JAIPUR ENGINEERING COLLEGE AND RESEARCH CENTRE

Year & Sem – I Year & I Sem

Subject –Engineering Mathematics-I

Unit -3

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CONTENTS (TO BE COVERED)

Half Range Fourier Series

Half-Range Series

When it is required to represent a function $f(x)$ by a Fourier series in $0 \le x \le \pi$ or $0 \le x \le c$ (and not in $-\pi \le x \le \pi$ or $-c \le x \le c$) then $f(x)$ can be expressed either in series of Cosines,

$$
\left.\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c}
\end{aligned}\right\} \quad \text{---(i)}
$$

or in series of sines.

$$
\left.\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} b_n \sin nx \\
f(x) &= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c}
\end{aligned}\right\} \quad \text{---(ii)}
$$

As $f(x)$ is not defined in $-\pi \le x < 0$ or $-c \le x < 0$, It can be choosen arbitrarily in these Interval.

If we choose $f(-x) = f(x)$ we get an even function for which $b_n = 0$

So Fourier series is a cosine series In such a case

$$\begin{cases} a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\,dx \,, \quad a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\cos nx\,dx \quad \text{— (3)} \\ \text{and in } 0 \le x \le c \\ a_0 = \dfrac{2}{c}\displaystyle\int_0^{\pi} f(x)\,dx \,, \quad a_n = \dfrac{2}{c}\displaystyle\int_0^{\pi} f(x)\cos\dfrac{n\pi x}{c}\,dx \end{cases}$$

In That case Fourier series is a cosine series as in (1)

Again If The function $f(x)$ in $-\pi \le x < 0$ or in $-c \le x < 0$ is chosen in such a way That

$$f(-x) = -f(x)$$

we get an odd function for which

$$a_0 = 0 = a_n$$

So the F.S is a sine series as in (2)

In such a case

$$\begin{cases} b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\sin nx\,dx\,, \quad 0 \le x \le \pi \\ b_n = \dfrac{2}{c}\displaystyle\int_0^{c} f(x)\sin \dfrac{n\pi x}{c}\,dx\,, \quad 0 \le x \le c \end{cases} \quad \text{--- (4)}$$

Series (1) and (2) are respectively known as half-range Cosine series (Note)

and half range sine series (Note)

Ex:— Find the half range Cosine series for The following function.

$$f(x) = (x-1)^2, \quad 0 < x < 1$$

Hence show That—

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Sol:— here half range is $0<x<1$ $\therefore$ $c=1$

$\therefore$ half range Cosine Series for $f(x)$ is written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1} \left(a_n \cos \frac{n\pi x}{c}\right) \quad \text{for } 0<x<c$$

$$\Rightarrow (x-1)^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\pi x) \text{ for } 0<x<1 \quad \text{--- (1)}$$

where $a_0 = \frac{2}{c}\int_0^c f(x)\,dx = 2\int_0^1 (x-1)^2 dx$

$$\therefore \quad a_0 = \left[\frac{(x-1)^3}{3}\right]_0^1 = \frac{2}{3} \quad \text{--- (2)}$$

and $\quad a_n = \frac{2}{c}\int_0^c f(x)\cos\left(\frac{n\pi x}{c}\right)dx$

$$= \frac{2}{1}\int_0^1 (x-1)^2 \cos n\pi x\, dx$$

$$= 2\left[(x-1)^2\frac{\sin n\pi x}{n\pi} + 2(x-1)\frac{\cos n\pi x}{(n\pi)^2} - 2\frac{\sin n\pi x}{(n\pi)^3}\right]_0^1$$

$$= 2\left[\frac{2}{(n\pi)^2}\right] = \frac{4}{n^2\pi^2}$$

$$\Rightarrow a_n = \frac{4}{n^2\pi^2} \quad - - - (3)$$

Now substituting (2) and (3) in (1) we get

$$f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \cos n\pi x$$

$$\text{or } (x-1)^2 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi x \quad - - (4)$$

$$\text{or } (x-1)^2 = \frac{1}{3} + \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos \pi x + \frac{1}{2^2}\cos 2\pi x + \cdots\right]$$

putting $x = 0$

$$1 = \frac{1}{3} + \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\Rightarrow \frac{2}{3} = \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\Rightarrow \frac{\pi^2}{6} = \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] \quad \text{--- (5)}$$

Also putting $x = 1$ in (4)

$$0 = \frac{1}{3} + \frac{4}{\pi^2}\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right]$$

or $$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - \cdots \quad \text{--- (6)}$$

Adding (5) and (6), we get

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \quad \text{Proved}$$

Ex:- Find the half range sine series for the function $f(x) = 2x - 1 \quad 0 < x < 1$

Sol:- since $c = 1$

The half range sine series for f(x) is given as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{c}\right); \quad 0 < x < c$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x; \quad 0 < x < 1 \quad \text{---(i)}$$

where $b_n = \frac{2}{c}\int_0^c f(x) \sin\left(\frac{n\pi x}{c}\right) dx$

$$= 2\int_0^1 (2x-1) \sin n\pi x \cdot dx$$

$$= 2\left[-(2x-1)\frac{\cos n\pi x}{n\pi} + \frac{2\sin n\pi x}{(n\pi)^2}\right]_0^1$$

$$= 2\left[-\frac{\cos n\pi}{n\pi} - \frac{1}{n\pi}\right] = -\frac{2}{n\pi}\left[1+\cos n\pi\right]$$

$$\Rightarrow b_n = -\frac{2}{n\pi}\left[1+(-1)^n\right]$$

$$\therefore b_n = 0 \quad \text{when } n \text{ is odd}$$

$$= \frac{-4}{n\pi} \quad \text{when } n \text{ is even} \qquad \text{— (2)}$$

using (2) in (1)

$$f(x) = \sum_{n=1}^{\infty} \left\{ \frac{-2}{n\pi} \left[1 + (-1)^n \right] \right\} \sin n\pi x$$

$$\Rightarrow \quad 2x - 1 = -\frac{4}{\pi} \left[\frac{\sin 2\pi x}{2} + \frac{\sin 4\pi x}{4} + \cdots \right]$$

$$\Rightarrow \quad 2x - 1 = -\frac{2}{\pi} \left[\sin 2\pi x + \frac{\sin 4\pi x}{2} + \cdots \right] \quad \underline{\text{Ans}}$$

Ex:- Find half range sine series for

$$f(x) = x(\pi - x) \quad 0 \le x \le \pi$$

Sol:- The half range sine series for $f(x)$ is given as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad 0 \le x \le \pi \quad \text{—(1)}$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

$$= \frac{2}{\pi}\int_0^{\pi} x(\pi - x) \sin nx \, dx$$

$$= \frac{2}{\pi}\left[-(\pi x - x^2)\frac{\cos nx}{n} + (\pi - 2x)\frac{\sin nx}{n^2} - 2\frac{\cos nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2\cos n\pi}{n^3} + \frac{2}{n^3}\right]$$

$$= -\frac{4}{\pi n^3}\left[(-1)^n - 1\right]$$

$$b_n = 0 \quad \text{when } n \text{ is even}$$
$$\quad = \frac{8}{\pi n^3} \quad \text{when } n \text{ is odd} \qquad \text{--- (2)}$$

using (2) in (1),

$$f(x) = \sum_{n=1}^{\infty} \left\{ -\frac{4}{\pi n^3}\left[(-1)^n - 1\right] \right\} \sin nx$$

$$\therefore\ x(\pi - x) = \frac{8}{\pi}\left[\sin x + \frac{\sin 3x}{3^3} + \cdots\right] \quad \underline{\text{Ans}}$$

Practice Problems

① Find the Fourier series to represent

$$f(x) = x^2 - 2 \qquad -2 < x < 2$$

$$a_0 = -\frac{4}{3}, \quad a_n = \frac{16}{n^2\pi^2}\cos n\pi \qquad \left\{ \begin{array}{l} \sin\frac{n\pi x}{2} = 0 \\ \text{when } x = 2 \text{ or } 0 \end{array} \right\}$$

② Obtain the Fourier series of the periodic function whose definition in one period is

$$f(x) = 0 \quad \text{for } -2 < x < 0$$
$$= 1 \quad \text{for } 0 < x < 2$$

Ans $f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \cdots\right)$

Refrences

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Thank you!