JECRC Foundation

JAIPUR ENGINEERING COLLEGE AND RESEARCH CENTRE

JAIPUR ENGINEERING COLLEGE AND RESEARCH CENTRE

Year & Sem – I Year & 1 Sem

Subject –Engineering Mathematics

Unit – II (SEQUENCES AND SERIES)

Presented by – (Dr. Tripati Gupta, Associate Professor)

VISION AND MISSION OF INSTITUTE

VISION OF INSTITUTE

To became a renowned centre of outcome based learning and work towards academic professional, cultural and social enrichment of the lives of individuals and communities .

MISSION OF INSTITUTE

- Focus on evaluation of learning, outcomes and motivate students to research aptitude by project based learning.
- Identify based on informed perception of Indian, regional and global needs, the area of focus and provide platform to gain knowledge and solutions.
- Offer opportunities for interaction between academic and industry .
- Develop human potential to its fullest extent so that intellectually capable and imaginatively gifted leaders may emerge.

Engineering Mathematics: Course Outcomes

Students will be able to:

CO1. Understand fundamental concepts of improper integrals, beta and gamma functions and their properties. Evaluation of Multiple Integrals in finding the areas, volume enclosed by several curves after its tracing and its application in proving certain theorems.

CO2. Interpret the concept of a series as the sum of a sequence and use the sequence of partial sums to determine convergence of a series. Understand derivatives of power, trigonometric, exponential, hyperbolic, logarithmic series.

Engineering Mathematics: Course Outcomes

CO3. Recognize odd, even and periodic function and express them in Fourier series using Euler's formulae.

CO4. Understand the concept of limits, continuity and differentiability of functions of several variables. Analytical definition of partial derivative. Maxima and minima of functions of several variables Define gradient, divergence and curl of scalar and vector functions.

CONTENTS (TO BE COVERED)

SEQUENCES AND SERIES

Sequence: A function whose domain is the set of natural numbers N and range, a subset of real numbers R is called a sequence.

A sequence is of the form $\{(1,x_1),(2,x_2),\dots,(n,x_n)\}$ where $x_1, x_2, \dots, x_n$ are real numbers. The real number x_n that the sequence associated with the positive integer n is called the image of n under the sequence.

Generally, it is denoted by $\{x_1, x_2, \ldots, x_n\}$.

Here $x_1, x_2, \ldots, x_n$ are the terms of the sequence and so x_n is the n^{th} term of the sequence.

A sequence has its n^{th} term denoted by $\{x_n\}$.

The Range: The range set is the set consisting of all distinct elements of a sequence, without repetition and without regard to the position of a term. Thus, the range may be a finite set or an infinite set.

Bounds of a Sequence:

(i) Bounded-above Sequence: A sequence $\{x_n\}$ is said to be bounded above if there exists a real number M such that $x_n \leq M \ \forall \ n \in \mathbb{N}$.

(ii) Bounded-below Sequence : A sequence $\{x_n\}$ is said to be bounded below if there exists a real number m such that

$x_n \geq m, \forall n \in N$.

(iii) Bounded Sequence

A sequence $\{x_n\}$ is said to be bounded if it is bounded above and below. M and m are the upper and the lower bounds of the sequence. for example

$x_n = \{(-1)^n ; n \in N\}$ is a bounded sequence

Convergence of a Sequence

A sequence $\{x_n\}$ is said to converge to a real number 'l' if for each $\epsilon > 0$ there exists a positive integer m (depending on ϵ) such that

$$|x_n - l| < \epsilon, \text{ for all } n \geq m.$$

Mathematically, we write

$$x_n \to l \text{ as } n \to \infty \text{ or } \lim_{n\to\infty} x_n = l$$

* A sequence $\{x_n\}$ is said to be divergent if $\lim_{n\to\infty} x_n$ is not a finite quantity i.e.

if $\lim_{n\to\infty} x_n = +\infty$ or $-\infty$

<u>Examples</u>: (i) Let a sequence $\{x_n\} = \{n^2\}$

Then $\lim_{n\to\infty} n^2 = \infty$ (which is not finite)

Hence $\{x_n\}$ is divergent.

(ii) Let $\{x_n\} = \left\{\frac{1}{2^n} ; n \in N\right\}$

Then $\lim_{n\to\infty} \frac{1}{2^n} = 0$ (a finite quantity)

Hence, the sequence $\{x_n\}$ is convergent.

Oscillatory Sequence

A sequence $\{x_n\}$ which neither converges to a finite number nor diverges to ∞ or $-\infty$ is said to be an oscillatory sequence.

Example: $\{x_n\} = \{(-1)^n\}$ oscillates finitely between -1 and 1 and the sequence $\{x_n\} = \{n(-1)^n\}$ oscillates infinitely between $-\infty$ and ∞.

Note:

(i) Every convergent sequence has a unique limit.

(ii) Every convergent sequence is bounded.

Monotonic Sequence

A sequence $\{x_n\}$ is said to be monotonic increasing if $x_{n+1} \geqslant x_n, \forall n$.

A sequence $\{x_n\}$ is said to be ~~de~~ monotonic decreasing if $x_{n+1} \leq x_n, \forall n$.

Thus, a sequence $\{x_n\}$ is said to be monotonic if it is either monotonic increasing or decreasing.

A sequence $\{x_n\}$ is strictly increasing if $x_{n+1} > x_n, \forall n$ and strictly decreasing if $x_{n+1} < x_n, \forall n$.

Examples: (i) $\left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$ is a monotonic decreasing sequence.

(ii) $\{x_n\} = \left\{\frac{n}{n+1}\right\}$ is a monotonic increasing sequence.

Infinite Series

If $<u_n>$ be a sequence of real numbers then the sum of the infinite number of terms of this sequence i.e. the expression $u_1 + u_2 + \cdots + u_n + \cdots$ is defined as an infinite series and is denoted by

$$\sum_{n=1}^{\infty} u_n \text{ or } \sum u_n .$$

A sequence $\langle S_n \rangle$ where S_n denotes the sum of the first n terms of the series,

Thus, $S_n = u_1 + u_2 + \cdots + u_n \ \forall n$.

The sequence $\langle S_n \rangle$ is called the 'sequence of partial sums of the series' and the partial sums, $S_1 = u_1$, $S_2 = u_1 + u_2$, $S_3 = u_1 + u_2 + u_3$ and so on.

Positive term Series

The Series $\sum u_n$ is called a positive-term series if each term of this series is positive i.e.

$$\sum u_n = u_1 + u_2 + \cdots + u_n + \cdots$$

Alternating Series

A series whose terms are alternatively positive and negative i.e.

$$\sum u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

Convergent Series

A series $\sum u_n$ is said to be Convergent if the sum of the first n terms of the series tends to a finite and unique limit as n tends to infinity, i.e. if $\lim_{n \to \infty} S_n =$ finite and unique then the series $\sum u_n$ is Convergent.

Divergent Series

A series $\sum u_n$ is said to be divergent if the sum of the first n terms of the series tends to $+\infty$ or $-\infty$ as $n \to \infty$ i.e. if $\lim_{n\to\infty} S_n = +\infty$ or $-\infty$, then the series $\sum U_n$ is divergent.

Oscillatory Series

The oscillatory series are two types:

(i) Oscillate finitely

A series Σu_n is said to oscillate finitely if the sum of its first n terms tends to a finite but not unique limit as n tends to infinity, i.e., if $\lim_{n \to \infty} S_n$ = finite but not unique the Σu_n oscillates finitely.

(ii) Oscillate Infinitely

A series Σu_n is said to oscillate infinitely if the sum of its first n terms oscillates infinitely, i.e. if $\lim_{n\to\infty} S_n = +\infty$ or $-\infty$ both then the series Σu_n oscillates infinitely.

Note: (i) The convergency or divergency of a series is not affected by altering, adding, or neglecting a finite number of its terms.

(ii) The convergency or divergency of a series is not affected by the multiplication of all terms of the series by a fixed number

A Necessary Condition for Convergence

A necessary condition for a positive-term series $\sum u_n$ to converge is that $\lim_{n\to\infty} u_n = 0$

Note: (i) The above condition is not sufficient.

Example: $\sum u_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \left(\frac{1}{n}\right) + \cdots$

where $u_n = \frac{1}{n}$, $\therefore \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

(ii) If $\lim_{n\to\infty} u_n = 0$, we are not sure that whether the series Σu_n is Convergent or not but if $\lim_{n\to\infty} u_n \neq 0$, then the series Σu_n is divergent.

Cauchy's Fundamental Test for Divergence

If $\lim_{n \to \infty} u_n \neq 0$, the series $\sum u_n$ is divergent.

Example 1: Test the convergence of the series

$$1+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\cdots+\frac{n}{n+1}+\cdots\infty$$

Solution: $u_n = \frac{n}{n+1}$

$$\therefore \lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{n}{n+1}$$

$$= \lim_{n\to\infty}\frac{1}{1+\left(\frac{1}{n}\right)} = 1 \neq 0$$

Hence, by Cauchy's test Σu_n is divergent.

Example 2: Discuss the convergence of the series

$$\sum_{n=0}^{\infty} (-1)^n.$$

Solution: $\sum u_n = \sum (-1)^n$

$S_n = (-1)^0 + (-1)^1 + (-1)^2 + \cdots$ to n terms

$= 1 - 1 + 1 - 1 + 1 - 1 \cdots$ to n terms

$= 1$ or 0 accordingly as n odd or even.

$\therefore \lim_{n\to\infty} S_n = 1$ or 0 i.e. finite but not unique.

Hence, the series $\sum u_n$ is a finitely oscillating series.

Geometric Series

The series $1+x+x^2+x^3+\cdots\infty$ is

(i) Convergent if $|x|<1$

(ii) Divergent if $x\geq 1$

(iii) Oscillatory if $x\leq -1$

Proof: (i) when $|x|<1$ then $\lim_{n\to\infty} x^n = 0$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-x^n}{1-x} = \frac{1-0}{1-x} = \frac{1}{(1-x)}$$

$$= \text{a finite quantity}$$

Hence, the series is convergent.

(ii) (a) when $x>1$, $\lim_{n\to\infty} x^n = \infty$, then

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{x^n-1}{x-1} = \infty$$

Hence the series is divergent.

(b) when $x=1$, the series becomes $1+1+1+1+\cdots\infty$

$$S_n = 1+1+1+\cdots \text{ n times} = n$$

$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} n = \infty$; Hence the series is divergent.

(iii) (a) when $x < -1$, let $x = -r$, $r > 1$

$$x^n = (-r)^n = (-1)^n r^n$$

$$\therefore \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1 - x^n}{1 - x} = \lim_{n \to \infty} \frac{1 - (-1)^n r^n}{1 - (-r)}$$

$$= +\infty \text{ if } n \text{ is odd}$$

$$= -\infty \text{ if } n \text{ is even}$$

Hence, the series is oscillatory.

(b) when $x=-1$, the series becomes $1-1+1-1+1-\cdots\infty$

$\therefore S_n = 1-1+1-1+\cdots$ n times

$\lim_{n\to\infty} S_n = 0$ if n is even

$= 1$ if n is odd

Hence, the series is oscillatory finitely.

Alternating Series

A series whose terms are alternatively positive and negative is called an alternating series.

Example: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \infty$

Leibnitz Test

If the alternating series $u_1 - u_2 + u_3 - u_4 + \ldots$

$(u_n > 0 \ \forall n)$ is such that

(i) $u_{n+1} \leq u_n \ \forall n$ and

(ii) $\lim_{n \to \infty} u_n = 0$

then the series converges.

Example: Test the series $\frac{2}{1^3} - \frac{3}{2^3} + \frac{4}{3^3} - \frac{5}{4^3} + \dots$

Solution: In the given series, we have

(i) the terms are alternately, +ve and −ve

(ii) the terms are continually decreasing

(iii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n+1}{n^3}$

$$= \lim_{n\to\infty} \frac{1+1/n}{n^2} = \frac{1+0}{\infty} = 0 \text{ (finite)}$$

Hence, the given alternating series by 'Leibnitz's test' is convergent.

Example: Test the series $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\cdots$

Solution: The given series can be written as

$$1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+\frac{1}{2^4}-\cdots$$

In this series, we find that

(i) the terms are alternatively +ve and -ve.

(ii) the terms are continually decreasing

(iii) $\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{1}{2^{n-1}} = 0$

Hence, the given series is convergent.

Example: Test the convergency of the series

$$1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots\cdots$$

Solution: $u_n=\frac{1}{\sqrt{n}}$, $u_{n-1}=\frac{1}{\sqrt{n-1}}$

In the given series, we have

(i) the terms are alternatively +ve and -ve.

(ii) $u_n < u_{n-1}$

(iii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Hence, the alternating series is convergent.

Example: Test the Convergency of the series

$$\frac{\sqrt{2}-\sqrt{1}}{1}-\frac{\sqrt{3}-\sqrt{2}}{2}+\frac{\sqrt{4}-\sqrt{3}}{3}-\frac{\sqrt{5}-\sqrt{4}}{4}+\cdots$$

Solution: Here, $u_n = \left[\frac{\sqrt{n+1}-\sqrt{n}}{n}\right]$

In the given series, we have

(i) the terms of the series are alternatively +ve and -ve.

(ii) the terms are continually decreasing as $u_n > u_{n+1}$ for all n.

(iii) $$\lim_{n\to\infty}\left[\frac{\sqrt{n+1}-\sqrt{n}}{n}\right] = \lim_{n\to\infty}\frac{1}{n}\left[\sqrt{n}\cdot\sqrt{1+\frac{1}{n}}-\sqrt{n}\right]$$

$$= \lim_{n\to\infty}\frac{1}{\sqrt{n}}\left[\left(1+\frac{1}{n}\right)^{1/2}-1\right]$$

$$= \lim_{n\to\infty}\frac{1}{\sqrt{n}}\left[\left(1+\frac{1}{2}\cdot\frac{1}{n}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\frac{1}{n^2}+\cdots\right)-1\right]$$

$$= \lim_{n\to\infty}\frac{1}{\sqrt{n}}\left[\frac{1}{2n}-\frac{1}{8n^2}+\cdots\right]=0$$

Hence, the alternating series is convergent.

Example: Test the convergence of the series

$$2-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\cdots$$

Solution: Here $u_n = \frac{n+1}{n}$.

In the given series, we have

(i) the terms are alternately +ve and −ve

(ii) the terms are decreasing order i.e $u_n < u_{n-1}$

(iii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1 \neq 0$

Hence, the third condition of the alternating series not satisfied, so the series is not convergent.

However, we can write the given series as

$(1+1)-(1+\frac{1}{2})+(1+\frac{1}{3})-(1+\frac{1}{4})+\cdots$

or $(1-1+1-1+\cdots)+(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots)$

The series in the IInd bracket is convergent, since the value of it is $\log(1+1)$ i.e. $\log 2$.

But, the series in the Ist bracket is an oscillating series whose value is either 0 or 1 accordingly as n is even or odd.

$\therefore$ the sum of n terms of the given series as $\lim_{n\to\infty}$ is $(0+\log 2)$ or $(1+\log 2)$ accordingly as n is even or odd i.e. $\log 2$ or $(1+\log 2)$ if n is even or odd.

Hence, by the definition, the given series is oscillating.

Alternating Convergent series

There are two types of alternating Convergent series (i) Absolutely Convergent series
(ii) Conditionally Convergent series

Absolutely Convergent series:

If $u_1+u_2+\cdots$ be such $|u_1|+|u_2|+|u_3|+\cdots$ be Convergent then $u_1+u_2+u_3+\cdots$ is called absolutely Convergent.

Conditionally Convergent series:

If $|u_1|+|u_2|+\cdots$ be divergent and $u_1+u_2+u_3+\cdots$ be Convergent then $u_1+u_2+u_3+\cdots$ is called Conditionally Convergent.

Example: The series $\sum u_n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \cdots$ is absolutely Convergent, because

$\sum |u_n| = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$ is an infinite geometric series of positive terms with Common ratio $\frac{1}{2} < 1$.

Example: The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \infty$ is Conditionally Convergent. because

$\sum |u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is not Convergent by p-series test. $\sum |u_n| = \sum \frac{1}{n} = \sum \frac{1}{n^p}$

so $p = 1$. Hence it is divergent.

Important test for Convergence of Infinite Series

The theorems and rules that have been considered and discussed in previous sections, makes us enable to determine the convergence of an infinite series in general. But, it is always not possible to find the sum of n terms of series (i.e. S_n).

Therefore we require some another way to test the convergence of an infinite series without using S_n. In this section we will discuss some 'tests' which enable us to determine the convergence independent of evaluation of S_n. These tests are as follows

1. p-Series Test 2. Comparison Test

3. D' Alembert Ratio Test 4. Raabe's Test

5. Gauss Test 6. Cauchy's Root Test

7. Logarithmic Test 8. De Morgan's and Bertrand's Test

9. Cauchy's Integral Test 10. Cauchy Condensation Test.

Flow chart for Tests the Convergence of positive term series

Apply Comparison test

↓ if it fails, then apply

D' Alembert ratio test

↓ if it fails, then apply

Raabe's Test

↓ if it fails, then apply

Cauchy's Root Test

↓ if it fails, then apply

Logarithmic Test

↓ if it fails, then apply

Gauss Test

p- Series Test

The infinite series Σu_n of the form

$$\Sigma \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$ is

known as p- series and it is

(i) Convergent, if $p > 1$

(ii) Divergent, if $p \leq 1$.

Example: The series $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$

is divergent as $p = \frac{1}{2} < 1$. (Here p means Common Power)

Comparison Test

If Σu_n and Σv_n be two given series of positive terms such that

$$\lim_{n \to \infty} \frac{u_n}{v_n} = l \neq 0 \quad (l \text{ is finite non-zero})$$

then the two series Σu_n and Σv_n are either both convergent or both divergent.

Example: Test the convergence of the series

$$\frac{1}{2}+\frac{\sqrt{2}}{5}+\frac{\sqrt{3}}{8}+\cdots+\frac{\sqrt{n}}{3n-1}+\cdots$$

Solution: Here, the nth term of the given series

is $\frac{\sqrt{n}}{3n-1}$.

i.e. $u_n=\frac{\sqrt{n}}{3n-1}=\frac{\sqrt{n}}{3n\left(1-\frac{1}{3n}\right)}$

$$=\frac{1}{3\sqrt{n}\left(1-\frac{1}{3n}\right)}$$

Now, take an auxiliary series

$$\Sigma v_n=\Sigma\frac{1}{\sqrt{n}}$$

Then applying Comparison Test

$$\frac{u_n}{v_n}=\frac{1}{3\sqrt{n}\left(1-\frac{1}{3n}\right)}\div\frac{1}{\sqrt{n}}$$

$$= \frac{\frac{1}{\sqrt{n}}}{3\sqrt{n}\left(1-\frac{1}{3n}\right)}$$

$$= \frac{1}{3\left(1-\frac{1}{3n}\right)}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{3\left(1-\frac{1}{3n}\right)} = \frac{1}{3} \neq 0$$

$\because \lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{3}$ which in non zero finite quantity. Therefore $\sum u_n$ and $\sum v_n$ converge and diverge simultaneously.

But $\sum v_n$ is divergent (as $\sum v_n$ is of the form $\sum \frac{1}{n^p}$ and $p = \frac{1}{2} < 1$).

Consequently the given series

$\sum u_n = \sum \frac{\sqrt{n}}{3n-1}$ is also divergent.

Example: Test the Convergency of the series

$$\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \frac{5}{4^p} + \cdots$$

Solution: The n^{th} term of the given series is

$$u_n = \frac{n+1}{n^p} = \frac{n(1+1/n)}{n^p} = \frac{(1+1/n)}{n^{p-1}}$$

Let $v_n = \frac{1}{n^{p-1}}$

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1$$ (a finite quantity)

But $\sum v_n$ is Convergent if $p-1>1$ or $p>2$ and it is divergent if $p-1 \leq 1$ or $p \leq 2$.

Hence, the given series is Convergent if $p>2$ and divergent if $p \leq 2$.

Example: Test for Convergence of the series

$$\frac{1}{1\cdot 2\cdot 3}+\frac{3}{2\cdot 3\cdot 4}+\frac{5}{3\cdot 4\cdot 5}+\cdots$$

Solution: Let the given series be denoted by

ΣU_n.

where $U_n=\dfrac{(2n-1)}{n(n+1)(n+2)}$

$$=\frac{n(2-1/n)}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$\Rightarrow U_n=\frac{(2-1/n)}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

Let $V_n=\dfrac{1}{n^2}$

$$\therefore \lim_{n\to\infty}\frac{U_n}{V_n}=\lim_{n\to\infty}\frac{(2-1/n)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}=2 \text{ (finite number)}$$

But Σu_n and Σv_n Converge and diverge simultaneously. Since $\Sigma v_n = \Sigma \frac{1}{n^2}$ and it is of the form $\Sigma \frac{1}{n^p}$ with $p = 2 > 1$, therefore Σv_n is Convergent. Hence, by Comparison Test Σu_n is also Convergent.

Example: Test for Convergence of the series whose general term is given by

$$U_n = \sqrt{(n^2+1)} - \sqrt{(n^2-1)}$$

Solution: $U_n = \sqrt{(n^2+1)} - \sqrt{(n^2-1)}$

$$= n\left(1+\frac{1}{n^2}\right)^{1/2} - n\left(1-\frac{1}{n^2}\right)^{1/2}$$

$$= n\left[\left(1+\frac{1}{n^2}\right)^{1/2} - \left(1-\frac{1}{n^2}\right)^{1/2}\right]$$

$$= n\left[\left\{1+\frac{1}{2n^2}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\frac{1}{n^4}+\cdots\right\} - \left\{1-\frac{1}{2n^2}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\frac{1}{n^4} - \cdots\right\}\right]$$

$$= 2n\left[\frac{1}{2n^2}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\frac{1}{n^6}+\cdots\right]$$

$$= \frac{1}{n}+\frac{1}{8n^5}+\cdots$$

Let $\Sigma V_n = \frac{1}{n}$

$$\therefore \lim_{n\to\infty}\frac{U_n}{V_n} = \lim_{n\to\infty}\frac{\frac{1}{n}+\frac{1}{8n^5}+\cdots}{1/n}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{8n^4}+\cdots\right) = 1 \text{ (finite}$$

But $\sum v_n = \frac{1}{n} = \sum \frac{1}{n^p}$

Because of $p=1$, $\sum v_n = \sum \frac{1}{n}$ is divergent.

Hence, by Comparison test the given series

$\sum u_n$ is divergent.

Example: Test the series $\sum \sin\left(\frac{1}{n}\right)$

Solution: Let the given series be $\sum u_n$, where

$$u_n = \sin\left(\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{3!}\frac{1}{n^3} + \frac{1}{5!}\frac{1}{n^5} - \cdots$$

$$= \frac{1}{n}\left[1 - \frac{1}{3!}\frac{1}{n^2} + \frac{1}{5!}\frac{1}{n^4} - \cdots\right]$$

Let $\sum v_n = \sum \frac{1}{n}$; where $v_n = \frac{1}{n}$

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{n}\left[1 - \frac{1}{3!}\frac{1}{n^2} + \frac{1}{5!}\frac{1}{n^4} - \cdots\right]}{1/n}$$

$$= \lim_{n\to\infty}\left[1 - \frac{1}{6n^2} + \frac{1}{120n^4} - \cdots\right]$$

$$= 1 \quad \text{[which is a finite number]}$$

Hence, $\sum u_n$ and $\sum v_n$ converge or diverge together. But the auxiliary series

$\sum v_n = \sum 1/n$ is divergent as $p=1$.

Hence, the given series is divergent.

D' Alembert's Ratio Test

Let Σu_n be a series of positive terms,

such that

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lambda$$

then series Σu_n will be

(i) Convergent if $\lambda > 1$

(ii) Divergent if $\lambda < 1$

and (iii) fails if $\lambda = 1$

Example: Test the series $\frac{x^1}{1\cdot 2}+\frac{x^2}{2\cdot 3}+\frac{x^3}{3\cdot 4}+\cdots$

Solution: $u_n = \frac{x^n}{n(n+1)}$ so $u_{n+1} = \frac{x^{n+1}}{(n+1)(n+2)}$

Now $$\frac{u_n}{u_{n+1}} = \frac{x^n}{n(n+1)} \frac{(n+1)(n+2)}{x^{n+1}}$$

$$= \frac{(n+2)}{n\cdot x}$$

$$\Rightarrow \frac{u_n}{u_{n+1}} = \frac{1}{x}\left(1+\frac{2}{n}\right)$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1}{x}\left(1+\frac{2}{n}\right) = \frac{1}{x}$$

From the ratio test, we conclude that

the given series Σu_n is

(i) Convergent if $\frac{1}{x} > 1$ or $x < 1$ and

(ii) divergent if $\frac{1}{x} < 1$ or $x > 1$

(iii) If $x = 1$ then this test fails and

the given series becomes Σu_n, whose

nth term $u_n = \frac{1}{n(n+1)} = \frac{1}{n^2(1+1/n)}$

Let $v_n = \frac{1}{n^2}$

$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left(\frac{1}{1+1/n}\right) = 1$, which is finite quantity.

$\because \sum v_n = \sum 1/n^2 = \sum \frac{1}{n^p} \Rightarrow p = 2 > 1$, Hence by Comparison Test the given series $\therefore$

$\sum u_n$ is also Convergent as $\sum v_n$ Convergent as $p = 2 > 1$ (p series Test) $\Rightarrow \sum v_n$ is Convergent.

Example: Test the series $\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\frac{x^6}{5\sqrt{4}}+\dots$

Solution: If Σu_n be the given series then the n^{th} term is

$$u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}} \qquad \text{so } u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

Now
$$\frac{u_n}{u_{n+1}} = \frac{x^{2n-2}}{(n+1)\sqrt{n}} \cdot \frac{(n+2)\sqrt{n+1}}{x^{2n}}$$

$$= \frac{(n+2)}{(n+1)}\sqrt{\frac{n+1}{n}}\,\frac{1}{x^2}$$

$$= \left(\frac{n+2}{n+1}\right)\sqrt{\frac{n+1}{n}}\,\frac{1}{x^2}$$

$$\frac{u_n}{u_{n+1}} = \frac{(1+2/n)}{(1+1/n)}\sqrt{1+\frac{1}{n}}\,\frac{1}{x^2}$$

$$\therefore \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{x^2}$$

From the ratio test, we conclude that the given series Σu_n is convergent or divergent accordingly as $\frac{1}{x^2} > 1$ or < 1 i.e. for

(i) $x^2 < 1$ series is convergent.

(ii) $x^2 > 1$ series is divergent.

If $x^2 = 1$, then this test fails and the given series reduces to Σu_n whose n^{th} term $u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n^{3/2}(1+1/n)}$

Let $v_n = \frac{1}{n^{3/2}}$; then by Comparison test

$$\lim_{n\to\infty} \frac{U_n}{v_n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1 \text{ (finite quantity)}$$

Hence $\sum U_n$ and $\sum v_n$ Converge or diverge together but $\sum v_n = \sum \frac{1}{n^{3/2}} = \sum \frac{1}{n^p}$ so $p = \frac{3}{2} > 1$

Hence, the series is Convergent.

Thus the given series is Convergent if $x^2 \leq 1$ and divergent if $x^2 > 1$.

Example: Show that the series

$$1+\frac{x}{1!}(\log a)+\frac{x^2}{2!}(\log a)^2+\frac{x^3}{3!}(\log a)^3$$

$+\ldots\infty$ is Convergent.

Solution: If ΣU_n be the given series then we have the n^{th} term $U_n=\frac{x^{n-1}}{(n-1)!}(\log a)^{n-1}$

and so $U_{n+1}=\frac{x^n}{n!}(\log a)^n$

$$\therefore \frac{U_n}{U_{n+1}}=\frac{n}{x(\log a)}$$

$$\therefore \lim_{n\to\infty}\frac{U_n}{U_{n+1}}=\lim_{n\to\infty}\left[\frac{n}{x(\log a)}\right]=\infty>1 \text{ for all } x.$$

Hence by ratio test, the given series is Convergent.

: Raabe's Test

Let $\sum U_n$ be an infinite series of positive terms and let

$$\lim_{n\to\infty}\left[n\left(\frac{U_n}{U_{n+1}}-1\right)\right]=\lambda$$

Then the series is

(i) Convergent if $\lambda>1$ and

(ii) Divergent if $\lambda<1$

(iii) $\lambda=1$, this test fails.

Example: Test the convergence of the series

$$1+\frac{3}{7}x+\frac{3\cdot 6}{7\cdot 10}x^2+\frac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13}x^3+\cdots$$

Solution: The n^{th} term of the given series (neglected first term) is

$$u_n=\frac{3\cdot 6\cdot 9\cdot 12\cdots\cdots(3n)}{7\cdot 10\cdot 13\cdot 16\cdots\cdots(3n+4)}x^n$$

$$u_{n+1}=\frac{3\cdot 6\cdot 9\cdot 12\cdots(3n)(3n+3)}{7\cdot 10\cdot 13\cdot 16\cdots\cdots(3n+4)(3n+7)}x^{n+1}$$

Now $\dfrac{u_n}{u_{n+1}}=\left(\dfrac{3n+7}{3n+4}\right)\cdot\dfrac{1}{x}$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\left[\frac{3+7/n}{3+3/n}\right]\cdot\frac{1}{x}=\frac{1}{x}$$

By ratio test, the given series is convergent if $\frac{1}{x} > 1$ i.e. $x < 1$ and divergent if $\frac{1}{x} < 1$ or $x > 1$.

If $x = 1$, this test fails. Then

$$\frac{u_n}{u_{n+1}} = \left(\frac{3+7/n}{3+3/n}\right)$$

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{3+7/n}{3+3/n} - 1\right]$$

$$= n\left[\frac{4/n}{3+3/n}\right]$$

$$= n\left[\frac{4}{3n\left(1+\frac{1}{n}\right)}\right]$$

$$\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty}\left[\frac{4/3}{1+1/n}\right]$$

$$= \frac{4}{3} > 1$$

Thus, the given series is convergent if $x \le 1$ and divergent if $x > 1$.

Example: Test the convergence of the series

$$1 + a + \frac{a(a+1)}{1\cdot 2} + \frac{a(a+1)(a+2)}{1\cdot 2\cdot 3} + \cdots$$

Solution: Leaving the first term, the nth term of the series is

$$u_n = \frac{a(a+1)(a+2)\cdots(a+n-1)}{1\cdot 2\cdot 3\cdots n}$$

$$u_{n+1} = \frac{a(a+1)(a+2)\cdots(a+n-1)(a+n)}{1\cdot 2\cdot 3\cdots n(n+1)}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \left(\frac{n+1}{a+n}\right)$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\left(\frac{n+1}{n+a}\right) = \lim_{n\to\infty}\left(\frac{1+1/n}{1+a/n}\right)$$

$$= 1$$

Hence, the ratio test fails.

Now, $$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{n+1}{a+n} - 1\right)$$

$$= n\left(\frac{1-a}{a+n}\right)$$

$$= \frac{1-a}{1+a/n}$$

$$\therefore \lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}} - 1\right)\right] = \lim_{n\to\infty}\left[\frac{1-a}{1+a/n}\right] = (1-a)$$

From Raabe's test if $(1-a)>1$ or $a<0$, the series is convergent. if $(1-a)<1$ or $a>0$, the series is divergent and if $1-a=1$ or $a=0$, this test fails and the given series becomes $1+0+0+0+\dots$ the sum of whose first n terms is always equal to 1. Hence when $a=0$, the series is convergent.

Thus, the given series is convergent if $a\le 0$ and the series is divergent if $a>0$.

Cauchy's Root Test

Let ΣU_n be an infinite series of positive terms and let $\lim_{n \to \infty} [U_n]^{1/n} = \lambda$,

Then the series is

(i) Convergent if $\lambda < 1$, and

(ii) Divergent if $\lambda > 1$

(iii) If $\lambda = 1$, this test fails.

Example: Test the Convergence of the series

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\cdots$$

Solution: The n^{th} term of the given series is

$$u_n=\left[\frac{(n+1)^{n+1}}{n^{n+1}}-\frac{n+1}{n}\right]^{-n}$$

$$\therefore \lim_{n\to\infty}[u_n]^{1/n}=\lim_{n\to\infty}\left[\left\{\frac{(n+1)^{(n+1)}}{n^{n+1}}-\frac{n+1}{n}\right\}^{-n}\right]^{\frac{1}{n}}$$

$$=\lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)^{n+1}-\left(\frac{n+1}{n}\right)\right]^{-1}$$

(15)

$$= \lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)\times\left\{\left(\frac{n+1}{n}\right)^n-1\right\}\right]^{-1}$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left\{\left(1+\frac{1}{n}\right)^n-1\right\}\right]^{-1}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-1}\left[\left(1+\frac{1}{n}\right)^n-1\right]^{-1}$$

$$= 1\cdot(e-1)^{-1}$$

$$= \frac{1}{e-1} < 1 \qquad \left[\because 2<e<2 \text{ and we take } e=2.718\right]$$

Hence, by Cauchy's root test, the given series is convergent.

Example: Test the convergence of the series

$$\sum \frac{n^{n^2}}{(n+1)^{n^2}}$$

Solution: Here, $u_n = \dfrac{n^{n^2}}{(n+1)^{n^2}}$

$$\therefore \lim_{n\to\infty} [u_n]^{1/n} = \lim_{n\to\infty}\left[\frac{n^{n^2}}{(n+1)^{n^2}}\right]^{1/n}$$

$$= \lim_{n\to\infty}\left[\frac{n^n}{(n+1)^n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{\left(1+\frac{1}{n}\right)^n}\right] = \frac{1}{e} < 1.$$

Hence, by Cauchy's test, the given series is convergent.

Example: Test the convergence of the series

$$\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \cdots$$

Solution: Neglecting the first term, we obtain the n^{th} term of the given series is

$$U_n = \left(\frac{n+1}{n+2}\right)^n x^n$$

$$\therefore \lim_{n\to\infty} [U_n]^{1/n} = \lim_{n\to\infty} \left[\left(\frac{n+1}{n+2}\right)^n x^n\right]^{1/n}$$

$$= \lim_{n\to\infty} \left[\left(\frac{n+1}{n+2}\right) \cdot x\right]$$

$$= \lim_{n\to\infty} \left[\frac{(1+1/n)}{(1+2/n)} \cdot x\right] = x$$

$$= (1+2/n) \quad]$$

From Cauchy's root test, the given series is convergent if $x<1$, and divergent if $x>1$, and if $x=1$, this test fails.

Now, putting $x=1$ in the given series, we ge[t]

$$U_n = \left(\frac{n+1}{n+2}\right)^n$$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n$$

$$= \lim_{n\to\infty} \left(\frac{1+\frac{1}{n}}{1+2/n}\right)^n$$

$$= \lim_{n\to\infty} \frac{(1+1/n)^n}{(1+2/n)^n} = \frac{e}{e^2} = \frac{1}{e} \neq 0$$

Hence, the series is divergent.

Thus, the given series is convergent if $x<1$ and divergent if $x \geq 1$.

Logarithmic Test

If $\sum u_n$ be a series of positive terms such that

$$\lim_{n \to \infty} \left[n \log \left(\frac{u_n}{u_{n+1}} \right) \right] = \lambda$$

then

(i) if $\lambda > 1$, $\sum u_n$ is Convergent.

(ii) if $\lambda < 1$, $\sum u_n$ is divergent.

(iii) if $\lambda = 1$, further investigation is required.

Example: Test for the Convergence of the series

$$x + \frac{2^2 . x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 . x^4}{4!} + \cdots$$

Solution: Here $U_n = \frac{n^n x^n}{n!}$

so $U_{n+1} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}$

$$\Rightarrow \frac{U_n}{U_{n+1}} = \frac{n^n}{(n+1)^{n+1}} \cdot \frac{1}{x}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{x}$$

$$\lim_{n \to \infty} \frac{U_n}{U_{n+1}} = \lim_{n \to \infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{x} = \frac{1}{e} \cdot \frac{1}{x}$$

Therefore the given series Convergent if $\frac{1}{ex} > 1$ or $x < \frac{1}{e}$ (by D Alembert Ratio test) and series is divergent if $\frac{1}{ex} < 1$ or $x > \frac{1}{e}$.

when $x = \frac{1}{e}$ then further investigation is required. Now

$$\log\left(\frac{u_n}{u_{n+1}}\right) = \log\left(\frac{e}{\left(1+\frac{1}{n}\right)^n}\right)$$

$$= \log e - \log\left(1+\frac{1}{n}\right)^n$$

$$= 1 - n\log\left(1+\frac{1}{n}\right)$$

$$= 1 - n\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right]$$

$$= \frac{1}{2n} - \frac{1}{3n^2} + \cdots$$

$$\lim_{n\to\infty} n\log\left(\frac{u_n}{u_{n+1}}\right) = \lim_{n\to\infty}\left[\frac{1}{2} - \frac{1}{3n} + \cdots\right]$$

$$= \frac{1}{2} < 1$$

Hence the given series in divergent at $x = \frac{1}{e}$.

Gauss Test

If $\sum u_n$ is a series of positive terms such that

$$\frac{u_n}{u_{n+1}} = \alpha + \frac{\beta}{n} + \text{terms of higher of } n.$$

$\alpha > 0$.

(i) if $\alpha > 1$, then the series $\sum u_n$ converges and if $\alpha < 1$ then the series $\sum u_n$ diverges, whatever β may be

(ii) if $\alpha = 1$ then series $\sum u_n$ converges if $\beta > 1$ and diverges if $\beta \le 1$.

Example: Test the convergence of the series

$$1+\frac{2^2}{3^2}+\frac{2^2\cdot 4^2}{3^2\cdot 5^2}+\frac{2^2\cdot 4^2\cdot 6^2}{3^2\cdot 5^2\cdot 7^2}+\cdots$$

Solution: Here the general term of the given series is (neglecting first term)

$$u_n=\frac{2^2\cdot 4^2\cdot 6^2\cdots(2n-2)^2}{3^2\cdot 5^2\cdot 7^2\cdots(2n-1)^2}$$

$$u_{n+1}=\frac{2^2\, 4^2\cdot 6^2\cdots(2n-2)^2(2n)^2}{3^2\cdot 5^2\cdot 7^2\cdots(2n-1)^2(2n+1)^2}$$

$$\frac{u_n}{u_{n+1}}=\frac{(2n+1)^2}{4n^2}=\left(1+\frac{1}{2n}\right)^2$$

$$=1+\frac{1}{n}+\frac{1}{4n^2}$$

Now applying Gauss test; here $\alpha=1$, $\beta=1$

so the given series is divergent.

Example! Test for convergence of the series

$$1+\frac{a\cdot b}{1\cdot c}x+\frac{a(a+1)b(b+1)}{1\cdot 2\cdot c(c+1)}x^2$$

$$+\frac{a(a+1)(a+2)b(b+1)(b+2)}{1\cdot 2\cdot 3\cdot c(c+1)(c+2)}x^3+\cdots;\ a,b,c \text{ all being positive.}$$

Solution! Given series is an infinite series. The nature of series will not be changed if we neglect first term, then the general term of the series can be expressed as

$$x_n = \frac{a(a+1)\cdots(a+n-1)\, b(b+1)\cdots(b+n-1)}{c(c+1)\cdots(c+n-1)} \frac{x^n}{n!}$$

$$x_{n+1} = \frac{a(a+1)\cdots(a+n)\, b(b+1)\cdots(b+n)}{c(c+1)\cdots(c+n)} \frac{x^{n+1}}{(n+1)!}$$

$$\frac{x_n}{x_{n+1}} = \frac{(n+1)(c+n)}{(a+n)(b+n)} \cdot \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{c}{n}\right)}{\left(1+\frac{a}{n}\right)\left(1+\frac{b}{n}\right)} \frac{1}{x}$$

$$= \frac{1}{x}$$

Then according the Alembert Ratio test the given series in convergent if $\frac{1}{x} > 1$ or $x < 1$ and divergent if $x > 1$.

Test of convergence at $x = 1$

$$\frac{x_n}{x_{n+1}} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{c}{n}\right)}{\left(1+\frac{a}{n}\right)\left(1+\frac{b}{n}\right)}$$

$$= \left(1+\frac{1}{n}\right)\left(1+\frac{c}{n}\right)\left(1+\frac{a}{n}\right)^{-1}\left(1+\frac{b}{n}\right)^{-1}$$

$$= \left(1+\frac{1}{n}\right)\left(1+\frac{c}{n}\right)\left(1-\frac{a}{n}+\frac{a^2}{n^2}+\cdots\right)\left(1-\frac{b}{n}+\frac{b^2}{n^2}-\cdots\right)$$

$$= \left(1+\frac{1}{n}+\frac{c}{n}+\frac{c^2}{n^2}\right)\left(1-\frac{a}{n}+\frac{a^2}{n^2}-\cdots\right)\left(1-\frac{b}{n}+\frac{b^2}{n^2}-\cdots\right)$$

$$= 1 + \frac{1-c-a-b}{n} + \text{all the terms containing higher powers of } \frac{1}{n}$$

Applying Gauss test

Here $\alpha = 1$, $\beta = (1+c-a-b)$

so given series is convergent if $(1+c-a-b) > 1$

and divergent if $(1+c-a-b) < 1$ at $x = 1$.

De Morgan's and Bertrand's Test

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A series $\sum u_n$ of positive terms such that

$$\lim_{n \to \infty} \left[\left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) - 1 \right\} \log n \right] = \lambda$$

then (i) $\sum u_n$ is convergent if $\lambda > 1$

(ii) $\sum u_n$ is divergent if $\lambda < 1$.

Example: Test for Convergence of the series

$$1^p + \left(\frac{1}{2}\right)^p + \left(\frac{1\times 3}{2\times 4}\right)^p + \left(\frac{1\times 3\times 5}{2\times 4\times 6}\right)^p + \cdots$$

Solution: The given series is

$$1^p + \left(\frac{1}{2}\right)^p + \left(\frac{1\times 3}{2\times 4}\right)^p + \left(\frac{1\times 3\times 5}{2\times 4\times 6}\right)^p + \cdots$$

Neglecting first term of the series, we get the general term of series as

$$u_n = \left(\frac{1\times 3\times 5\times \cdots \times (2n-3)}{2\times 4\times 6\times \cdots \times (2n-2)}\right)^p$$

$$\therefore u_{n+1} = \left(\frac{1\times 3\times 5\times \cdots \times (2n-3)(2n-1)}{2\times 4\times 6\times \cdots \times (2n-2)(2n)}\right)^p$$

$$\frac{u_n}{u_{n+1}} = \frac{1}{\left(\frac{2n-1}{2n}\right)^p} = \frac{1}{\left(1-\frac{1}{2n}\right)^p} = 1$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$$

$\therefore$ D'Alembert's Test fails.

Now, we apply Raabe's Test

$$n\left(\frac{u_n}{u_{n+1}}-1\right)=n\left[\left(1-\frac{1}{2n}\right)^{-p}-1\right]$$

$$=n\left[1+\frac{p}{2n}+\frac{p(p+1)}{8n^2}+\cdots-1\right]$$

$$=\frac{p}{2}+\frac{p(p+1)}{8n}+\cdots$$

$$\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=\frac{p}{2}$$

If $\frac{p}{2}>1$ i.e. $p>2$, the series is convergent

and divergent if $\frac{p}{2}<1$ i.e. $p<2$.

This test fails if $\frac{p}{2}=1$ i.e. $p=2$.

If $\frac{p}{2} > 1$ i.e. $p > 2$, the series is convergent

and divergent if $\frac{p}{2} < 1$ i.e. $p < 2$.

This test fails if $\frac{p}{2} = 1$ i.e. $p = 2$.

Now let us apply De Morgan's Test. when $p = 2$

$$n\left[\frac{u_n}{u_{n+1}} - 1\right] = 1 + \frac{3}{4n} + \cdots$$

Now $\lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right]\log n$

$$= \lim_{n\to\infty}\left[1 + \frac{3}{4n} + \cdots - 1\right]\log n$$

$$= \lim_{n\to\infty} \frac{3}{4}\left[\frac{\log n}{n} - \cdots\right]$$

$$= 0 < 1 \qquad \left[\because \lim_{n\to\infty}\frac{\log n}{n} = 0\right]$$

$\therefore \sum u_n$ is divergent when $p = 2$.

Cauchy's Integral Test

A positive term series

$$f(1)+f(2)+f(3)+\cdots\cdots+f(n)+\cdots\cdots$$

where $f(n)$ decreases as n increases, converges or diverges according to the integral

$$\int_1^\infty f(x)\,dx$$

is finite or infinite.

Example: Examine the Convergence of

$$\sum_{n=2}^{\infty} \frac{1}{n \log n}$$

Solution: Here $f(x) = \frac{1}{x \log x}$

$$\int_2^{\infty} \frac{1}{x \log x} dx = \lim_{m \to \infty} \left[\log \log x\right]_2^m$$

$$= \lim_{m \to \infty} \left[\log \log m - \log \log 2\right] \to \infty$$

By Cauchy's Integral test, the series is divergent.

Example: Examine the convergence of $\sum_{n=1}^{\infty} n e^{-n^2}$

Solution: Here $f(x) = x e^{-x^2}$

Now $\int_1^\infty x e^{-x^2} dx = \lim_{m \to \infty} \left[\frac{e^{-x^2}}{-2} \right]_1^m$

$= \lim_{m \to \infty} \left[\frac{e^{-m^2}}{-2} + \frac{e^{-1}}{2} \right]$

$= \frac{e^{-1}}{2} = \frac{1}{2e}$, which infinite.

Hence, the given series in convergent.

Power series in x

A series of the form $\sum a_n x^n$ or

$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

is said to be a power series in x, where a_i's are independent of x.

Example: Find the values of x for which the series $x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \dots \infty$ converges.

Solution: $u_n = (-1)^{n-1} \frac{x^n}{n^2}$; $u_{n+1} = \frac{(-1)^n x^{n+1}}{(n+1)^2}$

$$\frac{u_n}{u_{n+1}} = -\frac{(n+1)^2}{n^2 x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = -\lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^2}{x} = -\frac{1}{x}$$

By D'Alembert's Test the given series is convergent for $|x|<1$ and divergent if $|x|>1$.

At $x=+1$, the series becomes

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$$

This is an alternately convergent series

At $x=-1$, the given series becomes

$$-1-\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{4^2}-\cdots$$

This is also convergent series; $p=2$.

Hence, the interval of convergence is $-1\le x\le 1$.

Taylor's series

A Taylor's series is a representation of function as an infinite terms that are calculated at a single point.

$$\sum u_n = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

where $f^n(a)$ denotes the n^{th} order derivative of f evaluated at a. Taylor's series are not convergent series in general. Some Convergent Taylor's series are

(i) Exponential Series

(ii) Logirathmic series

(iii) Trigonometric series.

Exponential Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \infty$$

Convergent for all values of x

Proof: $u_n = \dfrac{x^{n-1}}{(n-1)!}$, $u_{n+1} = \dfrac{x^n}{n!}$

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{n}{x} = \infty$$

Hence by D'Alembert's Test the exponential series is Convergent for all values of x.

[D'Alembert Ratio Test
$\lim_{n \to \infty} \left(\dfrac{u_n}{u_{n+1}} \right) = +\infty$, then $\sum u_n$ is Convergent].

Logarithmic Series

Series of the form

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots$$

is convergent for $-1 < x \leq 1$.

Proof: $u_n = (-1)^{n-1}\frac{x^n}{n}$, $u_{n+1} = (-1)^n \frac{x^{n+1}}{n+1}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left(\frac{-n+1}{n}\right)\frac{1}{x} = -\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)\frac{1}{x}$$

$$= -\frac{1}{x}$$

Thus, the series is convergent if $|x|<1$ and divergent $|x|>1$.

At $x=1$, the series becomes

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ which is convergent.

At $x=-1$, the series becomes

$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \cdots \infty$ which is divergent.

Trigonometric functions

The power series of circular functions are given by

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots \quad \forall x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots \quad \forall x$$

These series are absolutely convergent series.

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Proof: $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$

$$u_n = (-1)^n \frac{x^{2n}}{(2n)!}$$

Since all terms in the series are of alternative sign so it is an alternating series.

$$u_{n+1} = (-1)^{n+1} \frac{x^{2n+2}}{(2n+2)!}$$

$$u_n > u_{n+1} \quad \because \frac{x^{2n}}{2n!} > \frac{x^{2n+2}}{(2n+2)!}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 0 \quad \forall x$$

Given alternating series Convergent. [by Leibnitz Test]

$$|U_n| = \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$\lim_{n \to \infty} |U_n| = 0$$

$\Rightarrow \cos x$ is absolutely Convergent.

Similarly it can be shown that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

is absolutely Convergent series.

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Thank you!