JAIPUR ENGINEERING COLLEGE AND RESEARCH CENTRE

Year & Sem. – B. Tech I year, Sem.-I

Subject –Engineering Mathematics

Unit – 1

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Designation - Associate Professor

Department - Mathematics

VISION OF INSTITUTE

To become a renowned centre of outcome based learning, and work towards academic, professional, cultural and social enrichment of the lives of individuals and communities.

MISSION OF INSTITUTE

❖Focus on evaluation of learning outcomes and motivate students to inculcate research aptitude by project based learning.

❖Identify, based on informed perception of Indian, regional and global needs, the areas of focus and provide platform to gain knowledge and solutions.

❖Offer opportunities for interaction between academia and industry.

❖Develop human potential to its fullest extent so that intellectually capable and imaginatively gifted leaders may emerge in a range of professions.

Engineering Mathematics: Course Outcomes

Students will be able to:

CO1. **Understand fundamental concepts of improper integrals, beta and gamma functions and their properties. Evaluation of Multiple Integrals in finding the areas, volume enclosed by several curves after its tracing and its application in proving certain theorems.**

CO2. Interpret the concept of a series as the sum of a sequence, and use the sequence of partial sums to determine convergence of a series. Understand derivatives of power, trigonometric, exponential, hyperbolic, logarithmic series.

CO3. Recognize odd, even and periodic function and express them in Fourier series using Euler's formulae.

CO4. Understand the concept of limits, continuity and differentiability of functions of several variables. Analytical definition of partial derivative. Maxima and minima of functions of several variables Define gradient, divergence and curl of scalar and vector functions.

Unit: 1 Beta Function

Definition: Definite Integral

$$\boldsymbol{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad m > 0, n > 0}$$

It is also called the Eulerian function of the first kind.

Symmetry of Beta function

We have

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \int_0^1 x^{m-1}(1-(1-x))^{n-1}\,dx \quad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

$$\beta(n,m)$$

Hence $\boldsymbol{\beta(m,n) = \beta(n,m)}$

Transformation of Beta function

(a) $\beta(m,n) = 2\int_0^{\pi/2} sin^{2m-1}\theta \; cos^{2n-1}\theta \; d\theta$

Proof: We have

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad Putting\ x = sin^2\theta \ \therefore dx = 2sin\theta cos\theta d\theta$$

$$= \int_0^{\pi/2} sin^{2m-2}\theta(1-sin^2\theta)^{n-1}.\,2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} sin^{2m-1}cos^{2n-1}\,d\theta$$

(b) $\beta(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

Proof: We have

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad Putting\ x = \frac{y}{1+y}\,, 1-x = \frac{1}{1+y} \therefore dx$$
$$= \frac{1}{(1+y)^2}dy$$

We get

$$\beta(m,n) = \int_0^\infty \left(\frac{y}{1+y}\right)^{m-1} \left(\frac{1}{1+y}\right)^{n-1} \frac{1}{(1+y)^2}\,dy$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$$

Therefore we can say $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

Gamma function: Definition

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \qquad x > 0$$

It is also called second Eulerian Integral.

Properties of Gamma function:

(a) $\Gamma(x+1) = x\Gamma(x)$

Proof: $\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt$

Integrating by parts, we get

$$\Gamma(x+1) = \left[-t^x e^{-t}\right]_0^\infty + x\int_0^\infty t^{x-1} e^{-t} dt$$

The first part is zero using L'Hospital's rule. Hence we get:

$$\Gamma(x+1) = x\int_0^\infty t^{x-1}e^{-t}dt = x\Gamma(x), \qquad x > 0$$

Clearly $\Gamma(1) = \int_0^\infty e^{-t}dt = 1$

Hence $\Gamma(x+1) = x\Gamma(x) = x(x-1)\Gamma(x-1)$

$$= x(x-1)(x-2)\ldots 2.1\Gamma(1) = x!$$

So $\Gamma(x+1) = x!\ , \quad x > 0$

(c) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof: $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2}e^{-t}dt, \qquad t > 0$ **Putting t = x^2, we get**

$$= 2\int_0^\infty e^{-x^2}dx$$

Also we can write

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2}dx$$

$$\therefore \Gamma\left(\frac{1}{2}\right).\Gamma\left(\frac{1}{2}\right) = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy$$

Putting $x^2+y^2 = r^2$ and dxdy = rdθdr, we get:

$$\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = 4\int_0^{\Pi/2}\int_0^{\infty} e^{-r^2} r\,d\theta\,dr$$

$$= 2\int_0^{\pi/2}\left[-e^{-r^2}\right]_0^{\infty} d\theta = 2\int_0^{\pi/2} d\theta = \pi$$

Hence $\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = \pi \quad \therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

RELATION BETWEEN BETA AND GAMMA FUNCTIONS

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

Proof We know that

$$\Gamma(n) = k^n \int_o^\infty e^{-kx}\, x^{n-1} dx$$

Replacing k by z in (36), we have

$$\Gamma(n) = z^n \int_o^\infty e^{-zx}\, x^{n-1} dx$$

Multiplying both sides by $e^{-z}\, z^{m-1}$, we get

$$\Gamma n\, e^{-z}\, z^{m-1} = \int_o^\infty e^{-zx}\, x^{n-1}\, e^{-z}\, z^{m-1}\, dx$$

$$= \int_0^\infty z^{n+m-1}\, e^{-z(1+x)} x^{n-1} dx$$

Integrating both sides w.r.t. z from 0 to ∞,

$$\Gamma(n) \int_0^\infty e^{-z}\, z^{m-1} d_z = \int_0^\infty \left[\int_0^\infty z^{n+m-1} e^{-z(1+x)} dx\right] dz$$

$$\Gamma(n)\Gamma(m) = \int_0^\infty x^{n-1} \frac{\Gamma(m+n)}{(1+x)^{m+n}} dx$$

$$\Gamma(n)\Gamma(m) = \Gamma(m+n) \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$\Gamma(n)\Gamma(m) = \Gamma(m+n)\beta(m,n) \qquad \left[\because \beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx\right]$$

$\therefore$ $$\boldsymbol{\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$$

Example 1. Show that

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\sqrt{2}}{8\sqrt{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$$

Solution: Let $I = \int_0^1 \frac{dx}{\sqrt{1-x^4}}$

Putting $x^4 = sin^2\theta \ \ i.e. \ \ x = sin^{\frac{1}{2}}\theta, so\ that\ dx = \frac{1}{2}\ sin^{-\frac{1}{2}}\theta \cos\theta \ \ d\theta, when\ x = 0, \theta = 0\ and$ $when\ x = 1, \theta = \frac{\pi}{2}$

$$I = \int_0^{\pi/2} \frac{\frac{1}{2} sin^{-1/2}\theta cos\theta d\theta}{cos\theta} = \frac{1}{2}\int_0^{\pi/2} sin^{-1/2}\theta\ cos^0\theta\ d\theta$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{3}{4}\right)} = \frac{1}{4}\frac{\left(\Gamma\frac{1}{4}\right)^2\sqrt{\pi}}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{\pi}}{4} \cdot \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)}$$

$$= \frac{\sqrt{\pi}}{4} \cdot \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{\pi/\left(\sin\frac{1}{4}\pi\right)} = \frac{\sqrt{\pi}}{4}\frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{\pi/\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{8\sqrt{\pi}} \cdot \left[\Gamma\frac{1}{4}\right]^2$$

Example : Prove that

$$\int_0^{\pi/2} \sqrt{\tan\theta}\ d\theta = \int_0^{\pi/2} \sqrt{\cot\theta}\ d\theta = \frac{\pi}{\sqrt{2}}$$

Solution: Let $\int_0^{\pi/2} \sqrt{\tan\theta}\ d\theta = \int_0^{\pi/2} \sqrt{\tan\left(\frac{\pi}{2} - \theta\right)}\ d\theta = \int_0^{\pi/2} \sqrt{\cot\theta}\ d\theta$ (By property of definite integral)

$$= \int_0^{\pi/2} cos^{1/2}\theta\ sin^{-1/2}\theta\ d\theta = \frac{\Gamma\left[\dfrac{\frac{-1}{2}+1}{2}\right]\Gamma\left[\dfrac{\frac{1}{2}+1}{2}\right]}{2\Gamma\left[\dfrac{\frac{-1}{2}+\frac{1}{2}+2}{2}\right]} = \frac{\Gamma\left[\frac{1}{4}\right]\Gamma\left[\frac{3}{4}\right]}{2\Gamma[1]}$$

$$\frac{1}{2}\Gamma\left[\frac{1}{4}\right]\Gamma\left[1-\frac{1}{4}\right] = \frac{1}{2}\frac{\pi}{sin\frac{\pi}{4}} = \frac{\pi}{\sqrt{2}}$$

Suggested links from NPTEL & other Platforms:

- Advanced Engineering Mathematics: Erwin Kreyszig, Wiley plus publication
- Engineering Mathematics : CB Gupta, SR Singh, Mukesh Kumar, Mc Graw Hill
- https://nptel.ac.in/courses/111/105/111105122/

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Thank you!