Unit-IV

Classical Optimization Techniques

2.1 INTRODUCTION

In the present chapter, we shall discuss the classical optimization techniques with necessary and sufficient conditions for obtaining the optimum solution of unconstrained single and multivariable optimization problems. Constrained multivariable problems with equality and inequality constraints have also been discussed in detail with examples. The classical optimization techniques are very useful to obtain the optimal solution of problems involving continuous and differentiable functions. Such types of techniques are analytical in nature to obtain maximum and minimum points for unconstrained and constrained continuous objective functions. For equality constrained problems we use Lagrange's multiplier method and for inequality constrained, the Kuhn-Tucker conditions, for getting optimum solutions.

2.2 UNCONSTRAINED OPTIMIZATION PROBLEMS

2.2.1 Single Variable Optimization Problems

Let $f(x)$ be a continuous function of single variable x defined in interval $[a, b]$.

Local Maxima

A function $f(x)$ with single variable is said to have a local (relative) maxima at $x = x_o$ if

$$f(x_0) \geq f(x_0 + h)$$

for all sufficiently small positive and negative values of h.

Local Minima

A function $f(x)$ with single variable is said to have a local (relative) minima at $x = x_o$ if

$$f(x_0) \leq f(x_0 + h)$$

for all sufficiently small positive and negative values of h.

Step 1: Differentiate $f(x_1, x_2)$ partially with respect to x_1 and x_2, we get

$$\frac{\partial f}{\partial x_1} \quad \text{and} \quad \frac{\partial f}{\partial x_2}$$

Step 2: For extreme points, we have

$$\frac{\partial f}{\partial x_1} = 0$$

and
$$\frac{\partial f}{\partial x_2} = 0$$

Solving these equations we get some points as (a_1, b_1), (a_2, b_2),…etc.

Step 3: Differentiate again partially to get

$$r = \frac{\partial^2 f}{\partial x_1^2}, \quad s = \frac{\partial^2 f}{\partial x_1 \partial x_2} \quad \text{and} \quad t = \frac{\partial^2 f}{\partial x_2^2}$$

Step 4: At (a_1, b_1), calculate r and $rt - s^2$

Case I: If $rt - s^2 > 0$ and $r < 0$ then $f(x_1, x_2)$ is maximum at (a_1, b_1).

Case II: If $rt - s^2 > 0$ and $r > 0$ then $f(x_1, x_2)$ is minimum at (a_1, b_1).

Case III: If $rt - s^2 < 0$ then $f(x_1, x_2)$ has neither maxima nor minima at (a_1, b_1).

Case IV: If $rt - \mathrm{s}^2 = 0$ then $f(x_1, x_2)$ has point of inflexion at (a_1, b_1).

2.3.2 Working Rule to Find the Extreme Points of Functions of n-Variables

Let us consider $u = f(x_1, x_2, x_3, \ldots, x_n)$ as a function of $x_1, x_2, x_3, \ldots, x_n$.

Necessary Conditions

$$\frac{\partial f}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} = 0, \ldots, \frac{\partial f}{\partial x_n} = 0$$

Sufficient Conditions

The Hessian matrix at P for n variables will be

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Its leading minors are defined as

$$H_1 = \left| \frac{\partial^2 f}{\partial x_1^2} \right|,$$

$$H_2 = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} \end{vmatrix},$$

$$H_3 = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \dfrac{\partial^2 f}{\partial x_2 \partial x_3} \\ \dfrac{\partial^2 f}{\partial x_3 \partial x_1} & \dfrac{\partial^2 f}{\partial x_3 \partial x_2} & \dfrac{\partial^2 f}{\partial x_3^2} \end{vmatrix},$$

Hence, the following cases will arise:

Case I: If H_1, H_2, H_3,... are positive (i.e., H is positive definite) then $f(x_1, x_2, x_3, \ldots, x_n)$ has minimum at P.

Case II: If H_1, H_2, H_3,... are alternately negative, positive, negative (i.e., H is negative definite) then $f(x_1, x_2, x_3, \ldots, x_n)$ has maximum at P.

Case III: H_1 and H_3,... are not of same sign and $H_2 = 0$ (i.e., semidefinite or indefinite) then $f(x_1, x_2, x_3, \ldots x_n)$ has a saddle point at P.

SOLVED EXAMPLES

Example 1: Assume the following relationship for revenue and cost functions. Find out at what level of output x, where x is measured in tons per week, profit is maximum.

$$R(x) = 1000x - 2x^2 \text{ and } C(x) = x^3 - 59x^2 + 1315x + 5000.$$

Solution: The profit function is

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 1000x - 2x^2 - x^3 + 59x^2 - 1315x - 5000 \\ &= -x^3 + 57x^2 - 315x - 5000 \qquad \ldots(1) \end{aligned}$$

Differentiating both sides of (1) with respect to x, we get

$$\frac{dP}{dx} = -3x^2 + 114x - 315 \qquad \ldots(2)$$

For maxima and minima, we have

$$\frac{dP}{dx} = 0$$

$$\Rightarrow \qquad -3x^2 + 114x - 315 = 0$$
$$\Rightarrow \qquad x = 3, 35.$$

Differentiating both sides of (2) again with respect to x, we get

$$\frac{d^2P}{dx^2} = -6x + 114$$

At $x = 3$, $\dfrac{d^2P}{dx^2} = 96 > 0$, i.e., P is minimum at $x = 3$.

At $x = 35$, $\dfrac{d^2P}{dx^2} = 96 < 0$, i.e., P is maximum at $x = 35$.

Hence, the profit is maximum at $x = 35$ tons per week.

Example 2: The profit P earned, by a company, on some item is function of its units produced say x and is given by

$$P = 800x - 2x^2$$

If the company's expenditure or interest, rent and salary of the staff be Rs. 1 lakh, show that the company will always be in loss.

Solution: Given that

$$P = 800x - 2x^2 \qquad \ldots(1)$$

Differentiate both sides of (1) with respect to x, we get

$$\frac{dP}{dx} = 800 - 4x \qquad \ldots(2)$$

For maxima and minima, we have

$$\frac{dP}{dx} = 0$$
$$800 - 4x = 0$$
$$x = 200$$

Differentiate both sides of (2) again with respect to x, we get

$$\frac{d^2P}{dx^2} = -4 \text{ (negative)}$$

So, profit (P) is maximum for ($x = 200$).

The net profit = P – expenditure

$$= 800(200) - 2(200)^2 - 1,00,000$$
$$= -20,000.$$

Hence, the company will always be in loss.

Example 3: Assuming that the petrol burnt (per hour) in driving a motor boat varies as the cube of its velocity, show that the most economical speed when going against a current of c kmph is $\dfrac{3c}{2}$ km per hour.

Solution: Let the velocity of the boat be v km per hour, the current's velocity is c km per hour and the relative velocity of the boat is $(v - c)$ km per hour.

If the total distance travelled 'a' km then the time taken $= \dfrac{a}{v-c}$ hour.

According to given the petrol burnt in one hour is proportional to v^3, i.e. $= \lambda v^3$ where λ is a suitable positive constant.

The petrol burnt for distance 'a' is given by

$$P = \lambda v^3 . \frac{a}{v-c} \quad \ldots(1)$$

Differentiate both sides of (1) with respect to v, we get

$$\frac{dP}{dv} = \lambda\alpha\left[\frac{3v^2(v-c)-v^3}{(v-c)^2}\right] \quad \ldots(2)$$

$$= \lambda\alpha\left[\frac{2v^3-3cv^2}{(v-c)^2}\right]$$

For maxima and minima, we have

$$\frac{dP}{dv} = 0$$

$$\Rightarrow \quad \lambda\alpha\left[\frac{2v^3-3cv^2}{(v-c)^2}\right] = 0$$

$$\Rightarrow \quad 2v^3 - 3cv^2 = 0$$

$$\Rightarrow \quad v = 0, \frac{3c}{2}$$

Differentiate both sides of (2) again with respect to v, we get

$$\frac{d^2P}{dv^2} = \lambda\alpha\left[\frac{(6v^2-6vc)(v-c)^2-(2v^3-3cv^2).2(v-c)}{(v-c)^4}\right]$$

$$\frac{d^2P}{dv^2} = \lambda\alpha\left[\frac{6v(v-c)^2-2v^2(2v-3c)}{(v-c)^3}\right]$$

At $v = 0$, $\dfrac{d^2P}{dv^2} = 0$, i.e., $P = 0 \Rightarrow$ no petrol is burnt.

At $v = \dfrac{3c}{2}$, $\dfrac{d^2P}{dv^2}$ is positive, i.e., P is minimum at $v = \dfrac{3c}{2}$.

Hence, the most economical speed is $v = \dfrac{3c}{2}$ km per hour.

Example 4: A beam of length l is supported at one end. If w is the uniformly distributed load per unit length, the bending moment M at a distance x from the end is given by $M = \frac{1}{2}lx - \frac{1}{2}wx^2$.

Solution: Given the bending moment is

$$M = \frac{1}{2}lx - \frac{1}{2}wx^2 \qquad \ldots(1)$$

Differentiate both sides of (1) with respect to x, we get

$$\frac{dM}{dx} = \frac{1}{2}l - wx$$

For maxima, we have

$$\frac{dM}{dx} = 0$$

$$\Rightarrow \qquad \frac{1}{2}l - wx = 0$$

$$\Rightarrow \qquad x = \frac{l}{2w}$$

Differentiate both sides of (2) with respect to x, we get

$$\frac{d^2M}{dx^2} = -w\,(\text{negative})$$

So the bending moment M is maximum at $x = \frac{l}{2w}$ from the end.

At $$x = \frac{l}{2w}, \quad M = \frac{1}{2}l\left(\frac{l}{2w}\right) - \frac{1}{2}w\left(\frac{l}{2w}\right)^2 = \frac{l^2}{8w}.$$

Example 5: Find the maximum and minimum value of $y = 3x^5 - 5x^3$.

Solution: Given that $y = 3x^5 - 5x^3$ $\qquad \ldots(1)$

Differentiate both sides of (1) with respect to x, we get

$$\frac{dy}{dx} = 15x^4 - 15x^2 \qquad \ldots(2)$$

For maxima and minima, we have

$$\frac{dy}{dx} = 0$$

$$\Rightarrow \qquad 15x^4 - 15x^2 = 0$$

$$\Rightarrow \qquad 15x^2(x-1)(x+1) = 0$$

$$x = 0, 1, -1.$$

Differentiate both sides of (2) again with respect to x, we get

$$\frac{d^2y}{dx^2} = 60x^3 - 30x$$

At $x = 0$, $\dfrac{d^2y}{dx^2} = 0$ i.e., x is a point of inflexion.

At $x = 1$, $\dfrac{d^2y}{dx^2} = 30 > 0$, i.e., y is minimum at $x = 1$.

At $x = -1$ $\dfrac{d^2y}{dx^2} = -30 < 0$, i.e., y is maximum at $x = -1$.

Example 6: In a submarine telegraph cable, the speed of signaling varies as $x^2 \log\left(\dfrac{1}{x}\right)$ where x is the ratio of the radius of the cube to that of the covering. Show that the greatest speed is attained when this ratio is $1 : \sqrt{e}$.

Solution: Let u be the speed of signaling, then

$$u = x^2 \log\left(\frac{1}{x}\right), \lambda > 0, x \neq 0$$

$$u = -\lambda x^2 \log x \qquad \ldots(1)$$

Differentiate both sides of (1) with respect to x, we get

$$\frac{du}{dx} = -2\lambda x \log x - \lambda x^2(1/x)$$

$$= -\lambda[2x \log x + x] \qquad \ldots(2)$$

For maxima and minima, we have

$$\frac{du}{dx} = 0$$

$\Rightarrow$ $$-\lambda[2x \log x + x] = 0$$

$\Rightarrow$ $$\log x = -\frac{1}{2}$$

$\Rightarrow$ $$x = e^{-1/2} = 1/\sqrt{e}.$$

Differentiate both sides of (2) again with respect to x, we get

$$\frac{d^2u}{dx^2} = -\lambda[2x.(1/x) + 2 \log x]$$

$$= -\lambda[2 \log x + 3]$$

A $x = 1/\sqrt{e}$, $\dfrac{d^2u}{dx^2} = -2\lambda$ (negative), i.e., u is maximum.

Hence, u is maximum, ratio for $x = 1 : \sqrt{e}$.

Example 7: A rectangular sheet of metal has four equal square portions removed at the corners and the sides are then turned up so as to form an open rectangular box. Show that when the volume contained in the box is maximum, the depth will be

$$\frac{1}{6}[(a + b) - (a^2 - ab + b^2)^{1/2}]$$

Where a and b are sides of the original dimensions of the rectangular.

Solution: Let x be the length of each side of the squares removed at the corners. Then the dimensions of the box will be $(a - 2x)$, $(b - 2x)$ and x. Let V be the volume of the box, then we have

$$V = (a - 2x)(b - 2x)x$$
$$= 4x^3 - 2x^2(a + b) + abx \qquad \ldots(1)$$

Differentiate both sides of (1) with respect to x, we get

$$\frac{dv}{dx} = 12x^2 - 4x(a + b) + ab \qquad \ldots(2)$$

For maxima and minima, we have

$$\frac{dv}{dx} = 0$$

$$\Rightarrow \quad 12x^2 - 4x(a + b) + ab = 0$$

$$\Rightarrow \quad x = \frac{4(a+b) \pm \sqrt{16(a+b)^2 - 48ab}}{24}$$

$$\Rightarrow \quad x = \frac{1}{6}[(a + b) \pm (a^2 - ab + b^2)^{1/2}]$$

Differentiate both sides of (2) again with respect to x, we get

$$\frac{d^2V}{dx^2} = 24x - 4(a + b)$$

At $x = \frac{1}{6}[(a + b) \pm (a^2 - ab + b^2)^{1/2}]$, $\frac{d^2V}{dx^2} = \pm\sqrt{(a-b)^2 + ab}$

If $x = \frac{1}{6}[(a + b) - (a^2 - ab + b^2)^{1/2}]$ then $\frac{d^2V}{dx^2}$ is negative, i.e., V is maximum.

Example 8: The efficiency of a screw jack is given by $\eta = \tan\alpha\cot(\alpha + \phi)$ where ϕ is constant.

Prove that the efficiency is maximum at $\alpha = \frac{\pi}{4} - \frac{\phi}{2}$ and $\eta = \frac{1-\sin\phi}{1+\sin\phi}$.

Solution: The efficiency of a screw jack is

$$\eta = \tan\alpha\cot(\alpha + \phi) \qquad \ldots(1)$$

Differentiate both sides of (1) with respect to α, we get

$$\frac{d\eta}{d\alpha} = -\tan\alpha\,\mathrm{cosec}^2(\alpha + \phi) + \sec^2\alpha\cot(\alpha + \phi)$$
$$= \sec^2\alpha\cot(\alpha + \phi) - \tan\alpha\,\mathrm{cosec}^2(\alpha + \phi)$$
$$= \frac{1}{\cos^2\alpha\sin^2(\alpha+\phi)}[\sin(\alpha + \phi)\cos(\alpha + \phi) - \sin\alpha\cos\alpha] \qquad \ldots(2)$$

For maxima and minima, we have

$$\frac{d\eta}{d\alpha} = 0$$

$$\Rightarrow \quad \frac{1}{\cos^2\alpha \sin^2(\alpha+\phi)}[\sin(\alpha+\phi)\cos(\alpha+\phi) - \sin\alpha\cos\alpha] = 0$$

$$\Rightarrow \quad \sin 2(\alpha+\phi) - \sin 2\alpha = 0$$

$$\Rightarrow \quad 2(\alpha+\phi) = \pi - 2\alpha$$

$$\Rightarrow \quad \alpha = \frac{\pi}{4} - \frac{\phi}{2}$$

By (2), we have

$$\cos^2\alpha \sin^2(\alpha+\phi)\frac{d\eta}{d\alpha} = [\sin(\alpha+\phi)\cos(\alpha+\phi) - \sin\alpha\cos\alpha]$$

$$= \frac{1}{2}[\sin 2(\alpha+\phi) - \sin 2\alpha]$$

$$= \cos(2\alpha+\phi)\cos\phi. \quad \ldots(3)$$

Differentiate both sides of (3) again with respect to α, we get

$$\cos^2\alpha\sin^2(\alpha+\phi)\frac{d^2\eta}{d\alpha^2} + \frac{d}{d\alpha}\{\cos^2\alpha\sin^2(\alpha+\phi)\}.\frac{d\eta}{d\alpha} = \frac{d}{d\alpha}[\cos(2\alpha+\phi)\cos\phi]$$

$$\cos^2\alpha\sin^2(\alpha+\phi)\frac{d^2\eta}{d\alpha^2} + \frac{d}{d\alpha}\{\cos^2\alpha\sin^2(\alpha+\phi)\}.\frac{d\eta}{d\alpha} = -[\sin(2\alpha+\phi)\cos\phi]$$

At $\alpha = \frac{\pi}{4} - \frac{\phi}{2}$, $\frac{d\eta}{d\alpha} = 0$ then $\frac{d^2\eta}{d\alpha^2}$ is negative, i.e., η is maximum.

At $\alpha = \frac{\pi}{4} - \frac{\phi}{2}$ the value of η by (1), we have

$$\eta = \frac{\sin(2\alpha+\phi-\sin\phi)}{\sin(2\alpha+\phi-\sin\phi)}$$

$$\eta = \frac{\sin(2\alpha+\phi-\sin\phi)}{\sin(2\alpha+\phi-\sin\phi)}$$

$$\eta = \frac{1-\sin\phi}{1+\sin\phi}.$$

Example 9: Show that the right circular cylinder of given surface (including its ends) and maximum volume is such that its height is equal to twice its radius.

Solution: We know that

$$V = \pi r^2 h \quad \ldots(1)$$

and $S = 2\pi rh + 2\pi r^2$ (let constant according to given)

$$\Rightarrow \quad 2\pi rh = 2k^2\pi - 2\pi r^2$$

$$h = \frac{k^2 - r^2}{r} \quad \ldots(2)$$

From (1) and (2), we have

$$V = \pi r\,(k^2 - r^2) \qquad \ldots(3)$$

Differentiate both sides of (3) with respect to r, we get

$$\frac{dV}{dr} = \pi(k^2 - 3r^2) \qquad \ldots(4)$$

For maxima and minima, we have

$$\frac{dV}{dr} = 0$$

$$\Rightarrow \quad \pi(k^2 - 3r^2) = 0$$

$$\Rightarrow \quad r = \frac{k}{\sqrt{3}} \qquad \ldots(5)$$

Differentiate both sides of (4) with respect to r, we get

$$\frac{d^2V}{dx^2} = -6\pi r \text{(negative) i.e., } V \text{ is maximum.}$$

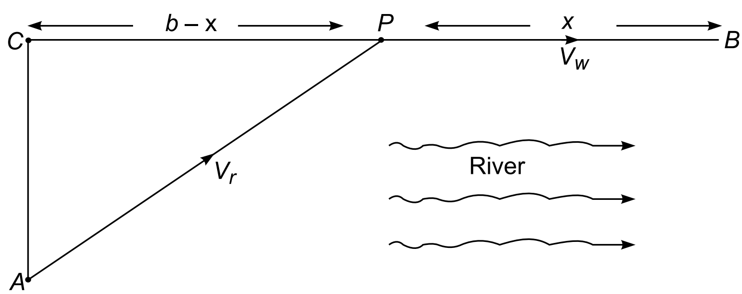
Using (2) and (5), we have

$$hr = k^2 - r^2$$

$$\Rightarrow \quad h = 2r.$$

Example 10: A person being in a boat 'a' miles from the nearest point of the beach, wishes to reach as quickly as possible, a point 'b' miles from that point along the sea shore. The ratio of his rate of walking to his rate of rowing is $\sec\alpha$. Prove that he should land at a distance $b - \alpha \cot\alpha$ from the place to be reached.

Solution: Let the person stand at A and reach on point B. Let C be the point where AC and BC meet.



Let $BC = b$ and let him row the distance AP in the boat and land at P where $PB = x$.

If t is the time of the journey, then

$$t = \frac{AP}{v_r} + \frac{BP}{v_w}$$

where $AP = \sqrt{a^2 + (b - x)^2}$ and $BP = x$

Given that $\dfrac{v_r}{v_w} = \sec\alpha.$

so
$$t = \frac{1}{v_r}\left[\sqrt{a^2+(b-x)^2} + \frac{x}{\sec\alpha}\right] \quad \ldots(1)$$

Differentiate both sides of (1) with respect to x, we get

$$\frac{dt}{dx} = \frac{1}{v_r}\left[\frac{-(b-x)}{\sqrt{a^2+(b-x)^2}} + \frac{1}{\sec\alpha}\right] \quad \ldots(2)$$

For maxima or minima, we have

$$\frac{dt}{dx} = 0$$

$$\Rightarrow \quad \frac{1}{v_r}\left[\frac{-(b-x)}{\sqrt{a^2+(b-x)^2}} + \frac{1}{\sec\alpha}\right] = 0$$

$$\Rightarrow \quad -(b-x)\sec\alpha + \sqrt{a^2+(b-x)^2} = 0$$

$$\Rightarrow \quad (b-x)\sec\alpha = \sqrt{a^2+(b-x)^2}$$

$$\Rightarrow \quad (b-x)^2\sec^2\alpha = a^2+(b-x)^2$$

$$\Rightarrow \quad (b-x)^2\tan^2\alpha = a^2$$

$$\Rightarrow \quad b-x = a\cot\alpha$$

$$\Rightarrow \quad x = b - a\cot\alpha$$

By (2), we have

$$v_r\sec\alpha\sqrt{a^2+(b-x)^2}\,\frac{dt}{dx} = -(b-x)\sec\alpha + \sqrt{a^2+(b-x)^2} \quad \ldots(3)$$

Differentiate both sides of (3) again with respect to x, we get

$$v_r\sec\alpha\sqrt{a^2+(b-x)^2}\,\frac{d^2t}{dx^2} + \frac{d}{dt}\left\{v_r\sec\alpha\sqrt{a^2+(b-x)^2}\right\} = \sec\alpha - \frac{(b-x)}{\sqrt{a^2+(b-x)^2}}$$

At $x = b - a\cot\alpha$, $\frac{dt}{dx} = 0$ and $\frac{d^2t}{dx^2}$ is positive.

i.e., t is minimum at $x = b - a\cot\alpha$.

Example 11: Prove that the minimum radius vector of the curve $\frac{a^2}{x^2}+\frac{b^2}{y^2} = 1$ is of length $(a+b)$.

Solution: It is given that

$$\frac{a^2}{x^2}+\frac{b^2}{y^2} = 1 \quad \ldots(1)$$

Changing the variable co-ordinate Cartesian to polars by taking $x = r\cos\theta$, $y = r\sin\theta$ in (1), we get

$$\frac{a^2}{r^2\cos^2\theta}+\frac{b^2}{r^2\sin^2\theta} = 1 \quad \ldots(2)$$

or $$r^2 = a^2 \sec^2\theta + b^2 \operatorname{cosec}^2\theta$$

Let $$R = a^2 \sec^2\theta + b^2 \operatorname{cosec}^2\theta \quad \ldots(3)$$

Differentiate both sides of (3) with respect to θ, we get

$$\frac{dR}{d\theta} = 2a^2 \sec\theta.\ \sec\theta\tan\theta + 2b^2 \operatorname{cosec}\theta(-\operatorname{cosec}\theta\cot\theta)$$

or $$\frac{dR}{d\theta} = 2a^2 \sec^2\theta\tan\theta - 2b^2 \operatorname{cosec}^2\theta\cot\theta \quad \ldots(4)$$

For maxima and minima, we have

$$\frac{dR}{d\theta} = 0$$

$$\Rightarrow \quad 2a^2 \sec^2\theta\tan\theta - 2b^2 \operatorname{cosec}^2\theta\cot\theta = 0$$

$$\Rightarrow \quad \tan^4\theta = \frac{b^2}{a^2}$$

$$\Rightarrow \quad \tan\theta = \sqrt{\frac{b}{a}}.$$

Differentiate both sides of (4) with respect to θ, we get

$$\frac{d^2R}{d\theta^2} = 2a^2[\sec^2\theta\sec^2\theta + 2\sec^2\theta\tan\theta.\ \tan\theta] - 2b^2[\operatorname{cosec}^2\theta(-\operatorname{cosec}^2\theta) - 2\operatorname{cosec}^2\theta\cot^2\theta]$$

$$= 2a^2[\sec^4\theta + 2\sec^2\theta\tan^2\theta] + 2b^2[\operatorname{cosec}^4\theta + 2\operatorname{cosec}^2\theta\cot^2\theta] > 0$$

i.e., R or r^2 is minimum at $\tan\theta = \sqrt{\frac{b}{a}}$

$$\Rightarrow \quad \sin\theta = \sqrt{\frac{b}{a+b}} \text{ and } \cos\theta = \sqrt{\frac{a}{a+b}}$$

By equation (2), we have

$$\frac{a^2}{r^2\left(\frac{a}{a+b}\right)} + \frac{b^2}{r^2\left(\frac{a}{a+b}\right)} = 1$$

$$\Rightarrow \quad r^2 = a(a+b) + b(a+b)$$
$$= (a+b)^2$$

At $\tan\theta = \sqrt{\frac{b}{a}}$, the value of r is $(a + b)$.

Example 12: ADC generator has integral resistance R ohms and has an open circuit voltage of V volts. Find the lead resistance r for which the power delivered by the generator is maximum.

Solution: We know that the ohm's law

$$V = i\,(R + r)$$

$$\Rightarrow \quad i = \frac{V}{R+r}$$

The power generated $P = i^2 r = \dfrac{V^2 r}{(R+r)^2}$ ...(1)

Here V, R being constant.

Differentiate both sides of (1) with respect to r, we get

$$\frac{dP}{dr} = V^2\left[\frac{(R+r)^2.1 - r.2(R+r)}{(R+r)^4}\right]$$

$$= V^2\left[\frac{(R+r)-2r}{(R+r)^3}\right]$$

$$= V^2\left[\frac{R-r}{(R+r)^3}\right] \quad \text{...(2)}$$

For maxima and minima, we have

$$\frac{dP}{dr} = 0$$

$$\Rightarrow \quad V^2\left[\frac{R-r}{(R+r)^3}\right] = 0$$

$$\Rightarrow \quad R = r$$

Differentiate both sides of (2) again with respect to r, we get

$$\frac{d^2P}{dr^2} = V^2\left[\frac{(R+r)^3.(-1)-(R-r).3(R+r)^2}{(R+r)^6}\right]$$

$$= V^2\left[\frac{-4R+2r}{(R+r)^4}\right]$$

At $R = r$, $\dfrac{d^2P}{dr^2} = \dfrac{-V^2}{8R^3}$ (negative) i.e., P is maximum for $r = R$.

Hence, $P_{max} = \dfrac{V^2}{4R}$

Example 13: A given quantity of metal is to be cast into a half cylinder, i.e., with rectangular base and semicircular ends. Show that in order to have minimum surface area, the ratio of the height of the cylinder to the diameter of semicircular ends is $\pi{:}\pi + 2$.

Solution: Suppose the volume of the half cylinder is

$$V = \frac{1}{2}\pi r^2 h \quad \text{...(1)}$$

where r and h are the radius and height of the half cylinder respectively

Surface area of rectangular base = $2rh$

Curved surface = πrh

Two semicircular ends = πr^2

The total surface area

$$S = 2rh + \pi rh + \pi r^2$$
$$= rh\,(2 + \pi) + \pi r^2 \qquad \ldots(2)$$

From equation (1), we have

$$h = \frac{2V}{\pi r^2}$$

Then
$$S = \frac{r.2V}{\pi r^2}(2 + \pi) + \pi r^2$$

$$= \frac{2V}{\pi r}(2 + \pi) + \pi r^2 \qquad \ldots(3)$$

Differentiate both sides of (3) with respect to r, we get

$$\frac{dS}{dr} = 2\pi r - \frac{2V}{\pi r^2}(\pi + 2) \qquad \ldots(4)$$

For maxima and minima we have

$$\frac{dS}{dr} = 0$$

$$\Rightarrow \qquad 2\pi r - \frac{2V}{\pi r^2}(\pi + 2) = 0$$

Using equation (1), we have

$$2\pi r - h(\pi + 2) = 0$$

$$\Rightarrow \qquad \frac{h}{2r} = \frac{\pi}{\pi + 2} \qquad \ldots(5)$$

Differentiate both sides of (4) again with respect to r, we get

$$\frac{d^2S}{dr^2} = 2\pi - \frac{4V}{\pi r^3}(\pi + 2)$$

$$= 6\,\pi \text{ (positive)} \quad \text{i.e., } S \text{ is maximum.}$$

Hence, surface area S is minimum at $\dfrac{h}{2r} = \dfrac{\pi}{\pi + 2}$

Example 14: Find the extreme points of the function $u = x^2 + 4y^2 + 4z^2 + 4xy + 4xz + 16yz$.

Solution: Given that

$$u = x^2 + 4y^2 + 4z^2 + 4xy + 4xz + 16yz \qquad \ldots(1)$$

Differentiate partially both sides of (1) with respect to x, y and z respectively, we get

$$\frac{\partial u}{\partial x} = 2x + 2y + 2z$$

$$\frac{\partial u}{\partial y} = 8y + 4x + 16z$$

and
$$\frac{\partial u}{\partial z} = 8z + 4x + 16y$$

For extreme points, we have

$$\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0 \text{ and } \frac{\partial u}{\partial z} = 0$$

$\Rightarrow$ $$2x + 2y + 2z = 0 \text{ or } 2(x + y + z) = 0$$

$$8y + 4x + 16z = 0 \text{ or } 4(x + 2y + 4z) = 0$$

and $$8z + 4x + 16y = 0 \text{ or } 4(x + 4y + 2z) = 0$$

Solving above these expression, we get $x = 0$, $y = 0$, $z = 0$.

Let the point P be (0, 0, 0)

Differentiate partially (2) and we get

$$\frac{\partial^2 u}{\partial x^2} = 2, \frac{\partial^2 u}{\partial y^2} = 8, \frac{\partial^2 u}{\partial z^2} = 8, \frac{\partial^2 u}{\partial x \partial y} = 4, \frac{\partial^2 u}{\partial y \partial x} = 4, \frac{\partial^2 u}{\partial y \partial z} = 16, \frac{\partial^2 u}{\partial z \partial y} = 16, \frac{\partial^2 u}{\partial x \partial z} = 4,$$

$$\frac{\partial^2 u}{\partial z \partial x} = 4.$$

The Hessian matrix of u (x, y, z) is

$$H = \begin{bmatrix} 2 & 4 & 4 \\ 4 & 8 & 16 \\ 4 & 16 & 8 \end{bmatrix}$$

The leading minors of H are

$$H_1 = |2| = 2, H_2 = \begin{vmatrix} 2 & 4 \\ 4 & 8 \end{vmatrix} = 0 \text{ and } H_3 = \begin{vmatrix} 2 & 4 & 4 \\ 4 & 8 & 16 \\ 4 & 16 & 8 \end{vmatrix} < 0.$$

Here H_1 and H_3 are not of same sign and H_2 = (i.e., semi-definite). Hence u has a saddle point at (0, 0, 0).

Example 15: Find the maximum and minimum value of u, where $u = \sin x \sin y \sin(x + y)$.

Solution: Given that

$$u = \sin x \sin y \sin(x + y) \qquad \ldots(1)$$

Differentiate partially (1) with respect to x and y both sides respectively, we get,

$$\frac{\partial u}{\partial x} = \sin y\, [\sin x \cos(x + y) + \cos x \sin(x + y)] \qquad \ldots(2)$$

and $$\frac{\partial u}{\partial y} = \sin x\, [\sin y \cos(x + y) + \cos y \sin(x + y)] \qquad \ldots(3)$$

For maxima and minima, we have

$$\frac{\partial u}{\partial x} = 0, \text{ and } \frac{\partial u}{\partial y} = 0$$

$\sin y[\sin x \cos(x + y) + \cos x.\ \sin(x + y)] = 0$

and $\sin x[\sin y \cos(x + y) + \cos y,\ \sin(x + y)] = 0$

Solving above these expressions, we get

$$\tan(x+y) = -\tan x \qquad \ldots(4)$$

and
$$\tan(x+y) = -\tan y \qquad \ldots(5)$$

Now, we have

$$\tan 2x = -\tan x = \tan(\pi - x)$$
$$2x = \pi - x$$
$$x = \pi/3 = y$$

Also

$$\sin y = 0 \quad \Rightarrow \quad y = 0$$

and
$$\sin x = 0 \quad \Rightarrow \quad x = 0$$

Thus, the stationary points are $(0, 0)$, $(\pi/3, \pi/3)$.

Differentiate partially (2) and (3) again, we get

$$r = \frac{\partial^2 u}{\partial x^2} = 2\sin y\cos(2x+y)$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = \sin 2(x+y)$$

and
$$t = \frac{\partial^2 u}{\partial y^2} = 2\sin x\cos(2y+x)$$

At $(0, 0)$, we get $r = 0$, $s = 0$, $t = 0$.

$$rt - s^2 = 0, \text{ i.e., } u \text{ has a saddle point at } (0, 0).$$

Now at $(\pi/3.\ \pi/3)$, we get $r = 2\sin(\pi/3)\cos\pi = -\sqrt{3}$.

$$s = \sin(4\pi/3) = -\sin(\pi/3) = -\frac{\sqrt{3}}{2}$$

and
$$t = 2\sin(\pi/3)\sin\pi = -\sqrt{3}$$

$\Rightarrow$
$$rt - s^2 = \frac{9}{4} \text{ is positive and } r < 0.$$

Hence, u is maximum at $(\pi/3, \pi/3)$.

Example 16: Find the extreme points $f(x_1, x_2) = 20x_1 + 26x_2 + 4x_1x_2 - 4x_1^2 - 3x_2^2$.

Solution: Given that

$$f(x_1, x_2) = 20x_1 + 26x_2 + 4x_1x_2 - 4x_1^2 - 3x_2^2 \qquad \ldots(1)$$

Differentiate partially (1) with respect to x_1 and x_2 both sides respectively, we get

$$\frac{\partial f}{\partial x_1} = 20 + 4x_2 - 8x_1 \qquad \ldots(2)$$

and
$$\frac{\partial f}{\partial x_2} = 26 + 4x_1 - 6x_2 \qquad \ldots(3)$$

For extreme points, we have

$$\frac{\partial f}{\partial x_1} = 0 \quad \text{and} \quad \frac{\partial f}{\partial x_2} = 0$$

$\Rightarrow \qquad 20 + 4x_2 - 8x_1 = 0$

And $\qquad 26 + 4x_1 - 6x_2 = 0$

Solving these, we get $x_1 = 7$, $x_2 = 9$.

Differentiating again partially (2) and we get

$$r = \frac{\partial^2 f}{\partial x_1^2} = -8$$

$$s = \frac{\partial^2 f}{\partial x_1 \partial x_2} = 4$$

and $$t = \frac{\partial^2 f}{\partial x_2^2} = -6$$

We have $\quad rt - s^2 = (-8)(-6) - (4)^2 = 32.$

At $\quad (7, 9)$, $rt - s^2 > 0$ and $r < 0$, i.e., f is maximum at $(7,9)$.

Example 17: Find the point (x_1, x_2, x_3) at which is functions

$f(x_1, x_2, x_3) = -x_1^2 - x_2^2 - x_3^2 + x_1x_2 + x_1 + 2x_3$ has optimum values.

Solution: Given that

$$f(x_1, x_2, x_3) = -x_1^2 - x_2^2 - x_3^2 + x_1x_2 + x_1 + 2x_3 \qquad \ldots(1)$$

Differentiate partially (1) with respect to x_1, x_2 and x_3 both sides respectively, we get

$$\frac{\partial f}{\partial x_1} = -2x_1 + 1$$

$$\frac{\partial f}{\partial x_2} = -2x_2 + x_3$$

and $$\frac{\partial f}{\partial x_3} = -2x_3 + x_2 + 2 \qquad \ldots(2)$$

For extreme points, we have

$$\frac{\partial f}{\partial x_1} = 0, \ \frac{\partial f}{\partial x_2} = 0 \text{ and } \frac{\partial f}{\partial x_3} = 0$$

$\Rightarrow \qquad -2x_1 + 1 = 0$

$\qquad -2x_2 + x_3 = 0$

and $\qquad -2x_3 + x_2 + 2 = 0$

Solving these we get $x_1 = ½$, $x_2 = 2/3$, $x_3 = 4/3$.

Let the point P be $\qquad = \left(\frac{1}{2}, \frac{2}{2}, \frac{4}{3}\right).$

Differentiate partially (2) again and we get

$$\frac{\partial^2 f}{\partial x_1^2} = -2, \ \frac{\partial^2 f}{\partial x_2^2} = -2, \ \frac{\partial^2 f}{\partial x_3^2} = -2, \ \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0, \ \frac{\partial^2 f}{\partial x_2 \partial x_1} = 0, \ \frac{\partial^2 f}{\partial x_1 \partial x_3} = 0, \ \frac{\partial^2 f}{\partial x_3 \partial x_1} = 0,$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_3} = 1, \ \frac{\partial^2 f}{\partial x_3 \partial x_2} = 1$$

At $\quad (a, a)$, we get $r = 2, s = 1, t = 2$.

Then $\quad rt - s^2 = 3 > 0$

Since at (a, a), $rt - s^2 > 0$ and $r > 0$, then u is minimum at (a, a).

The minimum value of $u = a.\, a + \dfrac{a^3}{a} + \dfrac{a^3}{a} = 3a^2$.

Example 19: Find the extreme points of the function $f(x_1, x_2) = -x_1^3 + 2x_2^3 + 3x_1^2 + 12x_2^2 + 24$. And determine their nature also.

Solution: Given that

$$f(x_1, x_2) = x_1^3 + 2x_2^3 + 3x_1^2 + 12x_2^2 + 24 \qquad \ldots(1)$$

Differentiate partially (1) with respect to x_1 and x_2 both sides respectively, we get

$$\frac{\partial f}{\partial x_1} = 3x_1^2 + 6x_1 \qquad \ldots(2)$$

and

$$\frac{\partial f}{\partial x_2} = 6x_2^2 + 24x_2 \qquad \ldots(3)$$

For extreme points, we have

$$\frac{\partial f}{\partial x_1} = \text{and } \frac{\partial f}{\partial x_2} = 0$$

$\Rightarrow \quad 3x_1^2 + 6x_1 = 0$

And $\quad 6x_2^2 + 24x_2 = 0$

Solving these, we get $x_1 = 0, -2$, $x_2 = 0, -4$.

Differentiating again partially (2) and we get

$$r = \frac{\partial^2 f}{\partial x_1^2} = 6(x_1 + 1)$$

$$s = \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$$

and

$$t = \frac{\partial^2 f}{\partial x_2^2} = 12(x_2 + 2)$$

At $\quad (0, 0)$ $rt - s^2 = 72(x_1 + 1)(x_2 + 2) = 72 > 0$ and $r > 0$ i.e., f is minimum at $(0, 0)$.

At $\quad (0, -4)$ $rt - s^2 = 72(x_1 + 1)(x_2 + 2) = -144 < 0 \Rightarrow$ no extreme point, i.e., f has a saddle point at $(0, -4)$.

At $\quad (-2, 0)$ $rt - s^2 = 72(x_1 + 1)(x_2 + 2) = -144 < 0 \Rightarrow$ no extreme point, i.e., f has a saddle point at $(-2, 0)$.

At $\quad (-2, -4)$ $rt - s^2 = 72(x_1 + 1)(x_2 + 2) = 144 > 0$ and $r < 0$ i.e., f is maximum at $(-2, -4)$.

Example 20: Find the dimension of a box of the largest volume that can be inscribed in a sphere of radius 3 meters.

Solution: Let the volume of the box be

$$V = 2x.\, 2y.\, 2z \quad \Rightarrow \quad V = 8xyz \qquad \ldots(1)$$

Changing the variables Cartesian to spherical polar co-ordinates as $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and using $r = 3$ (given).

$$V = 216\sin^2\theta\cos\theta\cos\phi\sin\phi$$
$$= 108\sin^2\theta\cos\theta\sin 2\phi \qquad \ldots(2)$$

Differentiate partially (2) with respect to θ and ϕ both sides respectively, we get

$$\frac{\partial V}{\partial \theta} = 108\sin 2\phi\,[\sin^2\theta(-\sin\theta) + 2\sin\theta\cos^2\theta]$$
$$= 108\sin 2\phi\sin\theta\{-\sin^2\theta + 2\cos^2\theta\} \qquad \ldots(3)$$

and $$\frac{\partial V}{\partial \phi} = 216\cos 2\phi\sin^2\theta\cos\theta \qquad \ldots(4)$$

For maxima and minima, we have

$$\frac{\partial V}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial V}{\partial \phi} = 0$$

$\Rightarrow$ $$108\sin 2\phi\sin\theta\{-\sin^2\theta + 2\cos^2\theta\} = 0$$

i.e., $$\sin\theta = 0 \text{ or } \tan\theta = \sqrt{2} \text{ or } \sin\phi = 0$$

i.e., $$\theta = 0,\ \tan^{-1}\sqrt{2},\ \phi = 0.$$

and $$216\cos 2\phi\sin^2\theta\cos\theta = 0$$

i.e., $$\theta = 0,\ \pi/4.$$

Because $\theta = 0$ or $\phi = 0$ gives $V = 0$, so we take only $\theta = \tan^{-1}\sqrt{2}$ and $\phi = \pi/4$.

Differentiate partially (3) again and we get

$$\frac{\partial^2 V}{\partial \theta^2} = 108\sin 2\phi[\sin\theta(-4\cos\theta\sin\theta - 2\sin\theta\cos\theta) + \cos\theta(2\cos^2\theta - \sin^2\theta)]$$

At $(\tan^{-1}\sqrt{2}, \pi/4)$, $$\frac{\partial^2 V}{\partial \theta^2} = 108\left[\frac{\sqrt{2}}{\sqrt{3}}\left(-6\times\frac{1}{\sqrt{3}}\times\frac{\sqrt{2}}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}}\left(1\left(\frac{1}{\sqrt{3}}\right)^2 - \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2\right)\right]$$
$$= -\frac{432}{\sqrt{3}}$$

$$\frac{\partial^2 V}{\partial \theta\,\partial \phi} = 216\cos 2\phi[\sin^2\theta(-\sin\theta) + \cos^2\theta(2\sin\theta)]$$
$$= 216\cos 2\phi\sin\theta[2\cos^2\theta - \sin^2\theta]$$

At $(\tan^{-1}\sqrt{2}, \pi/4)$, $$\frac{\partial^2 V}{\partial \theta\,\partial \phi} = 0$$

$$\frac{\partial^2 V}{\partial \phi^2} = -432\sin 2\phi[\sin^2\theta\cos\theta]$$

At $(\tan^{-1}\sqrt{2}, \pi/4)$, $$\frac{\partial^2 V}{\partial \phi^2} = -432\times\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2\times\left(\frac{1}{\sqrt{3}}\right) = -\frac{864}{3\sqrt{3}}$$

4. At $x = 400$.
6. At $x = 0$ give point of inflection; at $x = 1$ give maxima and value is 12; at $x = 2$ give minima and value is -11.
7. At $x = 6/5$ $f(x)$ give maxima.
8. Diameter = 3.93 meters, length = 4.12 meters.
10. At $x = 1$ give point of inflexion, at $x = 2$ give local maxima and $x = 3$ give local minima.
12. $y = x = C/4$, i.e., square.
13. At $h = C/\sqrt{3}$ volume $= \dfrac{2\pi c^3}{9\sqrt{3}}$.
15. Speed 40 km/hour.
17. The intensity is maximum for $\tan\theta = \sqrt{2}$ height $= 25\sqrt{2}$ meters.
20. Height = 40.423 mm, length = 216.154 mm, width = 129.154 mm and maximum volume $= 1128.5\ \text{cm}^3$.
21. At (0, 0), (0, a) and (a, 0) u is neither maxima nor minima. At $\left(\dfrac{a}{3}, \dfrac{a}{3}\right) = u$ is minimum if $a < 0$ and u is maximum if $a > 0$.
22. At (1, 2) u is minimum and at $(-1, -2)$ u is maximum.
23. At (0, 0) f is minimum and $\left(-\dfrac{4}{3}, -\dfrac{8}{3}\right)$ f is maximum.
24. At $\left(\sqrt{2}, \sqrt{-2}\right)$ u is maximum.
25. At $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$ f is maximum.
26. At $x = y = z = (2\, v_0)^{1/3}$ give minima.
27. At $\left(-\dfrac{2}{3}, -\dfrac{1}{3}, 1\right)$ u is minimum.
28. At (0, 0) u give saddle point.

2.4 CONSTRAINED MULTIVARIABLE OPTIMIZATION PROBLEMS WITH EQUALITY CONSTRAINTS

The optimization problem of a continuous and differentiable function subject to equality constraints:

Optimize (Max or Min) $Z = f(X)$

Subject to constraints (s.t.) $g_j(X) = 0$; $j = 1, 2, 3, \ldots, mg$

where
$$X = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_n \end{pmatrix}$$

Here m is less than or equal to n. There are several methods for solving this type of problem. Here we discuss only two methods:

1. Direct substitution method
2. Lagrange multipliers method

2.4.1 Direct Substitution Method

In this method, the value of any variable from the constraint set is put into the objective function. The problem reduces to unconstrained optimization problem and can be solved by unconstrained optimization method.

2.4.2 Lagrange Multiplier Method

Consider a general problem with n variables and m equality constraints:

Optimize $Z = f(X)$

s.t. $g_j(X) = 0$; $j = 1, 2, 3,\ldots, m$ $(m < n)$

$X \geq 0$

where
$$X = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_n \end{pmatrix}$$

Now we define a function

$$L(x_1, x_2,\ldots, x_n, \lambda_1, \lambda_2,\ldots, \lambda_m) = f(X) + \sum_{j=1}^{m} \lambda_j g_j (X) \qquad \ldots(1)$$

Here, $\lambda_1, \lambda_2, \ldots, \lambda_m$ are known as Lagrange's undetermined multipliers.

The necessary conditions for extreme of L are

$$\frac{\partial L}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0 \qquad \ldots(2)$$

and
$$\frac{\partial L}{\partial \lambda_j} = 0; \begin{pmatrix} i = 1, 2,\ldots n \\ j = 1, 2,\ldots m \end{pmatrix} \qquad \ldots(3)$$

Solving equation (4) and (5), we get

$$X = \begin{pmatrix} x_1^* \\ x_2^* \\ \cdots \\ \cdots \\ x_m^* \end{pmatrix} \text{ and } \lambda^* = \begin{pmatrix} x_1^* \\ x_2^* \\ \cdots \\ \cdots \\ x_m^* \end{pmatrix}$$

The sufficient condition for the function to have extreme at point X^*, is that the values of k obtained from equation

$$\begin{vmatrix} L_{11}-k & L_{12} & \dots & L_{1n} & g_{11} & g_{21} & \dots & g_{m1} \\ L_{21} & L_{22}-k & \dots & L_{2n} & g_{12} & g_{22} & \dots & g_{m2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ L_{n1} & L_{n2} & \dots & L_{nn}-k & g_{1n} & g_{2n} & \dots & g_{mn} \\ g_{11} & g_{12} & \dots & g_{1n} & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ g_{m1} & g_{m2} & \dots & g_{mn} & 0 & 0 & \dots & 0 \end{vmatrix} = 0$$

must be of the same sign. If all the eigen values of k are negative then it is a maxima and if all the eigen values k are positive then it is a minima. But if some eigen values are zero or of different sign then that is a saddle point. In above L_{ij} and g_{ij} denoted by $\frac{\partial^2 L}{\partial x_i \partial x_j}$ and $\frac{\partial g_j}{\partial x_i}$ respectively.

2.5. CONSTRAINED MULTIVARIABLE OPTIMIZATION PROBLEMS WITH INEQUALITY CONSTRAINTS

Let us consider a problem

Optimization (Max or Min) $Z = f(X)$ ...(1)

s.t. $g_j(X) \le 0$; $j = 1, 2, 3, \dots, m$ ...(2)

where
$$X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ \dots \\ x_n \end{pmatrix}$$

The inequality constraints in equation (2) can be converted into equality constraints by adding slack variables as

$$g_j(X) + y_j^2 = 0; j = 1, 2, 3, \dots, m$$

Now the problem becomes

Optimize (Max or Min) $Z = f(X)$

s.t.
$$G_j(X, Y) = g_j(X) + y_j^2 = 0; j = 1, 2, 3, \dots, m$$

where
$$X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ \dots \\ x_n \end{pmatrix} \text{ and } Y = \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ \dots \\ y_m \end{pmatrix}$$

Now it can be solved by Lagrange's multipliers method.

$$\frac{\partial L}{\partial \lambda_j} = 0$$

$\Rightarrow$ $$g_j(X) + y_j^2 = 0; j = 1, 2, 3,\ldots, m \qquad \ldots(6)$$

and $$\frac{\partial L}{\partial y_j} = 0$$

$\Rightarrow$ $$2\lambda_j y_j = 0; j = 1, 2, 3,\ldots, m \qquad \ldots(7)$$

From (6) and (7), we have

$$\lambda_j g_j(X) = 0$$

$\Rightarrow$ $$\lambda_j = 0$$

or $$g_j(X) = 0.$$

Case I: If $g_j(X) = 0$ at the optimum point then it is called the active constraint and we can find optimum solution.

Case II: If $\lambda_j = 0$ at the optimum point then it is called inactive constraints.

Note: If the given optimization problem is a minimization problem with constraints of the form $g_j(X) \geq 0$ then $\lambda_j \leq 0$ but if the problem is a maximization problem with constraints of the form $g_j(X) \leq 0$ then $\lambda_j \leq 0$. Let us consider some maximization or minimization problems given in the following terms.

(i) Maximize $Z = f(X)$

s.t. $g_j(X) \leq 0$; $j = 1, 2, 3,\ldots, m$

For the function $f(X)$ to have maxima, we have

$$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0; i = 1, 2, 3,\ldots, m$$

$$\lambda_j g_j(X) = 0; j = 1, 2, 3,\ldots, m$$

and $$\lambda_j \leq 0.$$

(ii) Maximize $Z = f(X)$

s.t. $g_j(X) \geq 0$; $j = 1, 2, 3,\ldots, m$

For the function $f(X)$ to have maxima, we have

$$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0; i = 1, 2, 3,\ldots, m$$

$$\lambda_j g_j(X) = 0; j = 1, 2, 3,\ldots, m$$

and $$\lambda_j \geq 0.$$

(iii) Minimize $Z = f(X)$

s.t. $g_j(X) \leq 0$; $j = 1, 2, 3,\ldots, m$

For the function $f(X)$ to have maxima, we have

$$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} = 0; i = 1, 2, 3, \ldots, m$$

$$\lambda_j g_j(X) = 0; j = 1, 2, 3, \ldots, m$$

and $$\lambda_j \leq 0.$$

(iv) Minimize $Z = f(X)$

s.t. $g_j(X) \geq 0; j = 1, 2, 3, \ldots, m$

For the function $f(X)$ to have maxima, we have

$$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} \; ; i = 1, 2, 3, \ldots, m$$

$$\lambda_j g_j(X) = 0; j = 1, 2, 3, \ldots, m$$

and $$\lambda_j \leq 0.$$

SOLVED EXAMPLES

Example 21: Find the optimum solution of the following constrained multivariable problem minimize $Z = x_1^2 + (x_2 + 1)^2 + (x_3 - 1)^2$ s.t. $x_1 + 5x_2 - 3x_3 = 6$.

Solution: Given that

$$Z = x_1^2 + (x_2 + 1)^2 + (x_3 - 1)^2 \qquad \ldots(1)$$

and $$x_1 + 5x_2 - 3x_3 = 6$$

$$\Rightarrow \qquad x_3 = \frac{x_1 + 5x_2 + 6}{3} \qquad \ldots(2)$$

Using (1) and (2), we get

$$Z = x_1^2 + (x_2 + 1)^2 + \frac{1}{9}(x_1 + 5x_2 - 9)^2 \qquad \ldots(3)$$

Differentiate partially both sides of (3) with respect to x_1 and x_2 respectively, we get

$$\frac{\partial Z}{\partial x_1} = 2x_1 + \frac{2}{9}(x_1 + 5x_2 - 9)$$

and $$\frac{\partial Z}{\partial x_2} = 2(x_2 + 1) + \frac{10}{9}(x_1 + 5x_2 - 9)$$

For maxima and minima, we have

$$\frac{\partial Z}{\partial x_1} = 0 \quad \text{and} \quad \frac{\partial Z}{\partial x_2} = 0$$

$$\Rightarrow \qquad 2x_1 + \frac{2}{9}(x_1 + 5x_2 - 9) = 0$$

and $$2(x_2 + 1) + \frac{10}{9}(x_1 + 5x_2 - 9) = 0$$

Solving these, we get

$$x_1 = \frac{2}{5} \text{ and } x_2 = 1$$

Differentiate again partially (4) and we get

$$r = \frac{\partial^2 Z}{\partial x_1^2} = 2 + \frac{2}{9} = \frac{20}{9}$$

$$t = \frac{\partial^2 Z}{\partial x_2^2} = 2 + \frac{50}{9} = \frac{68}{9}$$

and
$$\frac{\partial^2 Z}{\partial x_1 x_2} = \frac{10}{9}$$

At $\left(\frac{2}{5}, 1\right)$,
$$rt - s^2 = \left(\frac{20}{9}\right)\left(\frac{68}{9}\right) - \left(\frac{10}{9}\right)^2 = 1260 > 0 \text{ and } r > 0$$

i.e., Z is minimum at $\left(\frac{2}{5}, 1\right)$ the minimum value of Z is $\frac{28}{5}$.

Example 22: Find the dimensions of a box of large volume that can be inscribed in a sphere of radius a.

Solution: Suppose x, y and z are the dimensions of the box with respect to origin O and OX, OY, OZ are reference axes. The volume of box is

$$V = 8xyz \qquad \ldots(1)$$

Given that the box is to be inscribed in a sphere of radius 'a' i.e.,

$$x^2 + y^2 + z^2 = a^2 \qquad \ldots(2)$$

Eliminating z from (1) and (2), we get

$$V = 8xy\,(a^2 - x^2 - y^2)^{1/2} \qquad \ldots(3)$$

Differentiate partially (3) with respect to x and y both sides respectively, we get

$$\frac{\partial V}{\partial x_1} = 8y\left[x \cdot \frac{1}{2}(a^2 - x^2 - y^2)^{-1/2}(-2x) + (a^2 - x^2 - y^2)^{1/2}\right]$$

$$= 8y\left[-\frac{x^2}{(a^2 - x^2 - y^2)^{1/2}} + (a^2 - x^2 - y^2)^{1/2}\right]$$

$$= 8y\left[\frac{(a^2 - 2x^2 - y^2)}{(a^2 - x^2 - y^2)^{1/2}}\right] \qquad \ldots(4)$$

and
$$\frac{\partial V}{\partial y} = 8x\left[\frac{(a^2 - x^2 - 2y^2)}{(a^2 - x^2 - y^2)^{1/2}}\right] \qquad \ldots(5)$$

For maxima and minima, we have

$$\frac{\partial V}{\partial x_1} = 0 \text{ and } \frac{\partial V}{\partial y} = 0$$

$\Rightarrow$
$$8y\left[\frac{(a^2 - 2x^2 - y^2)}{(a^2 - x^2 - y^2)^{1/2}}\right] = 0$$

and
$$8x\left[\frac{(a^2 - x^2 - 2y^2)}{(a^2 - x^2 - y^2)^{1/2}}\right] = 0$$

The necessary conditions for extreme L are

$$\left.\begin{aligned}\frac{\partial L}{\partial x} &= 2x + 4\lambda = 0\\ \frac{\partial L}{\partial y} &= 2y + 2y\lambda = 0\\ \frac{\partial L}{\partial z} &= 2z + 2\lambda = 0\\ \frac{\partial L}{\partial \lambda} &= 4x + y^2 + 2z - 14 = 0\end{aligned}\right\} \quad \ldots(4)$$

Solving these we get $x = -2\lambda$, $z = -\lambda$ and $\lambda = -1$

$\Rightarrow$ $x = 2$, $z = 1$ and $y = \pm 2$, putting $x = -2\lambda$, $z = -\lambda$, $y = 0$ in (2), we get $\lambda = -1.4$.

Here $(2, 2, 1, -1)$, $(2, -2, 1, -1)$ and $(-2\lambda, 0, -\lambda, \lambda)$ or $(2.8, 0, 1.4, -1.4)$ are extreme points.

Differentiate again partially (4) and we get

$$\frac{\partial^2 L}{\partial x^2} = 2,\ \frac{\partial^2 L}{\partial x \partial y} = 0,\ \frac{\partial^2 L}{\partial x \partial z} = 0,\ \frac{\partial^2 L}{\partial y \partial x} = 0,\ \frac{\partial^2 L}{\partial y^2} = 2 + 2\lambda,\ \frac{\partial^2 L}{\partial y \partial z} = 2,\ \frac{\partial^2 L}{\partial z \partial x} = 0,\ \frac{\partial^2 L}{\partial z \partial y}$$

$$= 0,\ \frac{\partial^2 L}{\partial z^2} = 2,\ \frac{\partial g}{\partial x} = 4,\ \frac{\partial g}{\partial y} = 2y,\ \frac{\partial g}{\partial z} = 2.$$

The sufficient condition for the extreme point is

$$H = \begin{vmatrix} \frac{\partial^2 L}{\partial x^2} - k & \frac{\partial^2 L}{\partial x \partial y} & \frac{\partial^2 L}{\partial x \partial z} & \frac{\partial g}{\partial x} \\ \frac{\partial^2 L}{\partial yx \partial x} & \frac{\partial^2 L}{\partial y^2} - k & \frac{\partial^2 L}{\partial y \partial z} & \frac{\partial g}{\partial y} \\ \frac{\partial^2 L}{\partial z \partial x} & \frac{\partial^2 L}{\partial z \partial y} & \frac{\partial^2 L}{\partial z^2} - k & \frac{\partial g}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} & 0 \end{vmatrix} = 0$$

i.e.,
$$H = \begin{vmatrix} 2-k & 0 & 0 & 4 \\ 0 & 2+2\lambda-k & 0 & 2y \\ 0 & 0 & 2-k & 2 \\ 4 & 2y & 2 & 0 \end{vmatrix} = 0$$

or $4(2 - k)[-10 + 5k - 10\lambda - 2y^2 + y^2k] = 0$

At $(2, 2, 1, -1)$, equation (2) we have

$$(2 - k)(-10 + 5k + 10 - 8 + 4k) = 0$$

$$\Rightarrow \qquad k = 2,\ 8/9$$

i.e., values of k are positive then there is a minima.

At $(2, -2, 1, -1)$, equation (2) we have

$$(2 - k)(-10 + 5k + 10 - 8 + 4k) = 0$$

$$\Rightarrow \qquad k = 2, 8/9$$

Also values of k are positive then there is a minima.

At $(2.8, 0, 1.4, -1.4)$, from (2) we have

$$(2 - k)(-10 + 5k + 14 - 0 + 0) = 0$$

$$\Rightarrow \qquad k = 2, -4/5.$$

i.e., value of k are positive and negative (neither maxima and nor minima) i.e., saddle point.

Example 24: Minimize $f(x) = \dfrac{1}{2}(x_1^2 + x_2^2 + x_3^2)$

s.t. $g_1(x) = x_1 - x_2 = 0$

and $g_2(x) = x_1 + x_2 + x_3 - 1 = 0$ by Lagrange multiplier method.

Solution: It is given that

Minimize
$$f(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) \qquad \ldots(1)$$

s.t.
$$g_1(x) = x_1 - x_2 = 0 \qquad \ldots(2)$$

and
$$g_2(x) = x_1 + x_2 + x_3 - 1 = 0 \qquad \ldots(3)$$

The Lagrangian function L is

$$L(x_1, x_2, x_3; \lambda_1, \lambda_2) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) + \lambda_1(x_1 - x_2) + \lambda_2(x_1 + x_2 + x_3 - 1) \qquad \ldots(4)$$

The necessary conditions for extreme of L are

$$\left.\begin{aligned} \frac{\partial L}{\partial x_1} &= x_1 + \lambda_1 + \lambda_2 = 0 \\ \frac{\partial L}{\partial x_2} &= x_2 - \lambda_1 + \lambda_2 = 0 \\ \frac{\partial L}{\partial x_3} &= x_3 + \lambda_2 = 0 \\ \frac{\partial L}{\partial \lambda_1} &= x_1 - x_2 = 0 \\ \frac{\partial L}{\partial \lambda} &= x_1 + x_2 + x_3 - 1 = 0 \end{aligned}\right\} \qquad \ldots(5)$$

Solving these, we get

$$x_1 = x_2 = x_3 = \frac{1}{3}; \quad \lambda_1 = 0, \lambda_2 = -\frac{1}{3}$$

$$x + y \geq 80,$$
$$x + y + z \geq 120.$$
$$g_1 = x - 40,\ g_2 = x + y - 80,\ g_3 = x + y + z - 120 \qquad \ldots(6)$$

Using equations (4), (5) and (6), we have

$$L = x^2 + y^2 + z^2 + 20x + 10y + \lambda_1(x - 40 - y_1^2) + \lambda_2(x + y - 80 - y_2^2) + \lambda_3(x + y + z - 120 - y_3^2) \qquad \ldots(7)$$

The Kuhn-Tucker necessary conditions for minimization of L (with $g_j(X) \geq 0$) are

$$\frac{\partial L}{\partial x} = 0,\ \frac{\partial L}{\partial y} = 0,\ \frac{\partial L}{\partial z} = 0$$
$$\lambda_j g_j = 0; \quad j = 1, 2, 3$$
$$\lambda \geq 0; j = 1, 2, 3$$

Differentiate partially (7) and we get

$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= 2x + 20 + \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ \frac{\partial L}{\partial y} &= 2y + 10 + \lambda_2 + \lambda_3 = 0 \\ \frac{\partial L}{\partial z} &= 2z + \lambda_3 = 0 \end{aligned}\right\} \qquad \ldots(8)$$

$$\left.\begin{aligned} \lambda_1 g_1 &= \lambda_1(x - 40) = 0 \\ \lambda_2 g_2 &= \lambda_2(x + y - 80) = 0 \\ \lambda_3 g_3 &= \lambda_3(x + y + z - 120) = 0 \end{aligned}\right\} \qquad \ldots(9)$$

$$\lambda_1, \lambda_2, \lambda_3, \geq 0 \qquad \ldots(10)$$

If $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$ from (9) we have

$$\left.\begin{aligned} (x - 40) &= 0 \\ (x + y - 80) &= 0 \\ (x + y + z - 120) &= 0 \end{aligned}\right\}$$

Solving these we get

$$x = 40,\ y = 40,\ z = 40.$$

Using values of x, y and z. From equation (6), we get

$$\lambda_1 = -10,\ \lambda_2 = -10,\ \lambda_3 = -80$$

Hence, the condition $\lambda_j \leq 0$ from equation (10) is satisfied.

Hence, the optimum solution is

$$x = y = z = 40.$$

and minimize $f = (40)^2 + (40)^2 + (40)^2 + 20(40) + 10(4)$

$$= 6000.$$