JAIPUR ENGINEERING COLLEGE AND RESEARCH CENTRE

Year & Sem. – B. Tech I year, Sem.-I

Subject –Engineering Mathematics

Unit – 3

Presented by – Dr. Ruchi Mathur & Dr. Tripati Gupta

Designation - Associate Professor

Department - Mathematics

VISION OF INSTITUTE

To become a renowned centre of outcome based learning, and work towards academic, professional, cultural and social enrichment of the lives of individuals and communities.

MISSION OF INSTITUTE

❖**Focus on evaluation of learning outcomes and motivate students to inculcate research aptitude by project based learning.**

❖**Identify, based on informed perception of Indian, regional and global needs, the areas of focus and provide platform to gain knowledge and solutions.**

❖**Offer opportunities for interaction between academia and industry.**

❖**Develop human potential to its fullest extent so that intellectually capable and imaginatively gifted leaders may emerge in a range of professions.**

Engineering Mathematics: Course Outcomes

Students will be able to:

CO1. Understand fundamental concepts of improper integrals, beta and gamma functions and their properties. Evaluation of Multiple Integrals in finding the areas, volume enclosed by several curves after its tracing and its application in proving certain theorems.

CO2. Interpret the concept of a series as the sum of a sequence, and use the sequence of partial sums to determine convergence of a series. Understand derivatives of power, trigonometric, exponential, hyperbolic, logarithmic series.

CO3. Recognize odd, even and periodic function and express them in Fourier series using Euler's formulae.

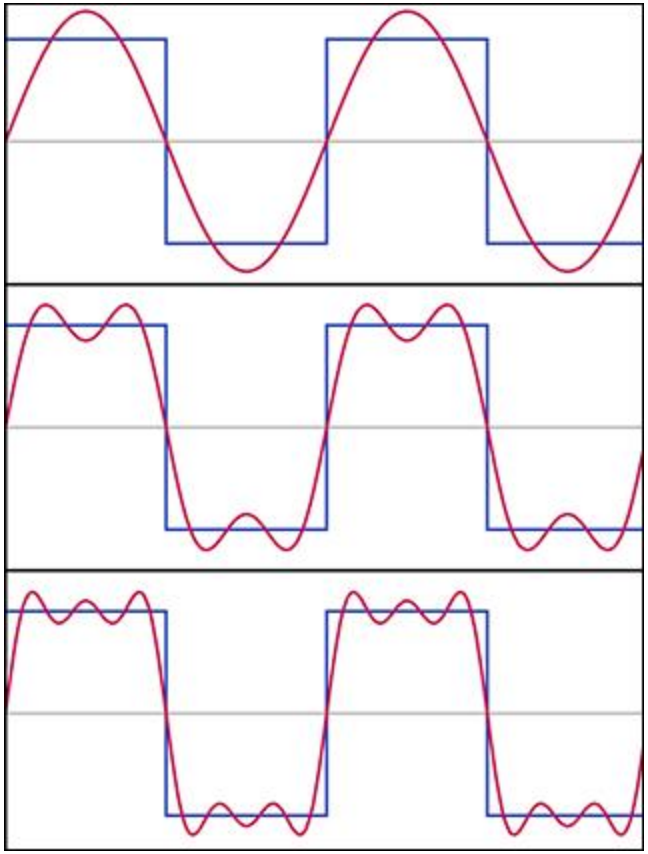
CO4. Understand the concept of limits, continuity and differentiability of functions of several variables. Analytical definition of partial derivative. Maxima and minima of functions of several variables Define gradient, divergence and curl of scalar and vector functions.

Lecture-I (Unit-3 Fourier Series)

Fourier series introduced in 1807 is one of the most important developments in Applied Mathematics. It is very useful in the study of heat conduction, electrostatics, mechanics etc.

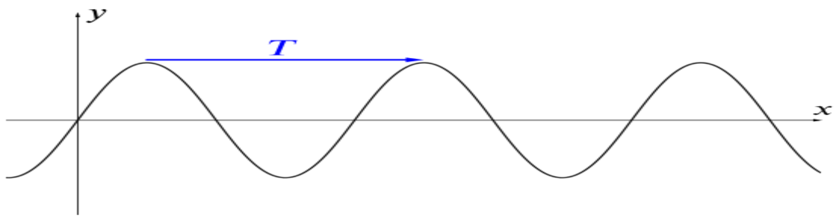
The Fourier series is an infinite series representation of periodic functions in terms of trigonometric sine and cosine functions.

Fourier Series is a very powerful method to solve ordinary and partial differential equations particularly with periodic functions appearing as non homogeneous terms.

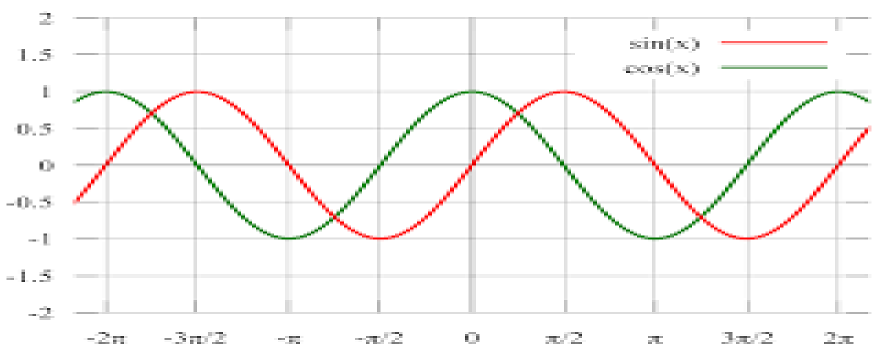


Periodic Function

A function f(t) is periodic if the function values repeat at regular intervals of the independent variable t. The regular interval is referred to as the period. See Figure 1. f(t) t period .If T denotes the period we have $f(t + T) = f(t)$ for any value of t.



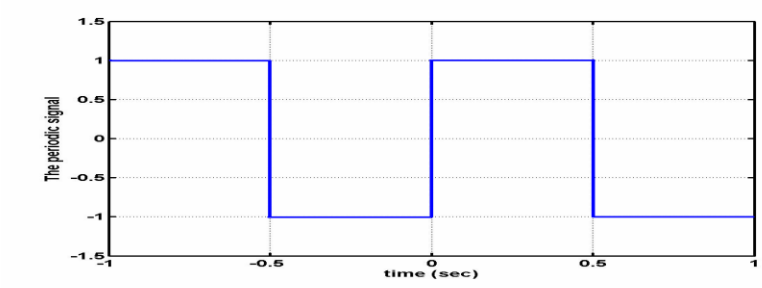
The most obvious examples of periodic functions are the trigonometric functions sin t and cost. both of which have period 2π



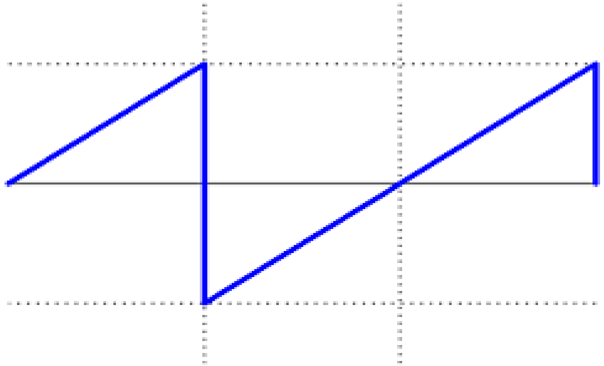
Non-sinusoidal periodic functions

The following are examples of non-sinusoidal periodic functions (they are often called “waves”).

Square Wave



Saw Tooth Wave



Dr. Ruchi Mathur & Dr. Tripati Gupta, JECRC,
Department of Mathematics

Mathematical Expression of Fourier Series

The Fourier series of the function f(x) is given by

$$f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + a_3 \cos 3x + b_3 \sin 3x + \cdots$$

$$or\ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\ nx + b_n \sin nx \quad \alpha < x < \alpha + 2\pi$$

where the constants $a_0, a_n\ and\ b_n$ are called Fourier coefficients.

Some important results: Let m and n be integers, m≠ 0 , n≠0 for m≠n

1. $\int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx\ dx = 0$

2. $\int_{\alpha}^{\alpha+2\pi} \sin mx \sin nx\ dx = 0$

3. $\int_{\alpha}^{\alpha+2\pi} \cos mx \sin nx\ dx = 0$

4. $\int_{\alpha}^{\alpha+2\pi} \cos mx\ dx = 0$

5. $\int_{\alpha}^{\alpha+2\pi} \sin mx\ dx = 0$

For m = n

1. $\int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx\ dx = \int_{\alpha}^{\alpha+2\pi} \cos^2 mx\ dx = \pi$

Dr. Ruchi Mathur & Dr. Tripati Gupta, JECRC,
Department of Mathematics

2. $\int_{\alpha}^{\alpha+2\pi} \sin mx \sin nx \, dx = \int_{\alpha}^{\alpha+2\pi} \sin^2 mx \, dx = \pi$

3. $\int_{\alpha}^{\alpha+2\pi} \cos mx \sin mx \, dx = 0$

Euler's formulae:

Let f (x) be a periodic function with period 2π defined in the interval $(\alpha, \alpha + 2\pi)$ of the form $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\ nx + b_n \sin nx$ … … … … …. (1)

where the constants are given by

$$a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x)dx \ldots \ldots .. (2)$$

$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) Cos\ nx\ dx \ldots \ldots .. (3)$ and

$$b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) Sin\ nx\ dx \ldots \ldots .. (4)$$

The formulae (2),(3) and (4) are known as Euler's formulae.

Dirchlit's conditions for Fourier Series

The sufficient conditions for the uniform convergence of a Fourier series are called Dirchlit's conditions.

All the functions that normally arise in engineering problems satisfy these conditions and hence, they can be expressed as Fourier Series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\, nx + b_n \sin nx$$

provided:

(1) Function f(x) is periodic , single valued and finite.
(2) Function f(x) has a finite number of discontinuities in any one period.
(3) Function f(x) has a finite number of maxima and minima.

When these conditions are satisfied , then Fourier series converge at f(x) at every point of continuity .

At the point of discontinuity, the sum of the series is equal to the mean of rght and left hand limits i.e.

$$\lim_{\epsilon \to 0} \frac{1}{2} [f(x+\epsilon) + f(x-\epsilon)]$$

Example1. Find the Fourier series expansion for the periodic Function

$$\boldsymbol{f(x) = x; 0 < x < 2\pi}$$

Solution: Consider the Fourier series $(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\, nx + b_n \sin nx)$ ..(1)

The Fourier Coefficients $a_{0\,,}a_n, b_n$ are as follows:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} xdx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = 2\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\, dx = \frac{1}{\pi}\left[\frac{x\sin nx}{n} - \left(-\frac{cos\;\; nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{\cos\, 2n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{1}{n^2\pi}[1-1] = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\, dx = \frac{1}{\pi}\left[\frac{x(-\cos\, nx)}{n} - \left(-\frac{sin\;\; nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{-2\pi\cos\, 2n\pi}{n}\right] = \frac{-2}{n}$$

Substituting these values of $a_{0\,,}a_n, b_n$ in (1) we get:

$$x = \pi + \sum_{n=1}^{\infty}\frac{-2}{n}\sin nx$$

$$x = \pi - 2(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x \ldots \ldots)$$

Example 2 : Find a Fourier series for a function $(x+x^2)$ in the interval $-\pi < x < \pi$.

Hence show that $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

Solution: Consider the Fourier series $(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cos\, nx + b_n \sin nx)$ ..(1)

The Fourier Coefficients $a_{0\,,}a_n, b_n$ are as follows:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)dx = \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\cos nx + \int_{-\pi}^{\pi} x^2 \cos nx\right]$$

First integral is zero as x cosnx is an odd function

$$\frac{2}{\pi}\left[\frac{\mathrm{x}^2 \sin nx}{n} - 2x\left\{\frac{-\cos nx}{n^2} + 2\frac{-\sin nx}{n^3}\right\}\right]_0^{\pi} = \frac{2}{\pi}\left[2\pi\frac{\cos n\pi}{n^2}\right] = \frac{4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\, dx$$

$x^2 \sin nx$ is an odd function therefore its integral is zero

$$\frac{2}{\pi}\left[\left(\frac{-x\cos nx}{n}\right)-\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi} = \frac{2}{\pi}\left[-\pi\frac{\cos n\pi}{n}\right] = -\frac{2}{n}(-1)^n$$

substituting in these values of a_0, a_n, b_n in (1) we get:

$$x + x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}(-1)^n\cos nx - \frac{2}{n}(-1)^n\sin nx\right]$$

$$x + x^2 = \frac{\pi^2}{3} + 4\left[\frac{-\cos x}{1} + \frac{\cos 2x}{2^2} + \ldots\right] - 2\left[\frac{-\sin x}{1} + \frac{\sin 2x}{2} + \ldots\right]$$

Summing the series at $x = \pi$

$$\lim_{\epsilon\to 0}\left(\frac{-\pi+\epsilon+(-\pi+\epsilon)^2+\pi-\epsilon+(\pi-\epsilon)^2}{2}\right) = \frac{\pi^2}{3} + 4\left[\frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2}\ldots\right]$$

$$\pi^2 = \frac{\pi^2}{3} + 4\left[\frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2}\ldots\right]$$

$$\frac{\pi^2}{6} = \left[\frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2}\ldots\right]$$

Suggested links from NPTEL & other Platforms:

- Advanced Engineering Mathematics: Erwin Kreyszig, Wiley plus publication
- https://www.youtube.com/watch?v=LGxE_yZYigI (NPTEL-NOC IITM)
- https://www.youtube.com/watch?v=SHx32HD8vDI

JECRC Foundation

JAIPUR ENGINEERING COLLEGE AND RESEARCH CENTRE

Thank you!