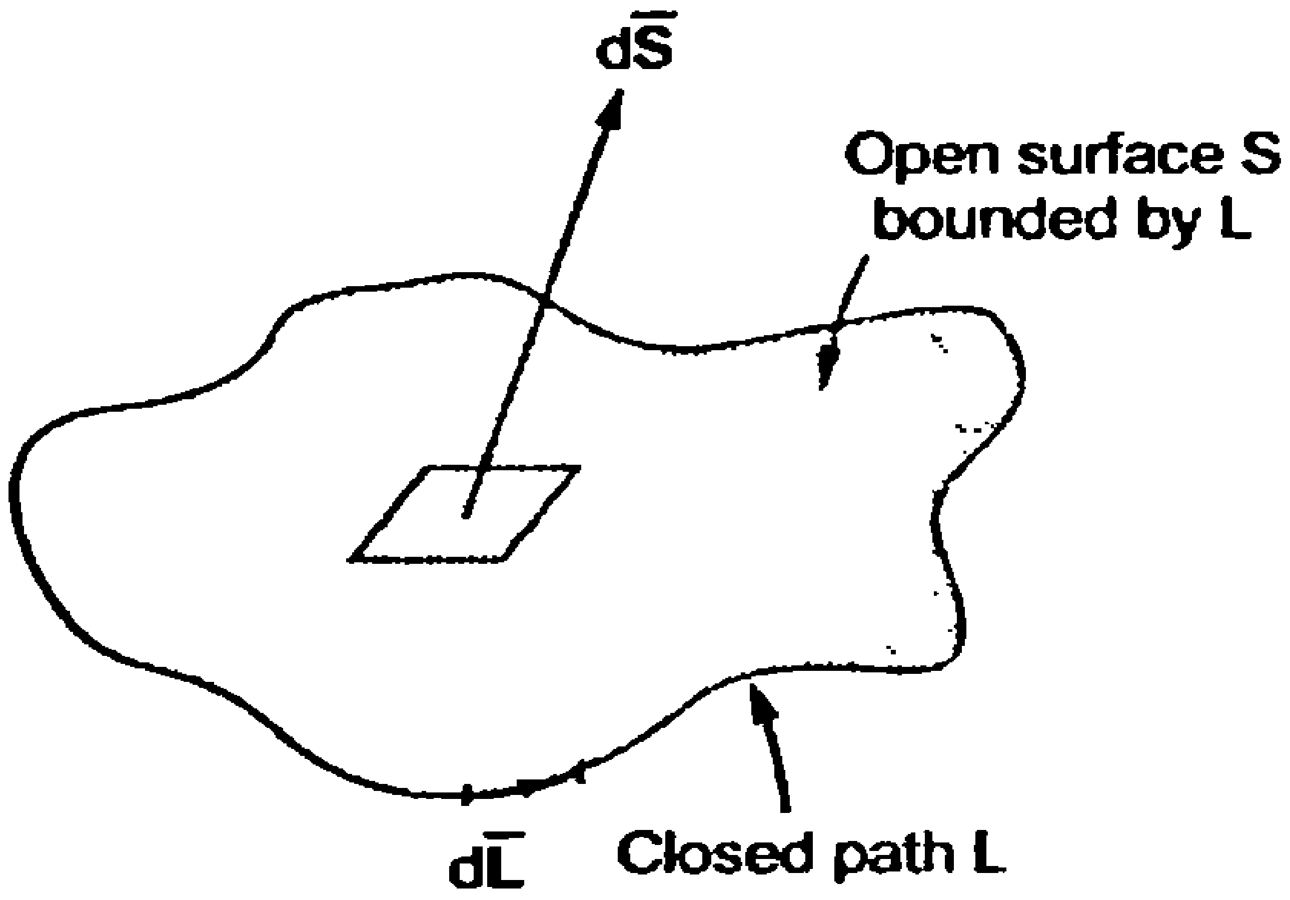
Stock's theorem – The Stock's theorem relates the line integral to a surface integral.

It states that- " **The line integral of $\bar{F}$ around a closed path L is equal to the integral of curl of $\bar{F}$ over the open surface S enclosed by the closed path L."**

Mathematically it is expressed as-

$$\oint_L \bar{F} \cdot d\bar{L} = \int_S (\nabla \times \bar{F}) \cdot d\bar{S}$$

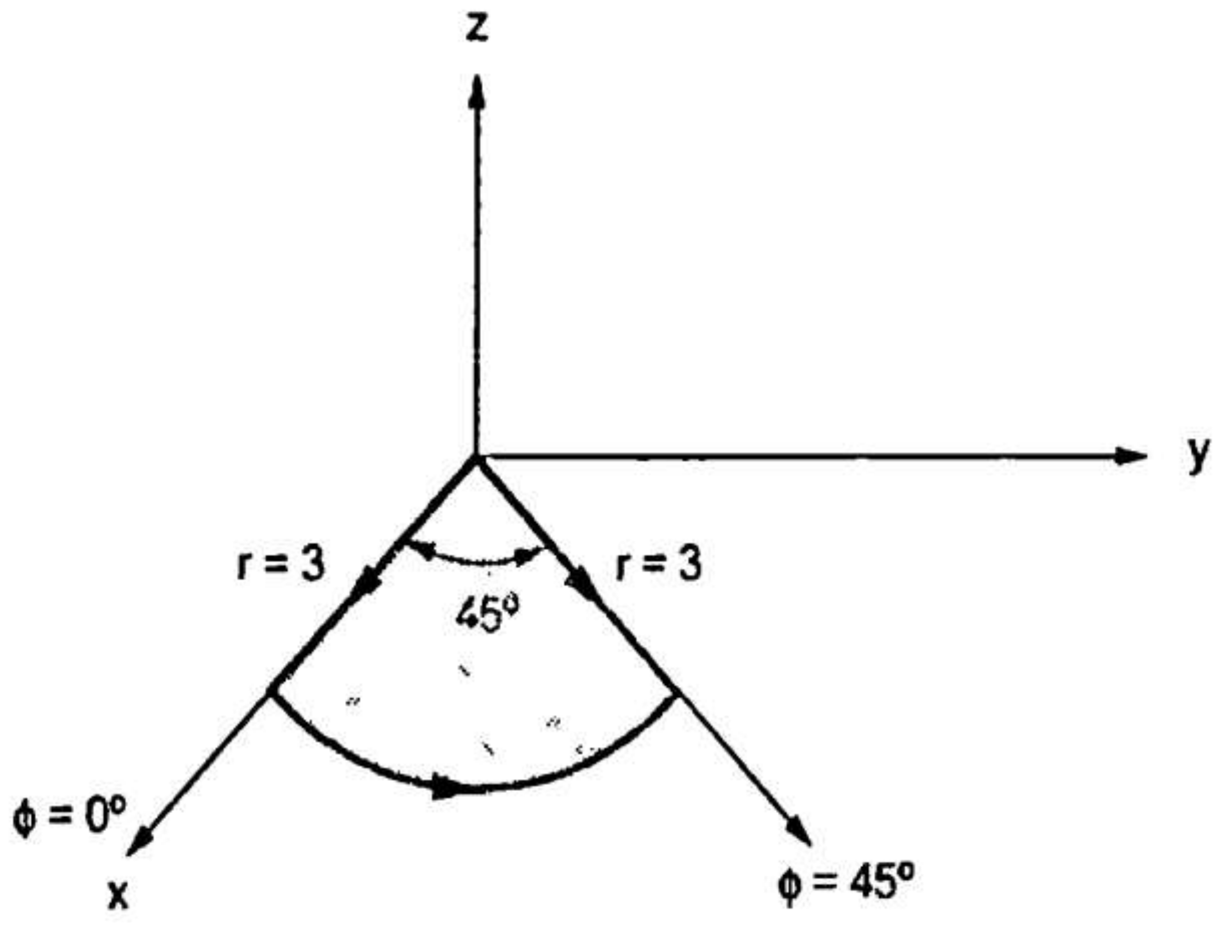
Where dL = perimeter of total surface S.



stock's theorem is applicable only when $\bar{F}$ and $\nabla \times \bar{F}$ are continuous on the surface S. The path L and the open surface S enclosed by path L for which stock's theorem is applicable are shown in fig.

Example 16- *Verify Stoke's theorem for a vector field* $\bar{F} = r^2 \cos\phi \, \bar{a}_r + z \sin\phi \, \bar{a}_z$ *around the path L defined by* $0 \le r \le 3$, $0 \le \phi \le 45°$ *and* $z = 0$ *as shown in the Fig.*

Solution-



From Stoke's theorem,

$$\oint_L \bar{F} \cdot d\bar{L} = \int_S (\nabla \times \bar{F}) \cdot d\bar{S}$$

The L.H.S. is already evaluated in previous example 13 , which is 2.636.

To evaluate R.H.S. , find $\nabla \times \bar{F}$

$$\nabla \times \bar{F} = \left[\frac{1}{r}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right]\bar{a}_r + \left[\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right]\bar{a}_\phi + \left[\frac{1}{r}\frac{\partial (r F_\phi)}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial \phi}\right]\bar{a}_z$$

$$F_r = r^2 \cos\phi, \quad F_\phi = 0, \quad F_z = z \sin\phi$$

$$\nabla \times \bar{F} = \left[\frac{1}{r} \times 0 - 0\right]\bar{a}_r + [0 - 0]\,\bar{a}_\phi + \left[\frac{1}{r}(0) - \frac{1}{r}(r)^2(-\sin\phi)\right]\bar{a}_z = r \sin\phi \, \bar{a}_z$$

$d\bar{S} = r\,dr\,d\phi\,\bar{a}_z$ as surface is in x-y plane i.e. z = 0 plane for which normal direction is $\bar{a}_z$.

$$\therefore \int_S (\nabla \times \bar{F}) \cdot d\bar{S} = \int_S (r \sin\phi\,\bar{a}_z) \cdot (r\,dr\,d\phi)\,\bar{a}_z = \int_{\phi=0}^{45^\circ} \int_{r=0}^{3} r^2 \sin\phi\,dr\,d\phi = \left[\frac{r^3}{3}\right]_0^3 [-\cos\phi]_0^{45^\circ}$$

$$= [9]\,[-0.707 - (-1)] = 9 \times 0.2928 = \mathbf{2.636}$$

Thus Stoke's theorem is verified.

Laplacian of a Scalar- The divergence of a vector and gradient of a scalar is discussed earlier. The composite operator of these two is called Laplacian of a scalar.

If V is a scalar field then, the Laplacian of scalar V is denoted as $\nabla^2 V$ and mathematically defined as the divergence of the gradient of V.

The operator ∇^2 is called laplacian operator.

- In Cartesian coordinate system-

$$\nabla^2 V = \nabla \cdot \nabla V = \left[\frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z\right] \cdot \left[\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z\right]$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$ ...In cartesian system

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2}$$ ...In cylindrical system

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$ ...In spherical system

Example-17 *Find the Laplacian of the scalar fields and comment on, which fields are harmonic.*

i) $W = x^2 y + xyz - yz^2$ ii) $U = rz \sin\phi + z^2 \cos^2\phi + r^2$

iii) $V = 2r \cos\theta \cos\phi$

Solution-

i) $W = x^2y + xyz - yz^2$

$$\nabla^2 W = \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2}$$

$$= \frac{\partial}{\partial x}(2xy + yz) + \frac{\partial}{\partial y}(x^2 + xz - z^2) + \frac{\partial}{\partial z}(xy - 2yz)$$

$$= 2y + 0 + 0 + 0 + 0 - 2y = 0$$

As $\nabla^2 W = 0$, the scalar field W is harmonic.

ii) $U = rz \sin\phi + z^2 \cos^2\phi + r^2$

$$\nabla^2 U = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}[r(z\sin\phi + 2r)] + \frac{1}{r^2}\frac{\partial}{\partial \phi}[rz\cos\phi - 2z^2 \sin\phi\cos\phi] + \frac{\partial}{\partial z}[r\sin\phi + 2z\cos^2\phi]$$

$\ldots\ 2\sin\phi\cos\phi = \sin 2\phi$

$$= \frac{1}{r}[z\sin\phi + 4r] + \frac{1}{r^2}[-rz\sin\phi - z^2 2\cos 2\phi] + [0 + 2\cos^2\phi]$$

$$= \frac{z}{r}\sin\phi + 4 - \frac{z}{r}\sin\phi - \frac{2z^2}{r^2}\cos 2\phi + 2\cos^2\phi$$

$$= 4 + 2\cos^2\phi - \frac{2z^2}{r^2}\cos 2\phi$$

As $\nabla^2 U \neq 0$, the scalar field U is not harmonic.

iii) $V = 2r\cos\theta\cos\phi$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right]\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial V}{\partial\theta}\right]+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}[r^2(2\cos\theta\cos\phi]+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}[\sin\theta(-2r\cos\phi\sin\theta)]$$

$$+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}(-2r\cos\theta\sin\phi)$$

$$= \frac{1}{r^2}[4r\cos\theta\cos\phi]+\frac{1}{r^2\sin\theta}[-2r\cos\phi\times 2\sin\theta\cos\theta]$$

$$+\frac{1}{r^2\sin^2\theta}[-2r\cos\theta\cos\phi]$$

$$= \frac{4\cos\theta\cos\phi}{r}-\frac{4\cos\theta\cos\phi}{r}-\frac{2\cos\theta\cos\phi}{r\sin^2\theta}=\frac{-2}{r}\cot\theta\,\mathrm{cosec}\,\theta\cos\phi$$

As $\nabla^2 V \neq 0$, the scalar field V is not harmonic.

REFERENCES

- Engineering Electromagnetic by william H. Hayt, Jr. John A. Buck
- Principals of Electromagnetics by Matthew N. O. Sadiku.
- CBS problems and solution series (Problems and solution of Engineering Electromagnetics).
- https://nptel.ac.in/courses/108/106/108106073/
- https://nptel.ac.in/courses/117/103/117103065/

JECRC Foundation

JAIPUR ENGINEERING COLLEGE AND RESEARCH CENTRE

Thank you!