

Advanced Engineering Mathematics

Optimization Techniques III Sem CS (3CS2-01)

Unit V

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UNIT-II

LINEAR PROGRAMMING PROBLEM

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INTRODUCTION AND DEFINATION

- Mathematical programming is used to find the **best or optimal solution** to a problem that requires a decision or set of decisions about how best to use a set of limited resources to achieve a state goal of objectives.
- Steps involved in mathematical programming

Conversion of stated problem into a mathematical model that abstracts all the essential elements of the problem.

- Exploration of different solutions of the problem.
- Finding out the most suitable or optimum solution.



- Linear programming requires that all the mathematical functions in the model be **linear functions**.
- The Linear Programming Model (1)

Requirement: Maximization of the linear function Z.

$$Z = c_1X_1 + c_2X_2 + c_3X_3 + \dots\dots\dots + c_nX_n \dots\dots\text{Eq (1)}$$

subject to the following constraints:

$$\begin{array}{r}
 a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1 \\
 a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2 \\
 \vdots \\
 a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_n \quad \dots\dots\text{Eq (2)} \\
 \text{all } x_j \geq 0
 \end{array}$$

Where a_{ij} , b_j and c_j are given constants.

The Linear Programming Model (2)

- The linear programming model can be written in more efficient notation as:

Maximize

$$Z = \sum_{j=1}^n c_j x_j$$

subject to:

$$\sum_{j=1}^n a_{ij} x_j \leq b_i$$

where

$$i = 1, 2, \dots, m$$

and

$$x_j \geq 0$$

where

$$j = 1, 2, \dots, n$$

....Eq (3)

The decision variables X_1, X_2, \dots, X_n represent levels of n competing activities.



Applications of linear programming

1. Scheduling school buses to minimize total distance traveled
2. Allocating police patrol units to high crime areas in order to minimize response time to 911 calls
3. Scheduling tellers at banks so that needs are met during each hour of the day while minimizing the total cost of labor
4. Selecting the product mix in a factory to make best use of machine- and labor-hours available while maximizing the firm's profit
5. Picking blends of raw materials in feed mills to produce finished feed combinations at minimum costs
6. Determining the distribution system that will minimize total shipping cost

Requirements of an LP problem

1. LP problems seek to maximize or minimize some quantity (usually profit or cost) expressed as an objective function
2. The presence of restrictions, or constraints, limits the degree to which we can pursue our objective
3. There must be alternative courses of action to choose from
4. The objective and constraints in linear programming problems must be expressed in terms of linear equations or inequalities



Formulating LP Problems

- The product-mix problem at Shader Electronics
 - ☑ Two products
 1. Shader X-pod, a portable music player
 2. Shader BlueBerry, an internet-connected color telephone
 - ☑ Determine the mix of products that will produce the maximum profit

Formulating LP Problems

Hours Required to Produce 1 Unit

X-pods Department	BlueBerrys (X_1)	(X_2)	Available Hours This Week
Electronic	4	3	240
Assembly	2	1	100
Profit per unit	\$7	\$5	

Decision Variables:

X_1 = number of X-pods to be produced

X_2 = number of Blue Berrys to be produced

Formulating LP Problems

- Objective Function:

$$\text{Maximize Profit} = 7X_1 + 5X_2$$

There are three types of constraints

- ☑ Upper limits where the amount used is \leq the amount of a resource
- ☑ Lower limits where the amount used is \geq the amount of the resource
- ☑ Equalities where the amount used is $=$ the amount of the resource

Formulating LP Problems

- **First Constraint:**

**Electronic
time used** **is \leq** **Electronic
time available**

$$4X_1 + 3X_2 \leq 240 \text{ (hours of electronic time)}$$

Second Constraint:

**Assembly
time used** **is \leq** **Assembly
time available**

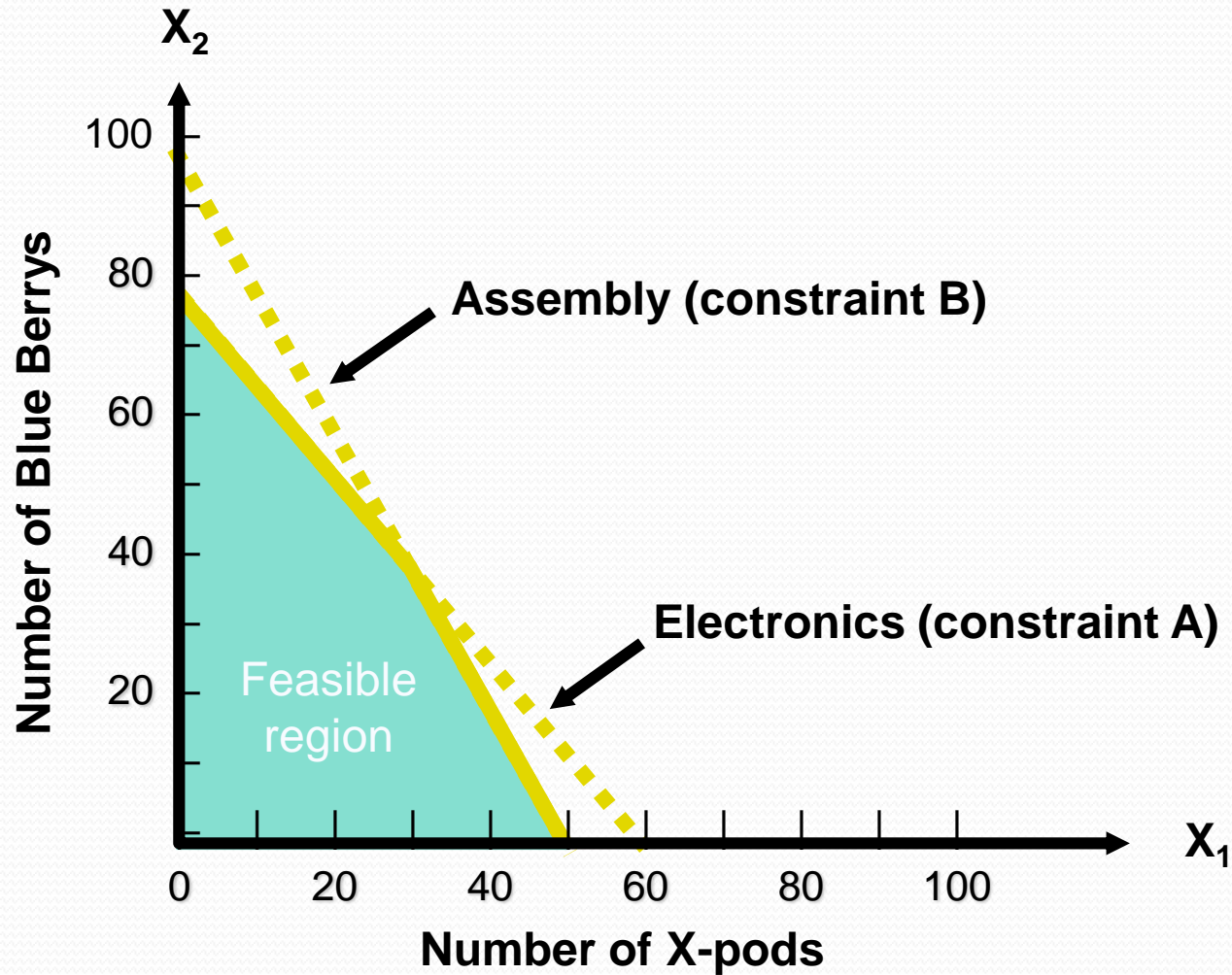
$$2X_1 + 1X_2 \leq 100 \text{ (hours of assembly time)}$$

Graphical method

Can be used when there are two decision variables

1. Plot the constraint equations at their limits by converting each equation to an equality
2. Identify the feasible solution space
3. Create an iso-profit line based on the objective function
4. Move this line outwards until the optimal point is identified

Graphical solution



Graphical Solution

Iso-Profit Line Solution Method

Choose a possible value for the objective function

$$\$210 = 7X_1 + 5X_2$$

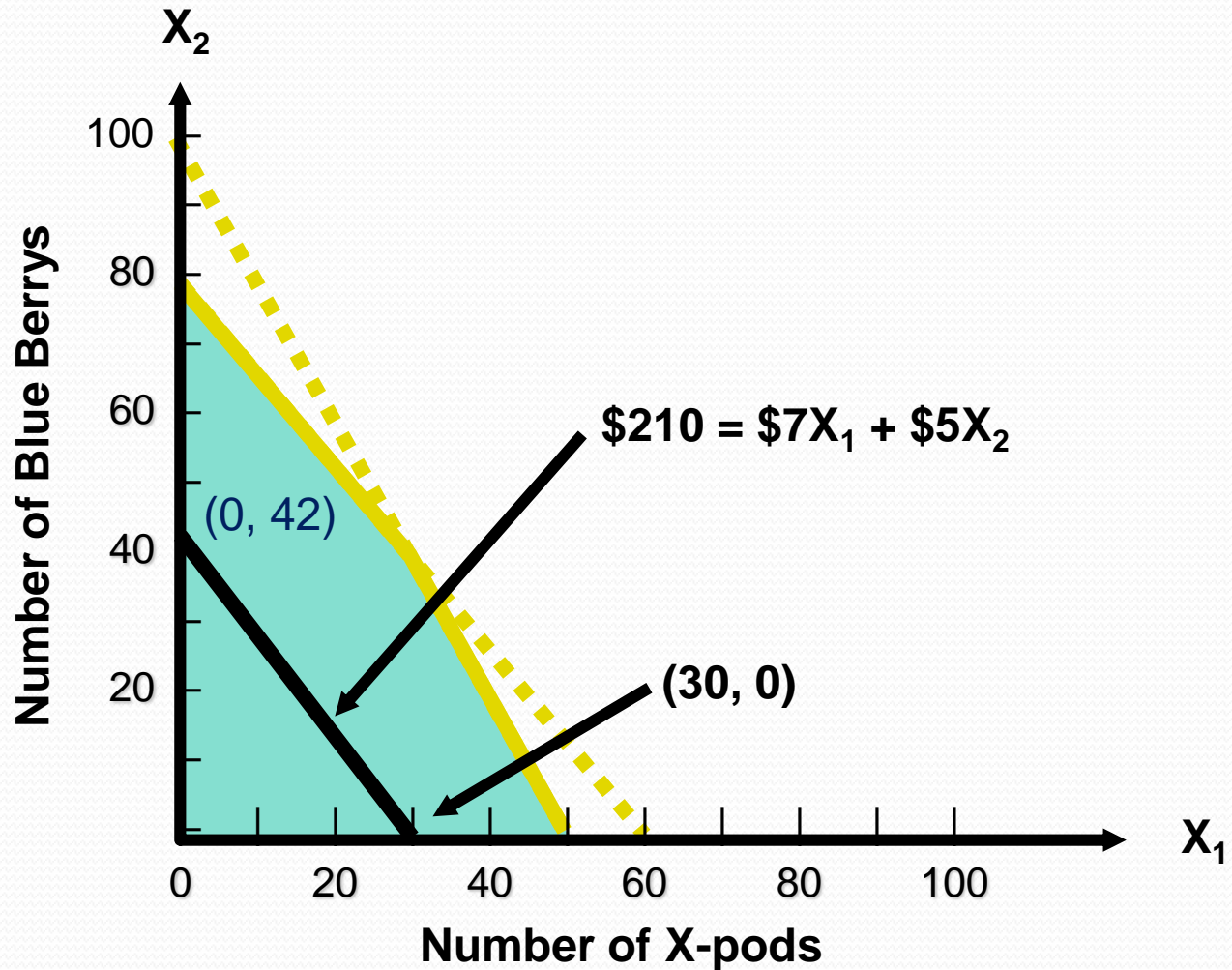
Solve for the axis intercepts of the function and plot the line

$$X_2 = 42 \quad X_1 = 30$$

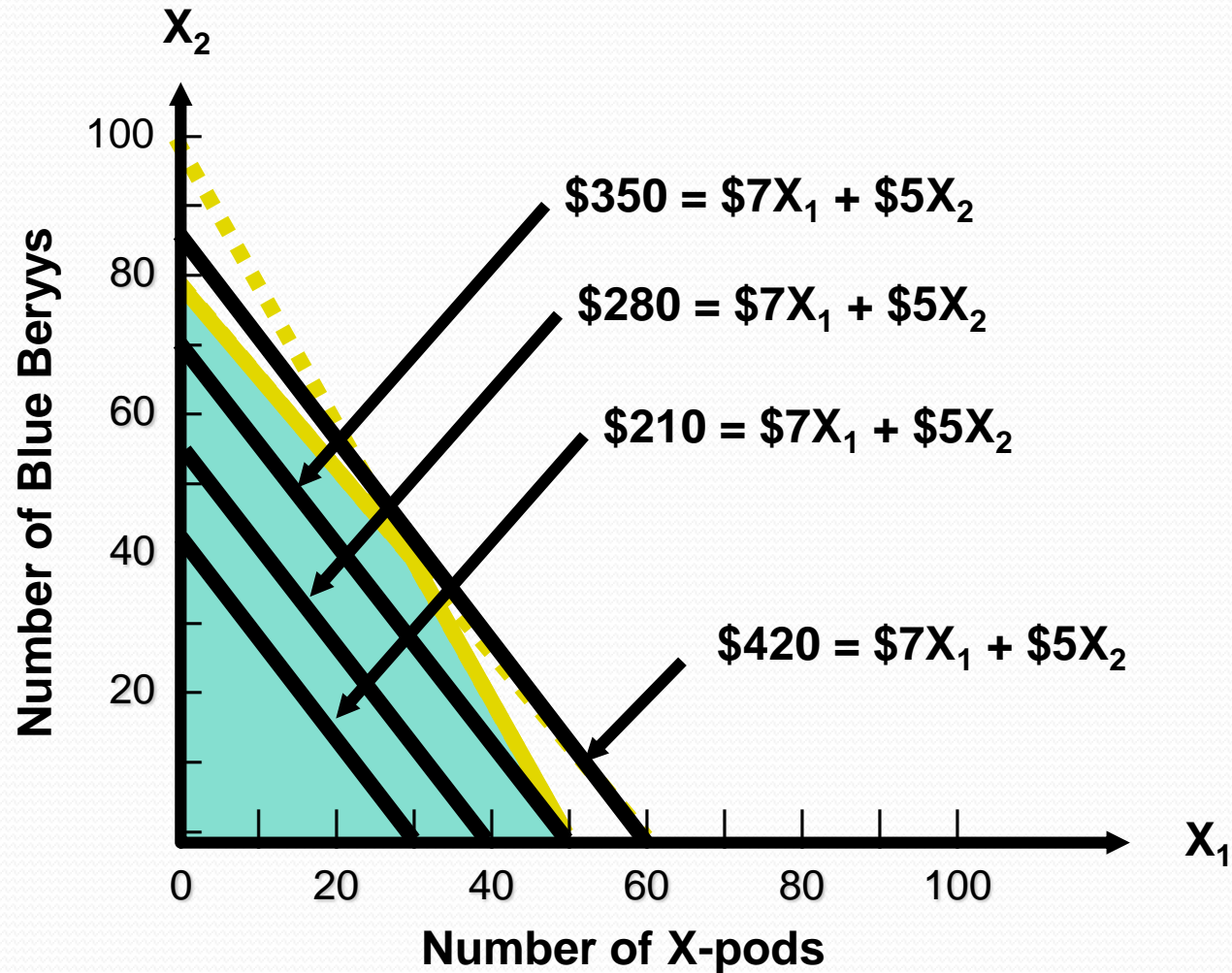
0 20 40 60 80 100

Number of X-pods

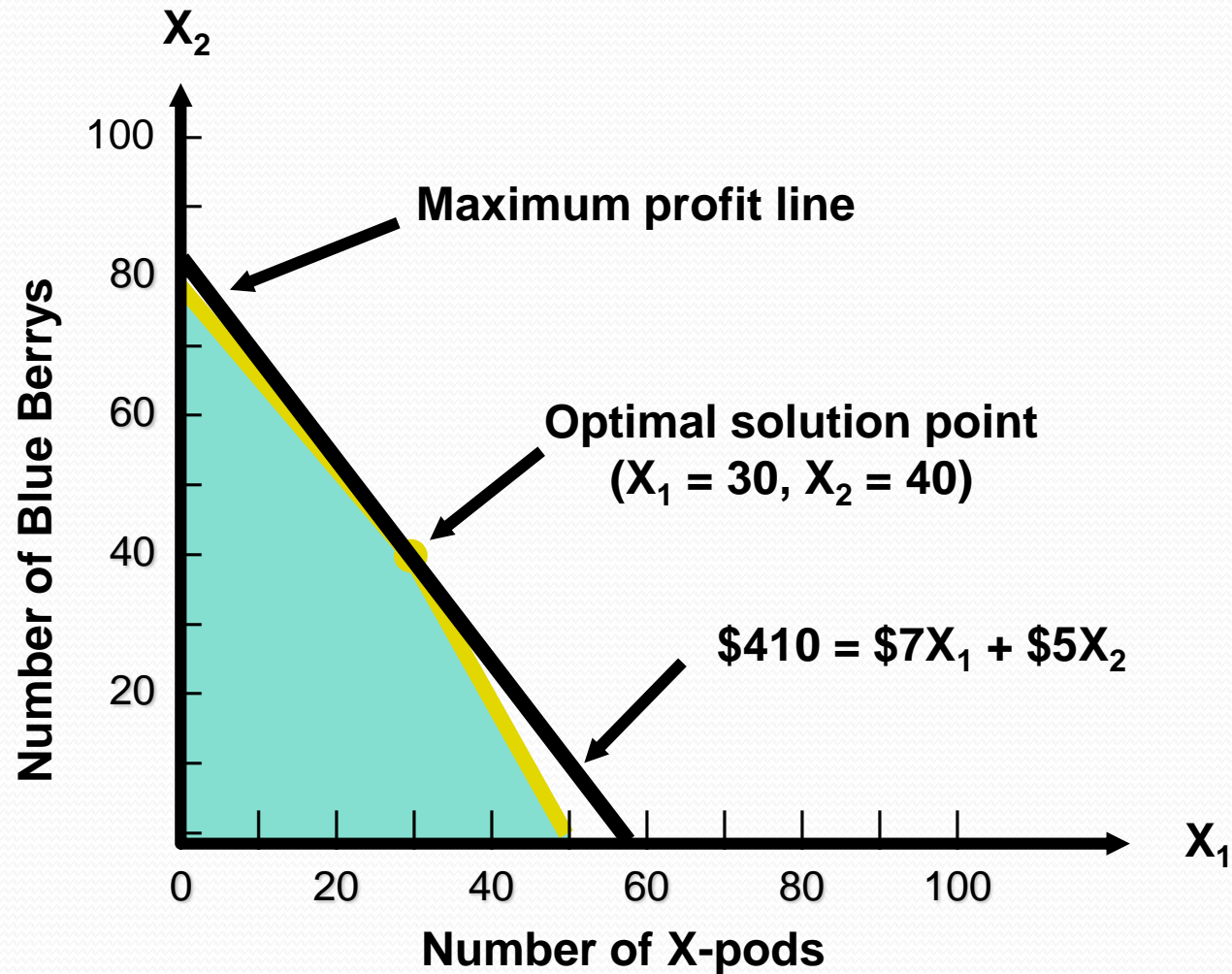
Graphical solution



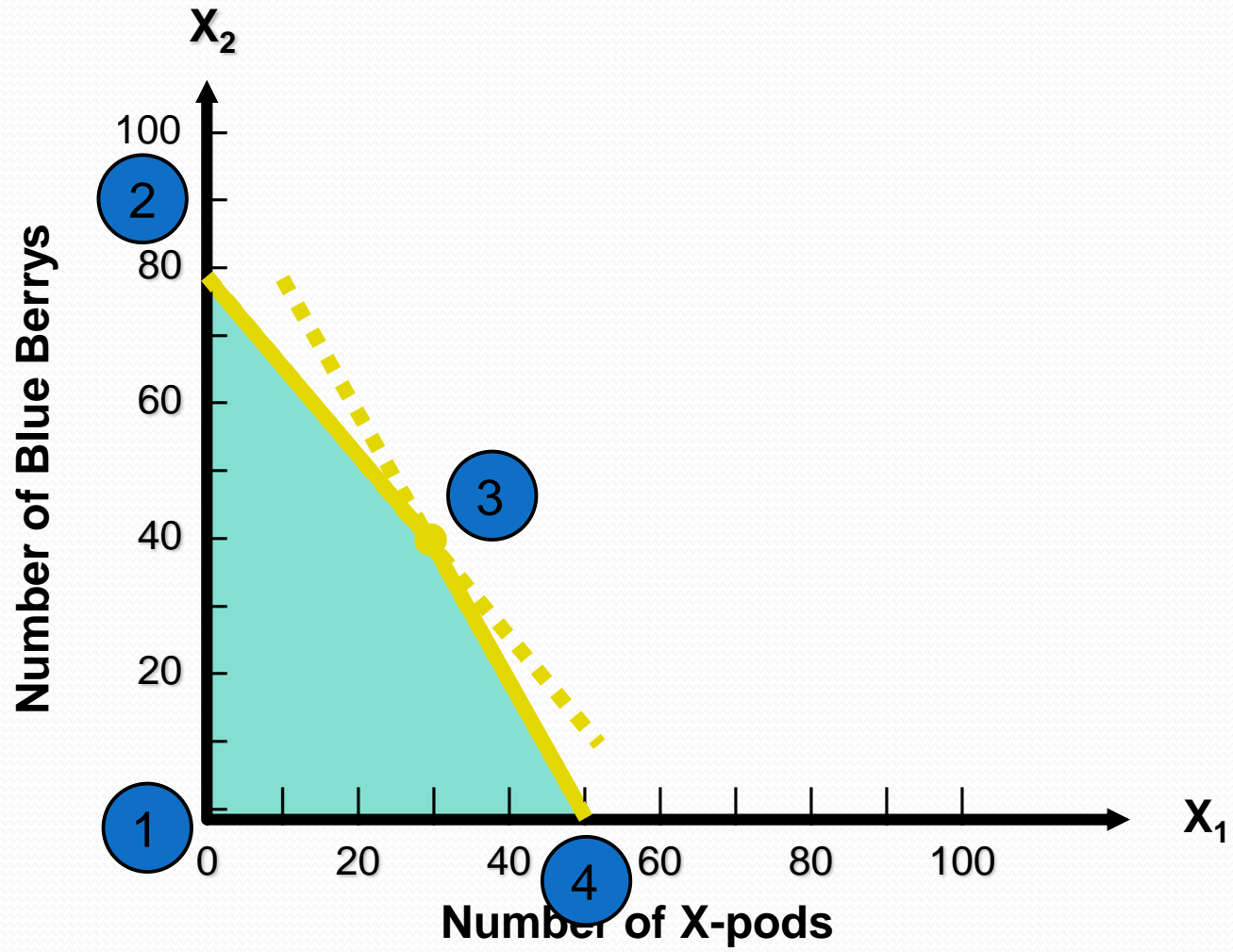
Graphical solution



Graphical solution



Corner point method



Corner-Point Method

- ✓ The optimal value will always be at a corner point
- ✓ Find the objective function value at each corner point and choose the one with the highest profit

Point ① $(X_1 = 0, X_2 = 0)$ Profit $\$7(0) + \$5(0) = \$0$

Point ② $(X_1 = 0, X_2 = 80)$ Profit $\$7(0) + \$5(80) = \$400$

Point ④ $(X_1 = 50, X_2 = 0)$ Profit $\$7(50) + \$5(0) = \$350$

Point ③ $(X_1 = 30, X_2 = 40)$ Profit $\$7(30) + \$5(40) = \$410$

The Simplex Method

- The geometric method of solving linear programming problems presented before. The graphical method is useful only for problems involving two decision variables and relatively few problem constraints.
- We use an algebraic method called the simplex method, which was developed by **George B. DANTZIG** (1914-2005) in 1947 while on assignment with the U.S. Department of the air force.

Standard Maximization Problems in Standard Form

A linear programming problem is said to be a standard maximization problem in standard form if its mathematical model is of the following form:

Maximize the objective function

$$Z_{\max} = P = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

Subject to problem constraints of the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n \leq b \quad , b \geq 0$$

With non-negative constraints

$$x_1, x_2, \dots, x_n \geq 0$$

Slack Variables

If a constraint has a sign \leq , then in order to make it an equality we have to add some positive quantity to the left hand side.

The positive variables which are added to left hand side of the constraints to convert into equations are called the **slack variables**.

Ex . $x_1 + x_2 + x_3 \leq 5$ $x_1, x_2, x_3 \geq 0$

Can be converted to

Ex . $x_1 + x_2 + x_3 + x_4 = 5$ $x_1, x_2, x_3, x_4 \geq 0$

Then x_4 is called the slack variable.

Surplus variables

- If constraint has a sign \geq then in order to make it an equality we have to subtract some positive quantity from left hand side .
- The positive variables which are subtracted from the left hand side of the constraints to convert them into equalities are called the surplus variables.

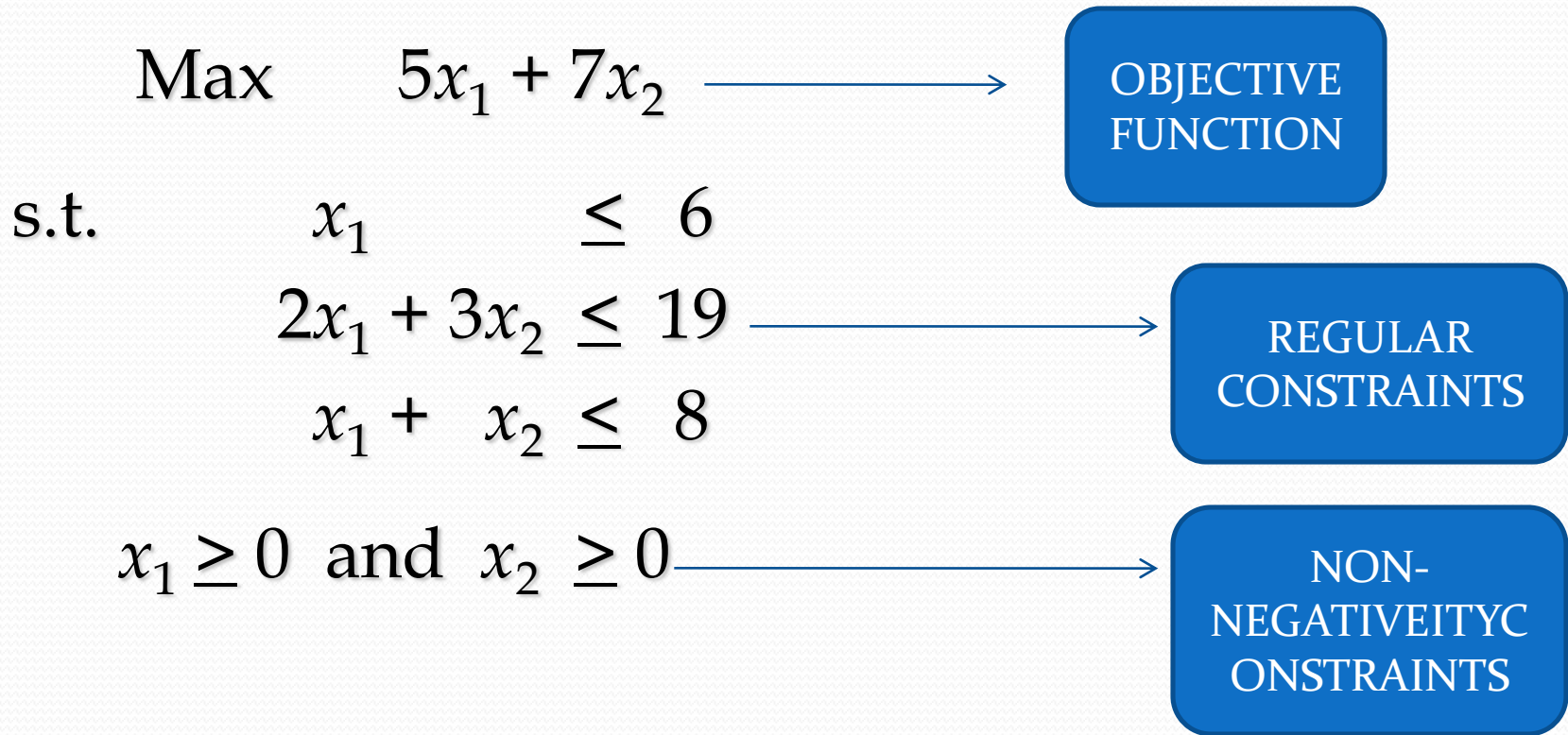
- **Ex.** $2x_1 + 5x_2 \geq 200$

- Can be converted to

$$2x_1 + 5x_2 - x_3 = 200$$

- then x_3 is the surplus variable.

Example 1: A Simple Maximization Problem



Working steps of simplex method

we follow the following steps:

Step-1 if the problem is not of maximization convert it to maximization problem.

As $\text{Min.}Z = - \text{Max.}Z$ assume $Z' = -Z$. Work on the problem with objective function as $\text{Max.} Z'$

Step-2 if any entry of vector $b = (b_1, b_2, \dots, b_n)^T$ is negative, multiply the corresponding constraint by (-1) to make it positive. (all entries of b should be non-negative)

Step-3 add (subtract) slack (surplus) variables, where ever needed to convert inequalities to equalities.

Step-4 if needed add artificial variables to get initial basic matrix $B = I_m$

Step-5 in objective function , slack and surplus variables are added with coefficients zero. The coefficient of artificial variables are decided by the method.

Step-6 calculate $z_j - c_j$ for every (column) vector a_j . If the most negative value of $z_j - c_j$ occur for j equal to k , then , k^{th} vector will be the entering vector in the new basis .

Step-7 calculate mini. Ratio $\left[\frac{(X_B)_i}{Y_{ik}}, Y_{ik} > 0 \right]$. If this value of i is r then β_r is the departing vector .

Step-8 from the new simplex table. Repeat steps 6,7 and 8 until $z_j - c_j \geq 0$ for all j is optimal.

Example of simplex method

- Solve the following L.P.P by simplex method

$$\text{Maximize } \mathbf{Z} = 5x_1 + 2x_2 + x_3$$

subject to the constraints

$$x_1 + 3x_2 - x_3 \leq 6,$$

$$x_2 + x_3 \leq 4,$$

$$3x_1 + x_2 \leq 7,$$

$$x_1, x_2, x_3 \geq 0.$$

- Step-1 convert into standard form

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n \leq b_1,$$

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n + x_{n+1} = b_1,$$

$$a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n \geq b_2,$$

$$a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n - x_{n+2} = b_2,$$



$$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n \leq b_m,$$

$$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n + x_{n+k} = b_m,$$

In our example problem

$$x_1 + 3x_2 - x_3 \leq 6,$$

$$x_1 + 3x_2 - x_3 + x_4 = 6,$$

$$x_2 + x_3 \leq 4,$$

$$x_2 + x_3 + x_5 = 4,$$

$$3x_1 + x_2 \leq 7,$$

$$3x_1 + x_2 + x_6 = 7,$$



$$x_1, x_2, x_3 \geq 0.$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$$

Here x_4, x_5, x_6 are slack variables

Simplex: Step 2

Step 2. Start with an initial basic feasible solution (b.f.s.) and set up the initial table.

In our example

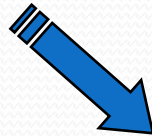
$$\text{Maximize } \mathbf{Z} = 5x_1 + 2x_2 + x_3$$

$$x_1 + 3x_2 - x_3 + x_4 = 6,$$

$$x_2 + x_3 + x_5 = 4,$$

$$3x_1 + x_2 + x_6 = 7,$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$$



c_B	Basis	c_j						Constants
		5	2	1	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	1	3	-1	1	0	0	6
0	x_5	0	1	1	0	1	0	4
0	x_6	3	1	0	0	0	1	7
\bar{c} row		5	2	1	0	0	0	$Z=0$

Step 2: Explanation

Adjacent Basic Feasible Solution

If we bring a non basic variable x_s into the basis, our system changes from the basis, x_b , to the following (same notation as the book):

$$\begin{array}{lcl} x_1 & + \bar{a}_{1s}x_s = & \bar{b}_1 \\ & & \\ & x_r & + \bar{a}_{rs}x_s = \bar{b}_r \\ & & \\ & x_m & + \bar{a}_{ms}x_s = \bar{b}_s \end{array} \qquad \begin{array}{l} x_i = \bar{b}_i - \bar{a}_{is} \quad \text{for } i=1, \dots, m \\ x_s = 1 \\ x_j = 0 \quad \text{for } j=m+1, \dots, n \text{ and } j \neq s \end{array}$$

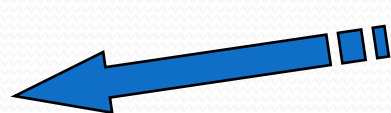
The new value of the objective function becomes:

$$Z = \sum_{i=1}^m c_i (\bar{b}_i - \bar{a}_{is}) + c_s$$

Thus the change in the value of Z per unit increase in x_s is

$$\bar{c}_s = \text{new value of } Z - \text{old value of } Z$$

$$c_s - \sum_{i=1}^m c_i \bar{a}_{is}$$

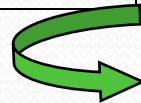


This is the Inner Product rule

Simplex: Step 3

Use the inner product rule to find the relative profit coefficients

c _B	Basis	c _j						Constants
		5	2	1	0	0	0	
		x ₁	x ₂	x ₃	x ₄	x ₅	x ₆	
0	x ₄	1	3	-1	1	0	0	6
0	x ₅	0	1	1	0	1	0	4
0	x ₆	3	1	0	0	0	1	7
\bar{c} row		5	2	1	0	0	0	Z=0



$$\bar{c}_j = c_j - c_B \bar{P}_j$$

$$\underline{c}_1 = 5 - 0(1) - 0(0) - 0(3) = 5 \rightarrow \text{largest positive}$$

$$\underline{c}_2 = \dots$$

$$\underline{c}_3 = \dots$$

Step 4: Is this an optimal basic feasible solution?

Simplex: Step 4

Apply the minimum ratio rule to determine the basic variable to leave the basis.

The new values of the basis variables:

$$x_i = \bar{b}_i - \bar{a}_{is} x_s \quad \text{for } i = 1, \dots, m$$

$$\max x_s = \min_{\bar{a}_{is} > 0} \left[\frac{\bar{b}_i}{\bar{a}_{is}} \right]$$

In our example:

c _B	Basis	c _j						Constants
		5	2	1	0	0	0	
		x ₁	x ₂	x ₃	x ₄	x ₅	x ₆	
0	x ₄	1	3	-1	1	0	0	6
0	x ₅	0	1	1	0	1	0	4
0	x ₆	3	1	0	0	0	1	7
\bar{c} row		5	2	1	0	0	0	Z=0

Row	Basic Variable	Ratio
1	x ₄	6
2	x ₅	-
3	x ₆	7/3

Simplex: Step 5

Perform the pivot operation to get the new tableau and the b.f.s.

c _B	Basis	c _j						Constants
		5	2	1	0	0	0	
		x ₁	x ₂	x ₃	x ₄	x ₅	x ₆	
0	x ₄	1	3	-1	1	0	0	6
0	x ₅	0	1	1	0	1	0	4
0	x ₆	3	1	0	0	0	1	7
\bar{c} row		5	2	1	0	0	0	Z=0

Key element is 3

New iteration:
find entering
variable:

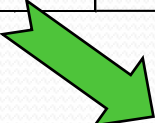
$$\bar{c}_j = c_j - c_B \bar{P}_j$$

$$c_B = (0 \ 0 \ 5)$$

$$c_2 = 2 - (0) 8/3 - (0) 1 - (5) 1/3 = 1/3$$

$$c_3 = 1 - (0) (-1) - (0) 1 - (5) 0 = 1$$

$$c_6 = 0 - (0) 0 - (0) 0 - (5) 1/3 = -5/3$$



c _B	Basis	c _j						Constants
		5	2	1	0	0	0	
		x ₁	x ₂	x ₃	x ₄	x ₅	x ₆	
0	x ₄	0	8/3	-1	1	0	0	11/3
0	x ₅	0	1	1	0	1	0	4
5	x ₁	1	1/3	0	0	0	1/3	7/3
\bar{c} row		0	1/3	1	0	0	-5/3	Z=35/3

Final Table



c_B	Basis	c_j						Constants
		5	2	1	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	0	8/3	-1	1	0	0	11/3
0	x_5	0	1	1	0	1	0	4
5	x_1	1	1/3	0	0	0	1/3	7/3
\bar{c} row		0	1/3	1	0	0	-5/3	$Z=35/3$

x_3 enters basis,
 x_5 leaves basis



c_B	Basis	c_j						Constants
		5	2	1	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	0	11/3	0	1	1	0	23/3
1	x_3	0	1	1	0	1	0	4
5	x_1	1	1/3	0	0	0	1/3	7/3
\bar{c} row		0	-2/3	0	0	-1	-5/3	$Z=47/3$


The optimal solution will be

$$x_1=7/3$$

$$x_2=0$$

$$x_3=4$$

$$\text{Max. } z=47/3$$



Artificial Variable Technique (The Big-M Method and Two phase Method)

- **Artificial variables** :- the simplex algorithm requires a starting basic feasible solution (BFS) . We find a starting BFS by using the slack variables as our basic variables. If an LPP has \geq or equality constraints , a starting BFS may not be readily apparent . To start , we need a canonical form:
- (1) if we have a \leq constraints with a non-negative right hand side , it will contain an obvious basic variable after introducing a slack variable.
- (2)if we have equality constraints , it contains no obvious variable .
- (3) if we have a \geq constraints with a non-negative right hand side , it contains no obvious basic variable even after introducing a surplus variable.

- For example :

$$2x + 3y \leq 5 \Leftrightarrow 2x + 3y + s = 5, (s, \textit{basic})$$

- If $2x + 3y = 5$, this is infeasible if $x=y=0$.

And $2x + 3y \geq 5 \Leftrightarrow 2x + 3y - s = 5, (s \geq 0)$ is infeasible if $x=y=0$.

- An artificial variable is a non-negative quantity added to the left side of exactly one constraint of an LPP in standard form. Enough artificial variables are added to the constraints so that the resulting coefficient matrix contains an $m \times m$ identity sub matrix. although they are initially basic variables when we apply the simplex method, artificial variables must be driven out of the basic so that their values are zero in the final solution. if this is not possible, then the problem is infeasible (i.e. its feasible region has no extreme points)

- For instance , if we have $4x_1 + 3x_2 \geq 12$ then introducing surplus variable we have $4x_1 + 3x_2 - s_1 = 12$ this equation will not contribute to BFS ,therefore , we need to introduce an artificial variable A_1 as $4x_1 + 3x_2 - s_1 + A_1 = 12$.
- There are two standard methods for handling artificial variables within the simplex method:
- The Big-M Method
- The Two-Phase Method
- These methods complete the study of the simplex method and allow us to solve any L.P.P to find its optimal solution or discover whether it is unbounded or infeasible.

Big –M Method

- Big –M Method is also called penalty method as for introducing the artificial variables , we have to pay penalties . Following are the steps for solving a problem having artificial variables using Big-M Method :
- (1) write the given LPP in standard form .
- (2) add an artificial variable A_i to the constraints identified as \geq or $=$ constraints. Also add the sign restriction $A_i > 0$.
- (3) if the LPP is a max problem , add (for each artificial variable) $-MA_i$ to the objective function where M denotes a very large positive number.

- (4) solve the transformed problem by the simplex . Since each artificial variable will be in the starting basis, all artificial variables must be eliminated from Z- row . In choosing the entering variable, remember that M is a very large positive number.
- For example, $4M-2$ is more positive than $3M+900$ and $-6M-5$ is more negative than $-5M-40$.
- (5) in the final simplex table observe that :
- (a) if all artificial variables are equal to zero in the optimal solution , we have found the optimal solution to the original problem.
- (b) if any artificial variables are positive in the optimal solution , the original problem is infeasible .

Example: Minimize $Z = 600X_1 + 500X_2$

subject to constraints,

$$2X_1 + X_2 \geq 80$$

$$X_1 + 2X_2 \geq 60 \text{ and } X_1, X_2 \geq 0$$

Step 1: Convert the LP problem into a system of linear equations.

We do this by rewriting the constraint inequalities as equations by subtracting new “surplus & artificial variables” and assigning them **zero & +M** coefficients respectively in the objective function as shown below.

So the Objective Function would be:

$$Z = 600X_1 + 500X_2 + 0.S_1 + 0.S_2 + MA_1 + MA_2$$

subject to constraints,

$$2X_1 + X_2 - S_1 + A_1 = 80$$

$$X_1 + 2X_2 - S_2 + A_2 = 60$$

$$X_1, X_2, S_1, S_2, A_1, A_2 \geq 0$$

Step 2: Obtain a Basic Solution to the problem.

We do this by putting the decision variables $X_1=X_2=S_1=S_2=0$, so that $A_1=80$ and $A_2=60$.

These are the initial values of *artificial variables*.

Step 3: Form the Initial Tableau as shown.

		C_j	600	500	0	0	M	M	
C_B	Basic Variable (B)	Basic Soln(X_B)	X_1	X_2	S_1	S_2	A_1	A_2	Min.Ratio (X_B /Pivotal Col.)
M	A1	80	2	1	-1	0	1	0	80
M	A2	60	1	2 [*]	0	-1	0	1	60 →
		Z_j	3M	3M	-M	-M	M	M	
		$C_j - Z_j$	600-3M	500-3M [↑]	M	M	0	0	

It is clear from the tableau that X2 will enter and A2 will leave the basis. Hence 2 is the key element in pivotal column. Now , the new row operations are as follows:

$$R2(\text{New}) = R2(\text{Old})/2$$

$$R1(\text{New}) = R1(\text{Old}) - 1 * R2(\text{New})$$

		C _j	600	500	0	0	M	
C _B	Basic Variable (B)	Basic Soln(X _B)	X ₁	X ₂	S ₁	S ₂	A ₁	Min.Ratio (X _B /Pivotal Col.)
M	A ₁	50	3/2 ☆	0	-1	1/2	1	100/3 →
500	X ₂	30	1/2	1	0	- 1/2	0	60
		Z _j	3M/2+250	500	- M	M/2-250	M	
		C _j - Z _j	350-3M/2↑	0	M	250-M/2	0	

It is clear from the tableau that X_1 will enter and A_1 will leave the basis. Hence 2 is the key element in pivotal column.

Now, the new row operations are as follows:

$$R_1(\text{New}) = R_1(\text{Old}) * 2/3$$

$$R_2(\text{New}) = R_2(\text{Old}) - (1/2) * R_1(\text{New})$$

		C_j	600	500	0	0	Min. Ratio (X_B/P ivotal Col.)
C_B	Basic Variable (B)	Basic Soln(X_B)	X_1	X_2	S_1	S_2	
600	X_1	100/3	1	0	-2/3	1/3	
500	X_2	40/3	0	1	1/3	-2/3	
	Z_j		600	500	-700/3	-400/3	
	$C_j - Z_j$		0	0	700/3	400/3	

Since all the values of $(C_j - Z_j)$ are either zero or positive and also both the artificial variables have been removed, an optimum solution has been arrived at with

$$X_1 = 100/3$$

$$X_2 = 40/3$$

$$Z = 80,000/3$$

Two –Phase Method

As the name suggests it contains two phases:

Phases-I In this phase , we change the objective function . All the cost values , except for the artificial variables, are zero . Artificial variables are given the cost values of (-1) each and the objective function is maximized . Constraints are taken as they are given in the equation. The problem is solved by simplex method . Once we get rid of any artificial variable from the basis , we do not even write it in the simplex table. When all the artificial variables are removed from the basis , and $z_j - c_j \geq 0 \forall j$ then the value of the new objective function will be zero. We end the phase-I.

Phase –II Starting table for this phase is the same as the last table of phase-I (i.e. we start with the feasible solution obtained in phase one). But the objective function is taken as the original given objective function in maximization form and we proceed to optimize it.

Example solve the following LPP by Two –Phase Method.

$$\text{Min. } z = x_1 + x_2$$

s.to

$$2x_1 + x_2 \geq 4$$

$$x_1 + 7x_2 \geq 7$$

$$x_1, x_2 \geq 0$$

Solution: we first convert the minimization problem to

maximization $Max.z^* = -Min.z$

s.to

$$2x_1 + x_2 - x_3 + x_5 = 4$$

$$x_1 + 7x_2 - x_4 + x_6 = 7$$

$$x_i \geq 0, i = 1 \text{ to } 6$$

Phase -I The problem of phase -I is

$$Max.z^* = 0.x_1 + 0.x_2 + 0.x_3 + 0.x_4 - x_5 - x_6$$

Here x_5, x_6 are artificial variables.

Simplex table(1)

C_j			0	0	0	0	-1	-1	
C_B	B.V	X_B	Y1	Y2	Y3	Y4	Y5	Y6	Mini ratio
-1	X5	4	2	1	-1	0	1	0	4/1
-1	X6	7	1	7	-1	0	0	1	7/7
	$Z_j - C_j$		-3	-8	1	1	0	0	

Simplex table(2)

	C_j		0	0	0	0	-1	-1	
-1	X_5	3	13/7	0	-1	1/7	1	-1/7	21/13
0	X_2	1	1/7	1	0	-1/7	0	1/7	7/1
	$Z_j - C_j$		-13/7	0	1	-1/7	0	8/7	
			↑				↓		
			Incoming				Out going		

13/7 is the key element

Simplex table(3)

	C_j		0	0	0	0	-1	-1
0	x_1	21/13	10	-7/13	1/13	7/13	-1/13	
0	x_2	10/13	0	1	1/13	-14/91	-1/13	14/91
	$Z_j - C_j$		0	0	0	0	1	1

All $Z_j - C_j \geq 0$ $j = 1$ to 6 . Max $Z_1 = 0$ and no artificial variable in the basis.

Phase –II table(1)

C_B	C_j B.V	X_B	-1 Y1	-1 Y2	0 Y3	0 Y4
-1	X1	21/13	1	0	-7/13	1/13
-1	X2	10/13	0	1	1/13	-14/13
	$Z_j - C_j$		0	0	6/13	1/13

$Z_j - C_j \geq 0 \forall j = 1, 2, 3, 4$. Thus the optimal solution is given by

$$X_1 = 21/13$$

$$X_2 = 10/13$$

$$\text{Min. } Z = 31/13$$

Review question

Q.1 solve the following L.P.P by simplex method.

$$\text{Max. } Z = 4x_1 + 5x_2$$

s.to

$$x_1 + x_2 \leq 3, 3x_1 + 4x_2 \leq 10, \text{ all } x_1, x_2 \geq 0$$

Q.2 solve the following L.P.P by big M- method.

$$\text{Max. } Z = x_1 + 2x_2 + 3x_3 - x_4$$

s.to

$$x_1 + 2x_2 + 3x_3 = 15,$$

$$2x_1 + x_2 + 5x_3 = 20$$

$$x_1 + 2x_2 + x_3 + x_4 = 10$$

$$, \text{ all } x_1, x_2, x_3, x_4 \geq 0$$

Review questions

Q.3 use two phase method to solve the following L.P.P.

$$\mathit{Mini}.Z = 10x_1 + 3x_2$$

s.to

$$x_1 + 2x_2 \geq 3,$$

$$x_1 + 4x_2 \geq 4,$$

$$\mathit{all}, x_1, x_2 \geq 0$$

Duality in linear programming

1. In physics and chemistry ,wave particle duality is a conceptualization that all objects in our universe exhibits properties of both waves and particles.
2. According to dictionary definition, dual means double. As applied to linear programming , duality implies that a double meaning can be attached to every linear programming problem. It can be thought of as a primal problem or as the dual problem. Unless we are told , it is not possible to determine which interpretation someone is placing on a given problem. Thus each LPP can be analyzed in two different ways (primal and its dual),but having equivalent solutions.

Each LPP (both maximization and minimization) stated in its original form has association with another LPP called **dual LPP** or in short **dual** , which is unique and is based on the same data. It is immaterial which one of the two is called as the **primal** and which one as **dual** as dual of the dual is primal:
if the general L.P problem is taken as

$$\text{Max. } Z_p = cX$$

s.to

$$AX \leq b$$

$$X \geq 0$$

Eq.(I)

where

$$c = (c_1, c_2, \dots, c_n)_{1 \times n}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{bmatrix}, A = \begin{bmatrix} a_{11}, a_{12} & \text{---} & a_{1n} \\ a_{21}, a_{22} & \text{---} & a_{2n} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ a_{m1}, a_{m2} & \text{---} & a_{mn} \end{bmatrix} \text{ and}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_m \end{bmatrix}_{m \times 1}$$

Then attached to it , there exist another problem,

$$\text{Min. } Z_D = b^T W$$

s.to

$$A^T W \leq c^T$$

$$W \geq 0$$

Eq.(II)

Where T , stands for the transpose of the matrix/vector and

$$W = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}_{m \times 1}$$

if (I) represent the primal problem, then (II) is the dual . In fact it is immaterial , which problem is designated as the primal , since the dual of a dual is primal. If the optimal solution of one is known , the optimal solution to the other can be obtained from it (without solving) . These two problem possess very closely related properties.

Note: if one of the constraints , in the primal , appear with an equality sign.(say i^{th}) , then the corresponding variable w_i will be unrestricted in sign .

i.e. if $a_{i1}x_1 + a_{i2}x_2 + \dots\dots\dots a_{in}x_n = b_i$ then w_i is unrestricted in sign . We can give the following relationship between the primal and its dual.

Relation between the primal and dual

Primal problem	Dual problem
1. Objective function maximize $Z_p = cX$	Minimize $Z_D = b^T W$
2. constraints: $AX \leq b$	$A^T W \geq c^T$
3. i^{th} Constraint is an inequality i^{th} constraint is an equation	$W_i \geq 0$ W_i is unrestricted in sign
4. $X_i \geq 0$ X_i is unrestricted in sign	j^{th} constraint is an inequality j^{th} constraint is an equation
5. Right hand side vector: b	Right hand side vector: c^T
6. Cost coefficients c	Cost coefficients b^T

Note: for every primal constraint there is a dual variable and for every dual constraint is a primal variable.

About the optimal solution.

1. Max $Z_P = \min Z_D$

2.(i) if the primal problem has an unbounded solution , then the dual has no feasible solution .

(ii) if the primal solution has no solution then dual has either no solution or an unbounded solution .

Example: if primal has no solution then dual also has no solution

Consider the primal problem as

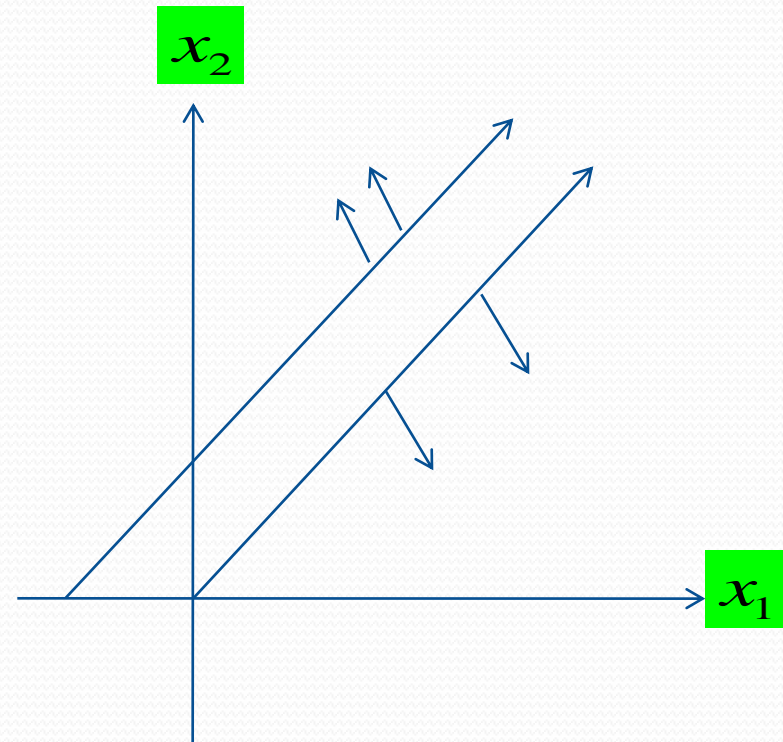
$$\text{Max. } Z = 3x_1 + 4x_2$$

s.to

$$x_1 - x_2 \leq -1$$

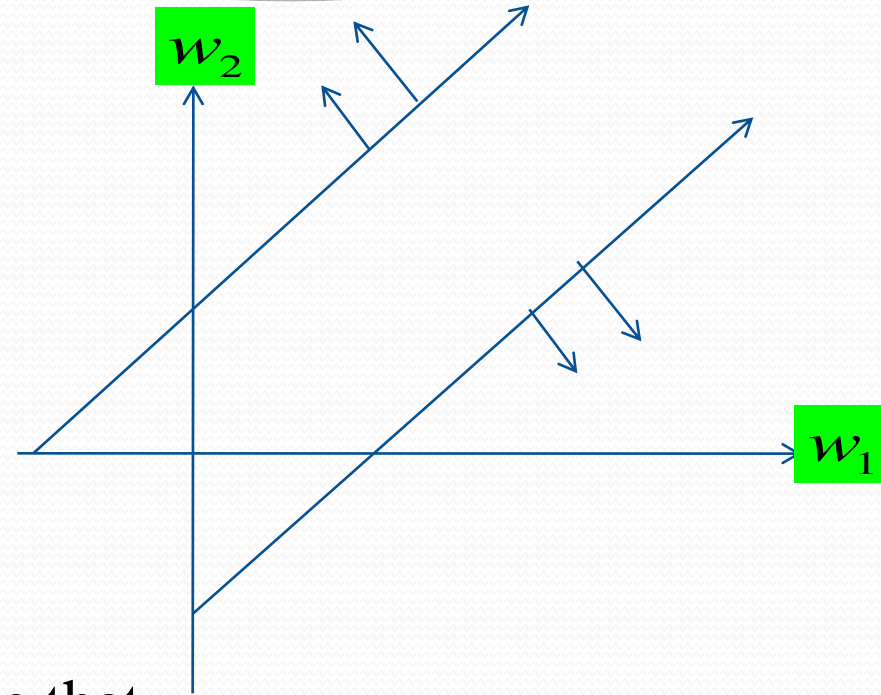
$$-x_1 + x_2 \leq 0$$

$$x_1, x_2 \geq 0$$



Its dual is

$$\begin{aligned} \text{Min. } Z_D &= -w_1 \\ \text{s.t.} \\ w_1 - w_2 &\geq 3 \\ -w_1 + w_2 &\geq 4 \\ w_1, w_2 &\geq 0 \end{aligned}$$



It is clear from the above example that
Primal and dual both can have no feasible solution

Example write the dual of the following L.P.P

$$\text{Max. } Z_p = 2x_1 + 4x_2$$

s.to

$$2x_1 + 3x_2 \leq 48$$

$$x_1 + 3x_2 \leq 42$$

$$x_1 + x_2 \leq 21$$

$$x_1, x_2 \geq 0$$

Solution: it is a maximization problem with all constraints having \leq Sign .

$$\text{Max. } Z_P = 2x_1 + 4x_2$$

s.to

$$2x_1 + 3x_2 \leq 48$$

$$x_1 + 3x_2 \leq 42$$

$$x_1 + x_2 \leq 21$$

$$x_1, x_2 \geq 0$$

The dual is

$$\text{Min. } Z_D = 48w_1 + 42w_2 + 21w_3$$

s.to

$$2w_1 + w_2 + w_3 \geq 2$$

$$3w_1 + w_2 + w_3 \geq 4$$

$$w_1, w_2, w_3 \geq 0$$



<https://www.youtube.com/watch?v=zDch07vVBxl>
<https://www.youtube.com/watch?v=8IRrgDoV8Eo>
<https://www.youtube.com/watch?v=aw7NBSje2iM>
<https://www.youtube.com/watch?v=MZ843Vvia0A>
<https://www.youtube.com/watch?v=oZqVDGFckZE>
<https://www.youtube.com/watch?v=UsqtzA9XmQE>
<https://www.youtube.com/watch?v=gmDwUCvOJQ8>

Conclusion

Conclusion of this unit is that the term linear programming define A particular class of optimization problems in which the constraints of the system can be expressed as linear equations or inequalities and the objective function is a linear function of the design variables.

Linear programming (LP) techniques are widely used to solve a number of military , economic , industrial and societal problems. The primary reasons for its wide use are the availability of commercial software to solve very large problems and the easy with which data variation (sensitivity analysis) can be handled through LP models.



Thank you

Estd. Yr. 2000



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OPTIMIZATION TECHNIQUES

III Sem CS (3CS2-01)

UNIT-V

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UNIT-III
TRANSPORTATION AND ASSIGNMENT
PROBLEM

CONTENTS OF UNIT-III:-

INTRODUCTION

SOLUTION OF TRANSPORTATION PROBLEM

OPTIMALITY TEST

ASSIGNMENT PROBLEM

CONCLUSION

What is a Transportation Problem?

- The transportation problem is a special type of LPP where the objective is to minimize the cost of distributing a product from a number of sources or origins to a number of destinations.
- Because of its special structure the usual simplex method is not suitable for solving transportation problems. These problems require special method of solution.

Application of Transportation Problem

- Minimize shipping costs
- Determine low cost location
- Find minimum cost production schedule
- Military distribution system

Introduction and definition

- **Distributing any commodity from any group of supply centers, called *sources*, to any group of receiving centers, called *destinations*, in such a way as to minimize the total distribution cost (shipping cost).**
- **Total supply must equal total demand.**
- **If total supply exceeds total demand, a *dummy destination*, whose demand equals the difference between the total supply and total demand is created. Similarly if total supply is less than total demand, a *dummy source* is created, whose supply**
- **equals the difference.**
- **All unit shipping costs into a dummy destination or out of a dummy source are 0.**



Example :1

		DESTINATIONS				Supply
		D ₁	D ₂	D ₃	D ₄	
Sources	S ₁	50	75	30	45	12
	S ₂	65	80	40	60	17
	S ₃	40	70	50	55	11
Demand		10	10	10	10	

Transportation Table:

	D ₁	D ₂	D ₃	D ₄	Supply
S ₁	50	75	30	45	12
S ₂	65	80	40	60	17
S ₃	40	70	50	55	11
Demand	10	10	10	10	

Shipment will be placed here

Unit Shipping Cost

Initial Solution of transportation problem

- 1. North-west Corner rule
- 1. Select the *remaining* variable in the upper left (northwest) corner and note the supply remaining in the row, s , and the demand remaining in the column, d .
- 2. Allocate the minimum of s or d to this variable. If this minimum is s , eliminate all variables in its row from future consideration and reduce the demand in its column by s ; if the minimum is d , eliminate all variables in the column from future consideration and reduce the supply in its row by d .
- REPEAT THESE STEPS UNTIL ALL SUPPLIES HAVE BEEN ALLOCATED.

	D ₁	D ₂	D ₃	D ₄	Supply
S ₁	50 10	75 2	30	45	12
S ₂	65	80 8	40 9	60	17
S ₃	40	70	50 1	55 10	11
Demand	10	10	10	10	

Total shipping cost = 2250

• 2. Lowest Cost entry method

- **1.** For the remaining variable with the lowest unit cost, determine the remaining supply left in its row, s , and the remaining demand left in its column, d (break ties arbitrarily).
- **2.** Allocate the minimum of s or d to this variable. If this minimum is s , eliminate all variables in its row from future consideration and reduce the demand in its column by s ; if the minimum is d , eliminate all variables in the column from future consideration and reduce the supply in its row by d .
- **REPEAT THESE STEPS UNTIL ALL SUPPLIES HAVE BEEN ALLOCATED.**

Least Cost Cell	Remaining Supply in the Row (s)	Remaining Demand in the Column (d)	Allocate	Modifications
1. X_{13} Cost = 30	12	10	$X_{13} = 10$	Eliminate column 3; reduce row 1 supply to $12 - 10 = 2$
2. X_{31} Cost = 40	11	10	$X_{31} = 10$	Eliminate column 1; reduce row 3 supply to $11 - 10 = 1$
3. X_{14} Cost = 45	2	10	$X_{14} = 2$	Eliminate row 1; reduce column 4 demand to $10 - 2 = 8$
4. X_{34} Cost = 55	1	8	$X_{34} = 1$	Eliminate row 3; reduce column 4 demand to $8 - 1 = 7$

	D ₁	D ₂	D ₃	D ₄	Supply
S ₁	50	75	30	45	12
S ₂	65	80	40	60	17
S ₃	40	70	50	55	11
Demand	10	10	10	10	

Total shipping cost = 2065

• 3. Vogel's Approximation Method (VAM)

- **1.** For each remaining row and column, determine the difference between the lowest two *remaining* costs; these are called the *row and column penalties*.
- **2.** Select the row or column with the largest penalty found in step 1 and note the supply remaining for its row, s , and the demand remaining in its column, d .
- **3.** Allocate the minimum of s or d to the variable in the selected row or column with the lowest remaining unit cost. If this minimum is s , eliminate all variables in its row from future consideration and reduce the demand in its column by s ; if the minimum is d , eliminate all variables in the column from future consideration and reduce the supply in its row by d .
- REPEAT THESE STEPS UNTIL ALL SUPPLIES HAVE BEEN ALLOCATED.

	D ₁	D ₂	D ₃	D ₄	Supply
S ₁	50	75	30	45	12
		2		10	
S ₂	65	80	40	60	17
		7	10		
S ₃	40	70	50	55	11
	10	1			
Demand	10	10	10	10	

Total shipping cost = 2030



Optimality Test on IBFS

**Methods : 1. Stepping Stone Method
2. MODI Method**

MODI Method Of Checking Optimality

Condition To Be Satisfied :

$$\text{Filled Cells} = m + n - 1$$

Where $m = \text{No. of Rows}$
 $n = \text{No. of columns}$

The Stepping-Stone Method

- **1. Find the current $C_{ij}-Z_{ij}$ values for each nonbasic variable and select the one with the most negative $C_{ij}-Z_{ij}$ value as the entering variable; if all $C_{ij}-Z_{ij}$ values are nonnegative, the current solution is optimal.**
- **2. Determine which basic variable reaches 0 first when the entering variable is increased.**
- **3. Determine a new basic solution and repeat the steps.**

Step 1: Determine the $C_{ij}-Z_{ij}$ Values for the Nonbasic Variables

- **1. If U_i is the dual variable associated with the i -th supply constraint, and V_j is the dual variable associated with the j -th demand constraint, then for shipments from node i to node j , one can find the corresponding Z_{ij} value by $Z_{ij} = U_i - V_j$. Thus the $C_{ij}-Z_{ij}$ value for variable X_{ij} is found by**
- $C_{ij} - Z_{ij} = C_{ij} - (U_i - V_j) = C_{ij} - U_i + V_j$

- **2. Given that there is a redundant equation among the $m + n$ constraints (and any of the $m + n$ constraints can be considered the redundant one), one can show that the U_i or V_j associated with the redundant equation is 0. Thus one U_i or V_j can arbitrarily be selected and set to 0. Arbitrarily choose $U_1 = 0$.**
- **3. Since the $C_{ij} - Z_{ij}$ values for *basic* variables are 0 (i.e., $C_{ij} - U_i + V_j = 0$ for basic variables), we can easily solve for the remaining values of the U_i 's and V_j 's from the $m + n - 1$ equations for the basic variables.**

- **4. Once the U_i 's and V_j 's have been determined, the $C_{ij}-Z_{ij}$ values for the nonbasic variables can be calculated by**
- **$C_{ij} - Z_{ij} = C_{ij} - U_i + V_j$**

Basic Cell	Basic $C_{ij}-Z_{ij}$ Values = 0	Substitute	Implies
X_{12}	$C_{12} - U_1 + V_2 = 0$	$75 - 0 + V_2 = 0$	$V_2 = -75$
X_{14}	$C_{14} - U_1 + V_4 = 0$	$45 - 0 + V_4 = 0$	$V_4 = -45$
X_{22}	$C_{22} - U_2 + V_2 = 0$	$80 - U_2 + (-75) = 0$	$U_2 = 5$
X_{23}	$C_{23} - U_2 + V_3 = 0$	$40 - 5 + V_3 = 0$	$V_3 = -35$
X_{32}	$C_{32} - U_3 + V_2 = 0$	$70 - U_3 + (-75) = 0$	$U_3 = -5$
X_{31}	$C_{31} - U_3 + V_1 = 0$	$40 - (-5) + V_1 = 0$	$V_1 = -45$

Non-basic cells:

Variable Cell

$C_{ij}-Z_{ij}$ Calculation

X_{11}	$C_{11} - U_1 + V_1 = 50 - 0 - 45 = 5$
X_{13}	$C_{13} - U_1 + V_3 = 30 - 0 - 35 = -5$
X_{21}	$C_{21} - U_2 + V_1 = 65 - 5 - 45 = 15$
X_{24}	$C_{24} - U_2 + V_4 = 60 - 5 - 45 = 10$
X_{33}	$C_{33} - U_3 + V_3 = 50 - (-5) - 35 = 20$
X_{34}	$C_{34} - U_3 + V_4 = 55 - (-5) - 45 = 15$

Note: X_{13} is the entering variable.

Step 2: Determine Which Current Basic Variable Reaches 0 First

	D ₁	D ₂	D ₃	D ₄	Supply
S ₁	50	-75	+30	45	12
S ₂	65	+80	-40	60	17
S ₃	40	70	50	55	11
Demand	10	10	10	10	

Note: 1. Cycle property

2. X_{12} is the leaving variable

Step 3: Determine the Next Transportation Table

	D ₁	D ₂	D ₃	D ₄	Supply
S ₁	50	75	30	45	12
			2	10	
S ₂	65	80	40	60	17
		9	8		
S ₃	40	70	50	55	11
	10	1			
Demand	10	10	10	10	

Total shipping cost = 2020

{Improvement = 2 (-5) = 10 }

2-(MODI) Method

- Find an initial basic feasible solution by some starting procedure. Then,
- 1. Set $U_1 = 0$. Solve for the other U_i 's and V_j 's by:
 - $C_{ij} - U_i + V_j = 0$ for basic variables.
 - Then calculate the $C_{ij} - Z_{ij}$ values for nonbasic variables by:
 - $C_{ij} - Z_{ij} = C_{ij} - U_i + V_j$
 - Choose the nonbasic variable with the most negative $C_{ij} - Z_{ij}$ value as the entering variable. If all $C_{ij} - Z_{ij}$ values are nonnegative, STOP; the current solution is optimal.
- 2. Find the cycle that includes the entering variable and some of the BASIC variables. Alternating positive and negative changes on the cycle, determine the “change amount” as the smallest allocation on the cycle at which a subtraction will be made.
- 3. Modify the allocations to the variables of the cycle found in step 2 by the “change amount” and return to step 1.

- **Note:** there must be $m + n - 1$ basic variables for the transportation simplex method to work!
- Add dummy source or dummy destination, if necessary
- (m of sources and n of destinations)

Example:

Building Brick Company (BBC) has orders for 80 tons of bricks at three suburban locations as follows: Northwood -- 25 tons, Westwood -- 45 tons, and Eastwood -- 10 tons. BBC has two plants, each of which can produce 50 tons per week.

How should end of week shipments be made to fill the above orders given the following delivery cost per ton:

	<u>Northwood</u>	<u>Westwood</u>	<u>Eastwood</u>
Plant 1	24	30	40
Plant 2	30	40	42

- Initial Transportation Table

Since total supply = 100 and total demand = 80, a dummy destination is created with demand of 20 and 0 unit costs.

	Northwood	Westwood	Eastwood	Dummy	Supply
Plant 1	24	30	40	0	50
Plant 2	30	40	42	0	50
Demand	25	45	10	20	

Example:

- Lowest cost entry method
 - Iteration 1: Tie for least cost (0), arbitrarily select x_{14} . Allocate 20. Reduce s_1 by 20 to 30 and delete the Dummy column.
 - Iteration 2: Of the remaining cells the least cost is 24 for x_{11} . Allocate 25. Reduce s_1 by 25 to 5 and eliminate the Northwood column.
 - Iteration 3: Of the remaining cells the least cost is 30 for x_{12} . Allocate 5. Reduce the Westwood column to 40 and eliminate the Plant 1 row.
 - Iteration 4: Since there is only one row with two cells left, make the final allocations of 40 and 10 to x_{22} and x_{23} , respectively.

Example:

- Initial table

	Northwood	Westwood	Eastwood	Dummy	Supply			
Plant 1	25	24	5	30	40	20	0	50
Plant 2		30	40	40	10	42	0	50
Demand		25	45	10	20			

Total transportation cost is \$2770

- Iteration 1

- MODI Method

1. Set $u_1 = 0$

2. Since $u_1 + v_j = c_{1j}$ for occupied cells in row 1, then
 $v_1 = 24, v_2 = 30, v_4 = 0.$

3. Since $u_i + v_2 = c_{i2}$ for occupied cells in column 2,
then $u_2 + 30 = 40$, hence $u_2 = 10.$

4. Since $u_2 + v_j = c_{2j}$ for occupied cells in row 2,

then

$$10 + v_3 = 42, \text{ hence } v_3 = 32.$$

- Iteration 1

- MODI Method (continued)

Calculate the reduced costs (circled numbers on the next slide) by $c_{ij} - u_i + v_j$.

<u>Unoccupied Cell</u>	<u>Reduced Cost</u>
(1,3)	$40 - 0 - 32 = 8$
(2,1)	$30 - 24 - 10 = -4$
(2,4)	$0 - 10 - 0 = -10$

• Since some of the reduced cost are negative, the current solution is not optimal.

• Cell (2,4) has the most negative; therefore, it will be the basic variable that must be occupied in the next iteration.

Iteration 1 Table:

	Northwood	Westwood	Eastwood	Dummy	u_i
Plant 1	25 24	5 30	+8 40	20 0	0
Plant 2	-4 30	40 40	10 42	-10 0	10
v_j	24	30	32	0	

- Iteration 1

- Stepping Stone Method

The stepping stone path for cell (2,4) is (2,4), (1,4), (1,2), (2,2). The allocations in the subtraction cells are 20 and 40, respectively. The minimum is 20, and hence reallocate 20 along this path. Thus for the next tableau:

$$x_{24} = 0 + 20 = 20 \text{ (0 is its current allocation)}$$

$$x_{14} = 20 - 20 = 0 \text{ (blank for the next tableau)}$$

$$x_{12} = 5 + 20 = 25$$

$$x_{22} = 40 - 20 = 20$$

The other occupied cells remain the same.

	Northwood	Westwood	Eastwood	Dummy	Supply
Plant 1	25 24	25 30	40	0	50
Plant 2	30	20 40	10 42	20 0	50
Demand	25	45	10	20	

Total transportation cost is $\$2570 = 2770 - 10(20)$

- Iteration 2

- MODI Method

The reduced costs are found by calculating the u_i 's and v_j 's for this tableau.

1. Set $u_1 = 0$.

2. Since $u_1 + v_j = c_{ij}$ for occupied cells in row 1, then
 $v_1 = 24, v_2 = 30$.

3. Since $u_i + v_2 = c_{i2}$ for occupied cells in column 2, then $u_2 + 30 = 40$, or $u_2 = 10$.

4. Since $u_2 + v_j = c_{2j}$ for occupied cells in row 2,

then

$$10 + v_3 = 42 \text{ or } v_3 = 32; \text{ and, } 10 + v_4 = 0 \text{ or } v_4 = -10.$$

- Iteration 2

- MODI Method (continued)

Calculate the reduced costs (circled numbers on the next slide) by $c_{ij} - u_i + v_j$.

<u>Unoccupied Cell</u>	<u>Reduced Cost</u>
(1,3)	$40 - 0 - 32 = 8$
(1,4)	$0 - 0 - (-10) = 10$
(2,1)	$30 - 10 - 24 = -4$

- **Since there is still negative reduced cost for cell (2,1), the solution is not optimal.**
- **Cell (2,1) must be occupied**

Iteration 2 Table

	Northwood	Westwood	Eastwood	Dummy	u_i
Plant 1	<div style="display: flex; justify-content: space-between;"> 25 24 </div>	<div style="display: flex; justify-content: space-between;"> 25 30 </div>	<div style="display: flex; justify-content: space-between;"> +8 40 </div>	<div style="display: flex; justify-content: space-between;"> +10 0 </div>	0
Plant 2	<div style="display: flex; justify-content: space-between;"> -4 30 </div>	<div style="display: flex; justify-content: space-between;"> 20 40 </div>	<div style="display: flex; justify-content: space-between;"> 10 42 </div>	<div style="display: flex; justify-content: space-between;"> 20 0 </div>	10
v_j	24	30	36	-6	

- Iteration 2

- Stepping Stone Method

The most negative reduced cost is $= -4$ determined by x_{21} . The stepping stone path for this cell is $(2,1), (1,1), (1,2), (2,2)$. The allocations in the subtraction cells are 25 and 20 respectively. Thus the new solution is obtained by reallocating 20 on the stepping stone path. Thus for the next tableau:

$$x_{21} = 0 + 20 = 20 \quad (0 \text{ is its current allocation})$$

$$x_{11} = 25 - 20 = 5$$

$$x_{12} = 25 + 20 = 45$$

$$x_{22} = 20 - 20 = 0 \quad (\text{blank for the next tableau})$$

The other occupied cells remain the same.

Northwood Westwood Eastwood Dummy Supply

Plant 1	5	24	45	30	40	0	50
Plant 2	20	30	40	10	42	20	0
Demand	25	45	10	20			

Total cost is \$2490 = 2570 - 4(20)

- Iteration 3

- MODI Method

The reduced costs are found by calculating the u_i 's and v_j 's for this tableau.

1. Set $u_1 = 0$

2. Since $u_1 + v_j = c_{1j}$ for occupied cells in row 1, then $v_1 = 24$ and $v_2 = 30$.

3. Since $u_i + v_1 = c_{i1}$ for occupied cells in column 1, then $u_2 + 24 = 30$ or $u_2 = 6$.

4. Since $u_2 + v_j = c_{2j}$ for occupied cells in row 2,

then

$$6 + v_3 = 42 \text{ or } v_3 = 36, \text{ and } 6 + v_4 = 0 \text{ or } v_4 = -6.$$

- Iteration 3

- MODI Method (continued)

Calculate the reduced costs (circled numbers on the next slide) by $c_{ij} - u_i + v_j$.

<u>Unoccupied Cell</u>	<u>Reduced Cost</u>
(1,3)	$40 - 0 - 36 = 4$
(1,4)	$0 - 0 - (-6) = 6$
(2,2)	$40 - 6 - 30 = 4$

Since all the reduced cost are nonnegative, the current solution is optimal

- Iteration 3 Table

Since all the reduced costs are non-negative, this is the optimal tableau.

	Northwood	Westwood	Eastwood	Dummy	u_i
Plant 1	5 24	45 30	+4 40	+6 0	0
Plant 2	20 30	+4 40	10 42	20 0	6
v_j	24	30	36	-6	

- Optimal Solution

<u>From</u>	<u>To</u>	<u>Amount</u>	<u>Cost</u>
Plant 1	Northwood	5	120
Plant 1	Westwood	45	1,350
Plant 2	Northwood	20	600
Plant 2	Eastwood	10	<u>420</u>

Total Cost = \$2,490



Review question

1. What is unbalanced transport problem?
2. What is degeneracy?
3. Solve the following transportation problem.

				Supply
	5	6	8	10
	10	12	6	20
	10	14	8	30
Demand	30	20	10	

- Example

- <https://www.youtube.com/watch?v=Q31jKiEXxdc>



UNIT-III

ASSIGNMENT PROBLEM



CONTENTS

- Introduction
- Hungarian Method of Assignment Problems.
- Special Cases in Assignment Problems.
- Exercises

The Assignment Problem (AP)

- **Assignment problem is a special case of Transportation problem with $m=n$ and $s_i=d_j$ for all i , and j .**
- **The Hungarian Algorithm**
- **solving the assignment problem of a least**
- **cost assignment of m workers to m jobs**

- **Assumptions:**
- **1. There is a cost assignment matrix for the m “people” to be assigned to m “tasks.” (If necessary dummy rows or columns consisting of all 0’s are added so that the numbers of people and tasks are the same.)**
- **2. All costs are nonnegative.**
- **3. The problem is a minimization problem.**



• The Hungarian Algorithm

• Initialization

- **1.** For each row, subtract the minimum number from all numbers in that row.
- **2.** In the resulting matrix, subtract the minimum number in each column from all numbers in the column.

- Iterative Steps

- **1.** Make as many 0 cost assignments as possible. If all workers are assigned, STOP; this is the minimum cost assignment. Otherwise draw the minimum number of horizontal and vertical lines necessary to cover all 0's in the matrix. (A method for making the maximum number of 0 cost assignments and drawing the minimum number of lines to cover all 0's follows.)
- **2.** Find the smallest value not covered by the lines; this number is the *reduction value*.
- **3.** Subtract the reduction value from all numbers not covered by any lines. Add the reduction value to any number covered by both a horizontal and vertical line.
- GO TO STEP 1.


- For small problems, one can usually determine the maximum number of zero cost assignments by observation. For larger problems, the following procedure can be used:
- **Determining the Maximum Number of Zero-Cost Assignments**
- **1.** For each row, if only one 0 remains in the row, make that assignment and eliminate the *row* and *column* from consideration in the steps below.
- **2.** For each column, if only one 0 remains, make that assignment and eliminate that *row* and *column* from consideration.

- **3.** Repeat steps 1 and 2 until no more assignments can be made. (If 0's remain, this means that there are at least two 0's in each remaining row and column. Make an arbitrary assignment to one of these 0's and repeat steps 1 and 2.)
- Again, for small problems, the minimum number of lines required to cover all the 0's can usually be determined by observation. The following procedure, based on network flow arguments, can be used for larger problems:

- **Drawing the Minimum Number of Lines to Cover All 0's**
- **1.** Mark all rows with no *assignments* (with a “.”).
- **2.** For each row just marked, mark each column that *has a 0* in that row (with a “.”).
- **3.** For each column just marked, mark each row that *has an assignment* in that column (with a “.”).
- **4.** Repeat steps 2 and 3 until no more marks can be made.
- **5.** Draw lines through *unmarked rows* and *marked columns*.

Example:

	A	B	C	D	E
1	10	4	6	10	12
2	11	7	7	9	14
3	13	8	12	14	15
4	14	16	13	17	17
5	17	11	17	20	19

 Row Subtraction

	A	B	C	D	E
1	6	0	2	6	8
2	4	0	0	2	7
3	5	0	4	6	7
4	1	3	0	4	4
5	6	0	6	9	8



Column Subtraction

	A	B	C	D	E
1	5	0	2	4	4
2	3	0	0	0	3
3	4	0	4	4	3
4	0	3	0	2	0
5	5	0	6	7	4

Minimum uncovered number

	A	B	C	D	E
1	3	0	0	2	2
2	3	2	0	0	3
3	2	0	2	2	1
4	0	5	0	2	0
5	3	0	4	5	2

Minimum uncovered number

	A	B	C	D	E
1	3	1	0	2	2
2	3	3	0	0	3
3	1	0	1	1	0
4	0	6	0	2	0
5	2	0	3	4	1

Optimal Solution

1-C	6
2-D	9
3-E	15
4-A	14
5-B	11
	<hr/>
	65



- **CONVERSION OF A MAXIMIZATION PROBLEM TO A MINIMIZATION PROBLEM**
- The Hungarian algorithm works only if the matrix is a cost matrix. A maximization assignment problem can be converted to a minimization problem by creating a lost opportunity matrix. The problem then is to minimize the total lost opportunity.

- Q solve the given assignment problem. given profit matrix

Profit Matrix:

	J1	J2	J3	J4
W1	67	58	90	55
W2	58	88	89	56
W3	74	99	80	22
(D)	0	0	0	0

- The lost opportunity matrix given below is derived by subtracting each number in the J1 column from 74, each number in the J2 column from 99, each number in the J3 column from 90, and each number in the J4 from 56.

- | | J1 | J2 | J3 | J4 |
|------------|-----------|-----------|-----------|-----------|
| W1 | 7 | 41 | 0 | 1 |
| W2 | 16 | 11 | 1 | 0 |
| W3 | 0 | 0 | 10 | 34 |
| (D) | 74 | 99 | 90 | 56 |

- The Hungarian algorithm can now be applied to this lost opportunity matrix to determine the maximum profit set of assignments.

Review questions

- Q.1 what is assignment problem ?
- Q.2 solve the following assignment problem

	→	A	B	C	D	E
↓	Machine jobs					
	1	10	20	4	24	8
	2	8	20	6	28	12
	3	12	26	8	32	12
	4	14	28	8	36	18
	5	16	28	12	36	24

- Q.3 solve the following assignment problems (unbalanced problems)

→

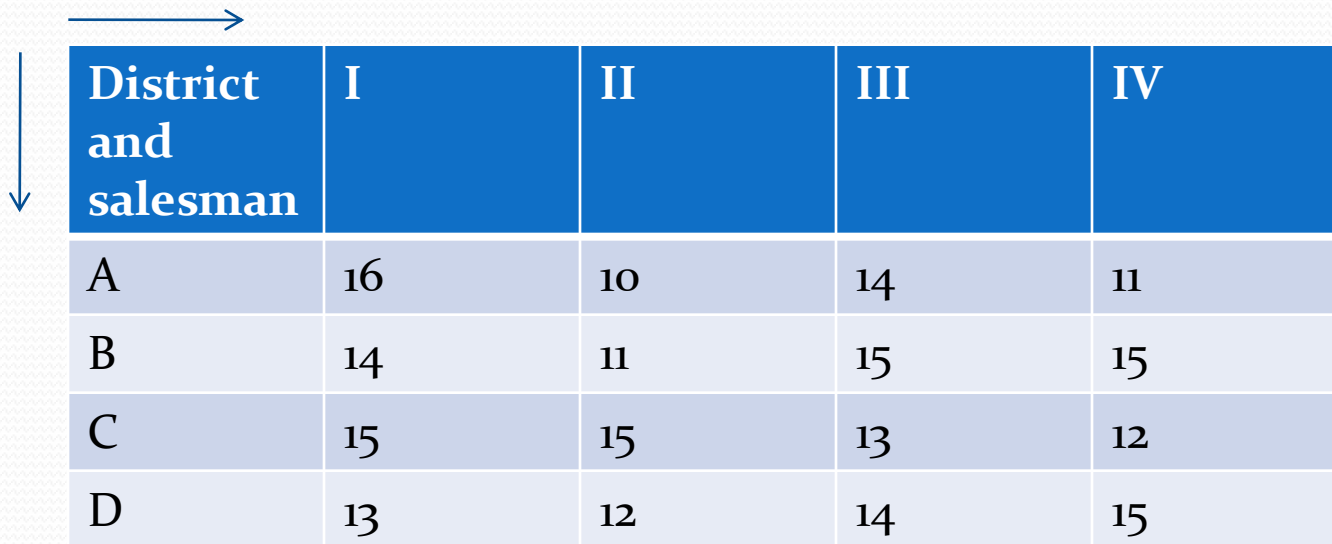
↓ Resources and workers	J ₁	J ₂	J ₃	J ₄
W ₁	7	5	8	4
W ₂	5	6	7	4
W ₃	8	7	9	8

Problems of maximization

- A maximization assignment problem can be converted to a minimization problem by creating a lost opportunity matrix using any of the following methods. The problem then is to minimize the total lost opportunity. The first method is by putting a negative sign before the values in the assignment matrix and then solves the sum as a minimization case using Hungarian methods.
- Second method is to locate the largest value in the given matrix and subtract each element in the matrix from this value .then one can solve this problem as a minimization case using the new modified matrix .

Example:

- A company has a team of four salespersons and there are four districts where the company wants to start its business .after taking into account the capabilities of salesman and nature of districts, the company estimates the profit per day in rupees for each salesman for each district given in the following table:



District and salesman	I	II	III	IV
A	16	10	14	11
B	14	11	15	15
C	15	15	13	12
D	13	12	14	15

Travelling salesman problem(TSP)

- Travelling salesman problem is a routine problem. It can be considered as a typical assignment problem with certain restrictions . Consider a salesman who is assigned the job of visiting n different cities . He knows(is given) the distance (or time) between all pairs of cities . He is asked to visit each of the cities only once. The trip should be continuous and he should come back to the city from where he started using the shortest route . It does not matter , from which city he starts. These restrictions imply:
- (i) no assignment should be made along the diagonal ,going directly from city i to j is not permitted . Therefore diagonal elements are generally taken very large i.e. ∞

Mathematical formulation

- A travelling salesman has to visit n cities and return to the starting point. He has to start from any one city and visit each city only once. The objective is to select the sequence which the cities are visited in such a way that the total travelling distance (or time) is minimized. Suppose he starts from the k th city and the last he visited is m . Let the cost of travel from i th city to j th city be C_{ij} . Then the objective function is

$$\text{Minimize } \sum_{i=1}^m \sum_{j=1}^n C_{ij} x_{ij}$$

- Subject to the constraints

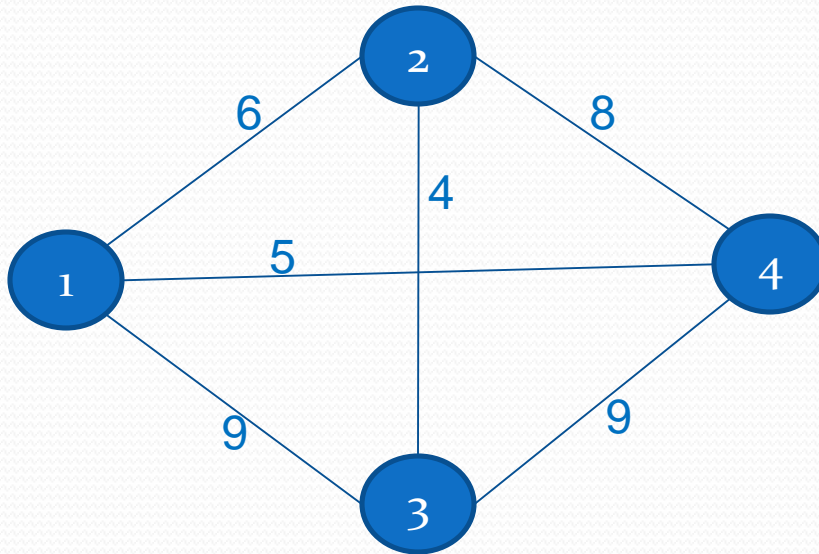
$$\begin{aligned} \sum_{i=1}^m x_{ij} &= 1, \text{ for } i = 1, 2, \dots, m, i \neq j, j \neq m \\ \sum_{j=1}^n x_{ij} &= 1, \text{ for } j = 1, 2, \dots, n, i \neq j, j \neq n \\ x_{mk} &= 1 \\ x_{ij} &= 0 \text{ or } 1 \end{aligned}$$

Working procedure

- Step-1 solve the problem as an assignment problem using the method used to solve the above examples.
- Step-2 (a) if the solution thus found out are cyclic in nature ,then that is the final solution.
(b) if it is not cyclic ,then go to step-3.
- Step-3 select the lowest entry in the table (other than zero). Delete the row and column of this lowest entry and again do the zero assignment in the remaining matrix.
- Step-4 check whether cyclic assignment is available.If not, include the next higher entry in the table and the procedure is repeated until a cyclic assignment is obtained.

Example

- Consider a four city TSP for which the cost between the city pairs are as shown in the figure below. Find the tour of the salesman so that the cost of travel is minimal.



Solution

- Construct the cost matrix as follows:

	1	2	3	4
1	∞	6	9	5
2	6	∞	4	8
3	9	4	∞	9
4	5	8	9	∞

Step-1

	1	2	3	4
1	∞	0	5	0
2	2	∞	4	8
3	9	4	∞	9
4	0	3	4	∞

- The optimal assignment is $1 \rightarrow 4, 2 \rightarrow 3, 3 \rightarrow 2, 4 \rightarrow 1$

Which is not cyclic.

Step-2 consider the lowest entry '2' of the cell (2,1). If there is a tie in selecting the lowest entry, then break the tie arbitrarily. Delete the II row and I column. Do the zero assignment in the remaining matrix. The resulting table is

∴-

	1	2	3	4
1	∞	0	4	0
2	2	∞	0	3
3	5	0	∞	4
4	0	3	0	∞

- Next optimal assignment is

$$1 \rightarrow 4, 2 \rightarrow 1, 3 \rightarrow 2, 4 \rightarrow 3$$

Which is cyclic.





<https://www.youtube.com/watch?v=q7VMzOTZvqM>

<https://www.youtube.com/watch?v=BUGIhEecipE>

Conclusion

- The transportation problem is a special class of the linear programming problem. It deals with the situation in which a commodity is transported from **sources(origins)** to **destinations**. The objective is to determine the amount of commodity to be transported from each source to each destination so that the total transportation cost is minimum.
- The Assignment problem is special from of the transportation problem with equal number of sources and destination and supply at each source and demand at each destination is limited to one unit.
- The name “assignment problem” originates from the classical problem where the objective is to assign a number of origins(job) to the equal number of destinations(persons) at minimum cost(or maximum profit) travelling salesman problem is a special type of assignment problem.



Thank you