5

PERT : Time Estimates

5.1. INTRODUCTION

PERT stands for Programme Evaluation and Review Technique, which can be applied to any field requiring planned, controlled and integrated work efforts to accomplish established goods. The method was basically developed by Navy special projects office in co-operation with Booz, Allen and Hamilton, a management consulating firm and Lockheed Missile System Division for evaluating the feasibility of existing schedule, on Polaris Missile Program and for reporting progress. A project is composed of many diversified activities which contribute to its completion according to a predetermined schedule. There may be many uncertainties associated with these activities. PERT is a technique that statistically presents knowledge about these activities and the activity uncertainties.

PERT may be elaborately *defined* as

1. A MANAGER'S TOOL : for defining a project and co-ordinating various operations involved in it.
2. A DIRECTION : what must be done to successfully accomplish the objective of the project.
3. A PROFILE : that aids the decision-maker, but does not make decision for him.
4. A WAY : for synchronizing various parts of the overall job.
5. A TECHNIQUE : that presents statistical information regarding the uncertainties about computation time of various activities associated with the project.

6. A METHOD : for focussing managerial attention in (*a*) latent problems that require quick decisions (*b*) procedure and adjustments regarding time, resources, or performance which may improve the capabilities of meeting target dates, *i.e.* a method of scheduling and budgeting resources as to accomplish a predetermined job on schedule, and (*c*) of minimising the production delays, interruptions and conflicts.

7. AN OUTSTANDING APPROACH : of expending the completion of project.

8. A COMMUNICATION FACILITY : in that it can report developments both favourable and unfavourable to manager, and in that it can keep the manager posted and informed.

The PERT system uses a network diagram consisting of *events* which must be established to reach project objectives. An *event* is that particular instant of time at which some specific part of a plan is to be achieved. It indicates a point in time and does not require any resources. The approach of event-orientation in network diagram grew out of the desire to report on the project progress *via* discernible management milestones.

5.2. UNCERTAINTIES : USE OF PERT

As soon as the work of network construction for a project is over, the item of concern is the determination of time required for the occurrence of each event. The planner is faced with questions concerning how long the project will take and when specific activities may be performed. Since the activities are to be performed in future, the time period required for the execution of each activity or job can only be estimated. This factor presents no problem if the project consists of activities with which the estimator or planner is thoroughly familiar, perhaps because of his wide personal experience. However, because of certain uncertainties, an exact estimation of time for completion of an activity may be difficult.

Two approaches may be used for the assessment of duration for activity completion. The first approach is the *deterministic approach* in which we may assume that we know enough about each job or operation, so that a single estimate of their durations is sufficiently accurate to give reasonable results. This approach is followed by CPM users. The second approach is the *non-deterministic approach* or the *probabilistic approach* in which one may only be able to state limits within which it is virtually certain that the activity duration will lie. Between these limits we must guess what is the probability of executing the activity. The second approach is followed by PERT planners.

PERT was developed and has been used most frequently in the research and development type projects, such as space industry, aerospace industry, defence products industry etc. PERT system is preferred for these projects or operations which are of non-repetitive nature or for those projects in which correct time determination for various activities cannot be made. PERT application is favourable in projects where much of their design and construction or production requires new developments in materials and technology. All this is to say that there is a large amount of uncertainties in the development of new systems. These uncertainties may be about the times required for developmental research, engineering designs, ultimate construction and may be for specific activity or sometimes about the configuration of end product itself. There is little past history on which to base network construction and time estimates. In such projects, management cannot be guided by past experience. They are referred to as once-through operations or projects. For example, the project of *launching a space craft* involves the work never done before. For such a project, the range of possible technical problems is immense. In such research and development projects, the time estimates made for use may be little more than guesses. PERT system is best suited for such projects.

5.3. TIME ESTIMATES

Time is the most essential and basic variable in PERT system of planning and control. We have seen that PERT is mostly used for research and development type projects which are referred to as once-through. In these projects, there is uncertainty about the times required for the completion of various activities. Exact estimation of times of completion for various activities is difficult. In the PERT network an estimate is made of not only the *most probable time* required to complete the activity, but some measure of uncertainty is also *incorporated* in this estimate to consider two more time estimates : the *pessimistic estimate* and the *optimistic estimate*.

Thus, to take the uncertainties into account, PERT planners make three kinds of time estimates :

(*i*) The optimistic time estimate,

(*ii*) The pessimistic time estimate, and

(*iii*) The most likely time estimate.

1. The Optimistic Time Estimate

This is the shortest possible time in which an activity can be completed, under ideal conditions. This particular time estimate represents the time in which we could complete the activity or job if everything went along perfectly, with no problems or adverse conditions. Better than normal conditions are assumed to prevail. This time estimate is denoted by t_O.

2. The Pessimistic Time Estimate

It is the best guess of the maximum time that would be required to complete the activity. This particular time estimate represents the time it might take us to complete a particular activity if every thing went wrong and abnormal situations prevailed. However, this estimate does not include possible effects of highly unusual catastrophies such as earthquakes, floods, fires etc. This time estimate is denoted by t_P.

3. The Most Likely Time Estimate

The most likely time or *most probable time* is the time that, in the mind of the estimator, represents the time the activity would most often require if normal conditions prevail. This time estimate lies between the optimistic and pessimistic time estimates. This time estimate reflects a situation where conditions are normal, things are as usual and there is nothing exciting. This time estimate is denoted by t_L.

These time estimates, though look simple, are not always easy to prepare. However, they give useful information about the expected uncertainties in an activity. These time estimates are usually expressed in days, weeks or months, and represent calender dates and not actual working days.

5.4. FREQUENCY DISTRIBUTION

As stated above, the three time estimates are very difficult to prepare, unless some guidance is available. The planner should base the estimations on available information and past experience. For example, consider a certain activity '*A*' under diverse conditions. The time required for the completion of this activity under each condition is known. Naturally, the time of completion will be short (optimistic time) if better than normal conditions exist. The number of cases when such normal conditions exist for completion of an activity *A* will be naturally small. Similarly, time of completion will be long (pessimistic time) if adverse conditions are there, and such cases will also be small in number.

If a curve is now plotted between the 'time' of completion and the number of jobs completed in that 'time', *a frequency distribution curve*, such as the one shown in Fig. 5.1 will be obtained. From the curve, it is clear that there are large number of cases of the activity that are completed in the *most likely time*. Point *P* corresponds to the optimistic time (t_O), point *R* corresponds to the pessimistic time (t_P) while point *Q* corresponds to the most likely time (t_L). Such a curve is also called *unimodal curve*, since it has single hump.

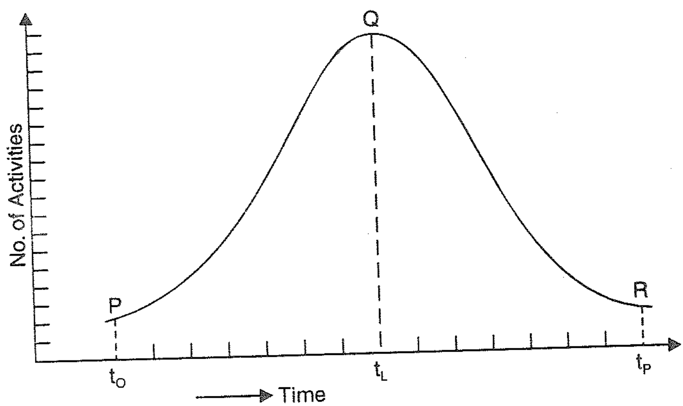


FIG. 5.1. FREQUENCY DISTRIBUTION CURVE.

The curve shown in Fig. 5.1 is symmetrical on either side of point *Q* ; such a curve is known as the *normal curve*. It is not necessary that a frequency distribution curve may be normal ; it may have *skew* due to which it is not symmetrical about the peak Q.

Fig. 5.2 (a) shows the frequency distribution curve for job A, having skew to the left ; the difference between t_L and t_O is only 1 day while the difference between t_P and t_L is $7 - 4 = 3$ days. Fig. 5.2 (c) shows the frequency distribution curve for activity B,

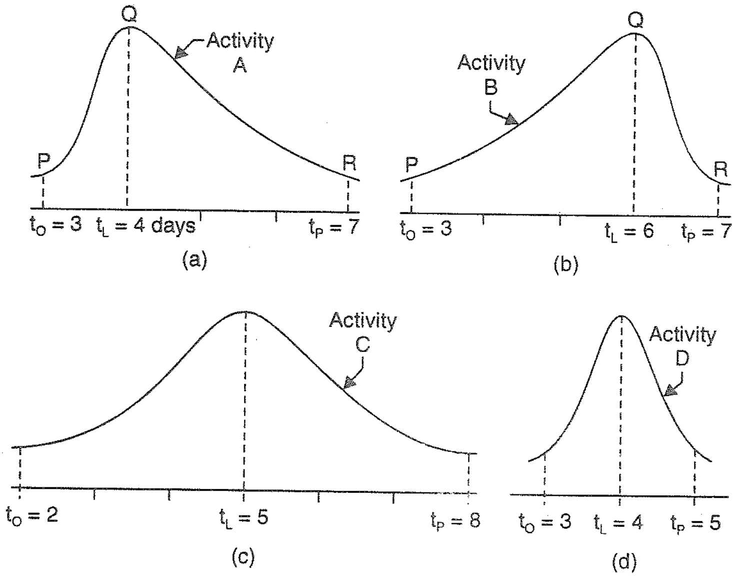


FIG. 5.2. FREQUENCY DISTRIBUTION CURVES.

having skew to the right ; the difference between t_L and t_O is equal to $6 - 3 = 3$ days while the difference between t_P and t_L is only 1 day. The frequency distribution curves for activities C and D, shown in Fig. 5.2 (c) and (d) respectively are symmetrical about the peak, and they are therefore *normal curves*. However, curve (c) has wider variation between t_P and t_O and has therefore *greater uncertainty* in time estimate. On the other hand, curve (D) has smaller variation between t_P and t_O and hence more reliable time estimates are expected. To conclude, *a wide range in time estimates represents, greater uncertainty and hence less confidence in our ability to correctly anticipate the actual time that the activity will require.*

To summarise, statistical data for varying durations of time that jobs of a particular type consumed in the past can be expressed in the form of a frequency distribution curve. The method of preparing a *frequency distribution curve* will now be explained with the help of an example.

Example 5.1. *In a certain project, the times required for digging 54 trenches of fixed dimensions are recorded below. The trenches were excavated by different parties, each consisting of the same number of persons. Plot the frequency distribution curve.*

Table 5.1

TIMES OF COMPLETION OF TRENCH (DAYS)

8	11	14	9	10	8
10	9	12	11	9	10
12	8	7	13	11	9
6	10	9	10	10	11
9	14	13	14	7	
11	16	10	9	13	
10	12	8	12	11	
13	16	11	15	8	
15	15	17	14	12	
12	10	13	9	11	

Solution. From Table 5.1, we find that the minimum time taken for completion of trench is 6 days which corresponds to the optimistic time (t_O), while the maximum time taken is 17 days which corresponds to the pessimistic time (t_P). The time varies between 6 days to 17 days. Table 5.2 gives the No. of trenches completed in 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 and 17 days respectively.

Table 5.2

Days of completion	*No. of trenches completed during these days*	*Days of completion*	*No. of trenches completed during these days*
6	1	12	6
7	2	13	5
8	5	14	4
9	8	15	3
10	9	16	2
11	8	17	1

The data of Table 5.2 can now be plotted to get the frequency distribution curve between No. of days of completion and No. of trenches completed during this period, as shown in Fig. 5.3. From the curve, the most likely time (t_L), corresponding to the peak of the curve, comes out to be 10 days. The frequency curve so obtained is not symmetrical ; it has skew to the left.

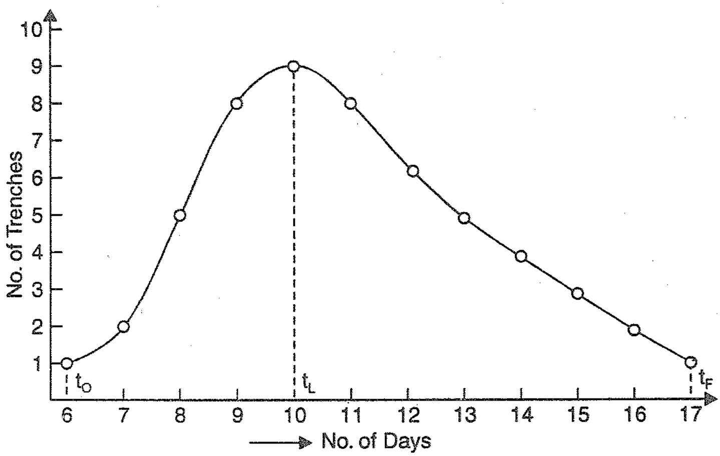


FIG. 5.3

5.5. MEAN, VARIANCE AND STANDARD DEVIATION

From the previous article we find that the frequency distribution curve can be drawn if data about varying durations of time taken for the completion of jobs of a particular type are available. If this curve is symmetrical, then it is called the *normal curve* ; otherwise it is said to have a *skew* which could be either to the left or to the right. Whatever may be the form of the curve, the following aspects of the characteristics of the distribution are important :

(*i*) Mean time or average time (called the mean of the distribution),

(*ii*) Deviation,

(*iii*) Variance and

(*iv*) Standard deviation.

Mean. Mean of the distribution may be defined by the algebraic sum of time durations taken by various jobs divided by the number of the jobs :

$$t_m = \frac{\Sigma t}{n} \qquad ...(5.1)$$

Deviation. Deviation is the difference between the time under consideration and the mean time. This difference may be either positive or negative.

Thus $$\delta = t - t_m \qquad ...(5.2)$$

where δ = deviation of any time t from the mean

t = time under consideration, for which deviation is being found.

Variance. Variance is the mean of the squared deviations. It is expressed by σ^2.

Thus, $$\sigma^2 = \frac{\Sigma\delta^2}{n} = \frac{\Sigma(t - t_m)^2}{n} \qquad ...(5.3)$$

Variance is calculated in the following steps :

(*i*) Obtain the *mean* of the distribution, by Eq. (5.1).

(*ii*) Determine the deviation of each time from the mean.

(*iii*) Find square of these individual deviations.

(*iv*) Find the mean of the squared deviations.

It is to be noted that though the deviations may be negative also, but their squares will always be positive. Hence variance will always be positive. It cannot have zero value unless each individual deviation is zero.

Variance is commonly used in statistics as measure of *variability* of the distribution.

Standard deviation. It is simply the square root of the variance. Standard deviation is denoted by symbol σ.

Thus, $$\sigma = \sqrt{\frac{\Sigma(t - t_m)^2}{n}} \qquad ...(5.4)$$

Let us now calculate mean, variance and standard deviation for the data of example 5.1. The computation are arranged in Table 5.3.

From Table 5.3, consisting of 54 time observations, we get

$$\Sigma t = 597$$

Table 5.3

Time taken (t)	*Deviation* $\delta = t - t_m$	δ^2	*Time taken (t)*	*Deviation* $\delta = t - t_m$	δ^2
8	– 3.06	9.364	17	+ 5.94	35.284
10	– 1.06	1.124	13	+ 1.94	3.764
12	+ 0.94	0.884	9	– 2.06	4.244
6	– 5.06	25.604	11	– 0.06	0.004
9	– 2.06	4.244	13	+ 1.94	3.764
11	– 0.06	0.004	10	– 1.06	1.124
10	– 1.06	1.124	14	+ 2.94	8.644
13	+ 1.94	3.764	9	– 2.06	4.244
15	+ 3.94	15.524	12	+ 0.94	0.884
12	+ 0.94	0.884	15	+ 3.94	15.524
11	– 0.06	0.004	14	+ 2.94	8.644
9	– 2.06	4.244	9	– 2.06	4.244
8	– 3.06	9.364	10	– 1.06	1.124
10	– 1.06	1.124	9	– 2.06	4.244
14	+ 2.94	8.644	11	– 0.06	0.004
16	+ 4.94	24.404	10	– 1.06	1.124
12	+ 0.94	0.884	7	– 4.06	16.484
16	+ 4.94	24.404	13	+ 1.94	3.764
15	+ 3.94	15.524	11	– 0.06	0.004
10	– 1.06	1.124	8	– 3.06	9.364
14	+ 2.94	8.644	12	+ 0.94	0.884
12	+ 0.94	0.884	11	– 0.06	0.004
7	– 4.06	16.484	8	– 3.06	9.364
9	– 2.06	4.244	10	– 1.06	1.124
13	+ 1.94	3.764	9	– 2.06	4.244
10	– 1.06	1.124	11	– 0.06	0.004
8	– 3.06	9.364	$\Sigma t = 597$		$\Sigma\delta^2 = 338.856$
11	– 0.06	0.004	$t_m = \frac{\Sigma t}{n} = \frac{597}{54} \approx 11.06$		$\sigma^2 = \frac{\Sigma\delta^2}{n} = \frac{338.856}{54} = 6.275$

$$\therefore \qquad t_m = \frac{\Sigma t}{n} = \frac{597}{54} \approx 11.06$$

Also, $\Sigma\delta^2 = 338.856$

$$\therefore \text{ Variance} \quad \sigma^2 = \frac{\Sigma\delta^2}{n} = \frac{338.856}{54} = \mathbf{6.275}$$

Hence standard deviation

$$= \sqrt{6.275} \approx \mathbf{2.5.}$$

From Fig. 5.1, we observed that the most likely time (t_L) was 10 days, while the mean time t_m is 11.06 days. The tallest peak of the distribution curve is called the *mode*, corresponding to the most likely time (t_L). Both *mean* and *mode* do not coincide because the distribution curve is not symmetrical about its peak. In the case of a symmetrical curve (*i.e.* normal distribution curve) the mean (centre of gravity) coincides with the mode.

5.6. PROBABILITY DISTRIBUTION

Probability is connected with *chance* and *uncertainty*. The three time estimates that the estimator selects either from his experience or from the *frequency distribution* has inherent uncertainties. In probability analysis, and in consequent probability distribution, we try to associate numbers with uncertainties. In the frequency distribution one studies the group behaviour, while in the probability distribution, we have the distribution of probability values for all possible outcomes. The probability measures are always between 0 to 1. If an event has probability of 1, it is certain to occur, while if the probability is zero it will not occur. Closer the probability value is to 1, more certain is the occurrence of the event.

Let us take an example of manufacture of steel trusses by a factory. Let us assume that the factory manufactures 50 trusses in all, under varying circumstances, and the duration of time taken are as follows :

5 trusses in 12 days each

12 trusses in 14 days each

13 trusses in 15 days each

8 trusses in 16 days each

12 trusses in 18 days each.

Let us now find the probability of manufacturing a truss in 12 days. This is evidently equal to the ratio of number of trusses manufactured in 12 days each to the total number of trusses manufactured. Thus, probability

$$= \frac{5}{50} = 0.1 \text{ or } 10\%.$$

Similarly, the probability of manufacturing the truss in 15 days

$$= \frac{15 + 12 + 13}{50} = 0.6 \text{ or } 60\%.$$

Thus, probability number can always be assigned to the estimated time, if sufficient data is available. Generally, the available data (frequency distribution) is used to plot *probability distribution.*

Probability distribution is the curve, with its height so standardised that the area under the curve is equal to unity. The height or the ordinate of the curve at any point x, is denoted by function $f(x)$, usually called the *probability density function.*

Thus $$\int_{-\infty}^{+\infty} f(x)\, dx = 1 \quad \text{...(5.5)}$$

Fig. 5.4 shows the probability curve. It is to be noted that the ordinate $f(x)$ to the curve at any point x does not give the probability. The probability of completion of work in 8 days is equal to the ratio of the shaded area (area to the left of 8 days) to the total area of the curve. Since the total area of the curve is equal to unity, the

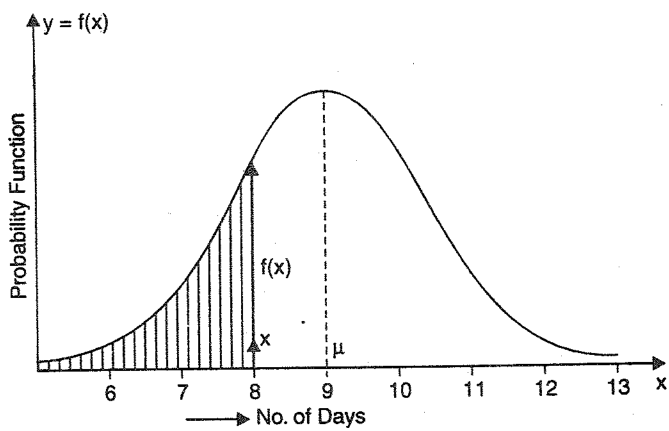


FIG. 5.4

probability of completion of the job in 8 days is equal to the shaded area itself.

Normal Probability Distribution

The probability curve is not necessarily symmetrical about its apex. If the curve is symmetrical, then it is known to have *normal* or Gaussian distribution, shown in Fig. 5.5.

The mean of the *normal probability distribution* is denoted by μ (*i.e.* $x = \mu$). It can be proved that :

(*a*) Approximately 68% of the values of the normal distribution lie within $\pm\,\sigma$ from the average, where σ is the standard deviation. This means that the shaded area of the curve (Fig. 5.5) between $x = \mu - \sigma$ to $x = \mu + \sigma$ is 68% of the total area.

(*b*) Approximately 95% of all the values lie within $\pm\,2\sigma$ from the average. This means that the area of the curve between $x = \mu - 2\sigma$ to $x = \mu + 2\sigma$ is 95% of the total area.

(*c*) Approximately 99.7% of all the values lie within $\pm\,3\sigma$ from the average. This means that the area of the curve between $x = \mu - 3\sigma$ to $x = \mu + 3\sigma$ is 99.7%.

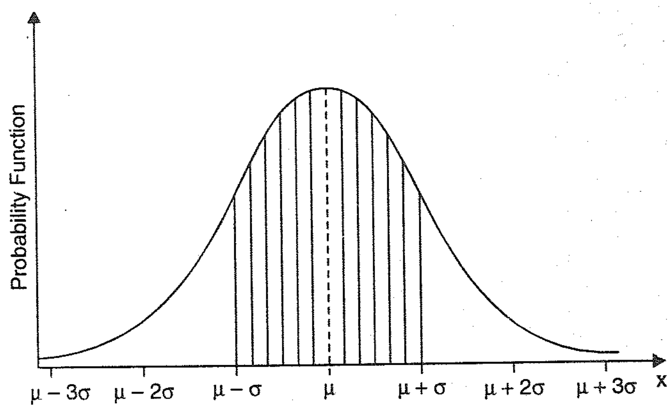


FIG. 5.5. NORMAL PROBABILITY DISTRIBUTION.

The last property (*c*) can be used to calculate the *standard deviation* directly if the minimum time (t_O) and maximum time (t_P) are known. Let us say that the minimum time is 6 days and maximum time is 18 days for the completion of a job. If 99.7% of all the values (*i.e.* possible completion times) are assumed to lie between 6 and 18 days then the distance between the extreme left value (6 days) and extreme right value (18 days) should be equal to $\pm\,3\sigma$ or 6σ in total. The standard deviation

$$= \frac{18-6}{6} = 2 \text{ days.}$$

Hence we conclude, in general, that standard deviation is given by

$$\sigma = \frac{t_P - t_O}{6} \qquad \text{...(5.6)}$$

or variance

$$\sigma^2 = \left(\frac{t_P - t_O}{6}\right)^2 \qquad \text{...[5.6 (a)]}$$

It is seen that the standard deviation is affected by the relative distance from the most optimistic estimate to the most pessimistic estimate. It is not influenced by the most likely estimate (t_L).

The above method of calculating standard deviation is *approximate*. A more exact method is by frequency distribution, explained in example 5.1. However, in PERT problems, the emphasis is one-time, non-repetitive projects for which there is no history of the activity. Hence we must base computations for σ on the given time estimates of the estimator. If the estimator feels them his range of t_O and t_P includes about all the possible values under the curve, then the standard deviation can be computed from Eq. 5.6 with reasonable accuracy.

5.7. THE BETA DISTRIBUTION

The beta distribution is a typical type of probability distribution, which fits well for PERT analysis. A beta distribution is the one which is not symmetrical about its apex. Fig. 5.6 shows two beta distributions, one having skew to the left (*beta distribution for optimistic estimator*) and the other having skew to the right (*beta distribution for the pessimistic estimator*).

The originators of PERT were interested in finding that type of probability distribution which satisfies the following conditions :

1. The distribution should have a small probability of reaching the most optimistic time (shortest time).

2. The distribution should have a small probability of reaching the most pessimistic time (longest time).

3. The distribution should have one and only one most likely time (*i.e.* unimodal) which would be free to move between the two extremes mentioned in 1 and 2 above.

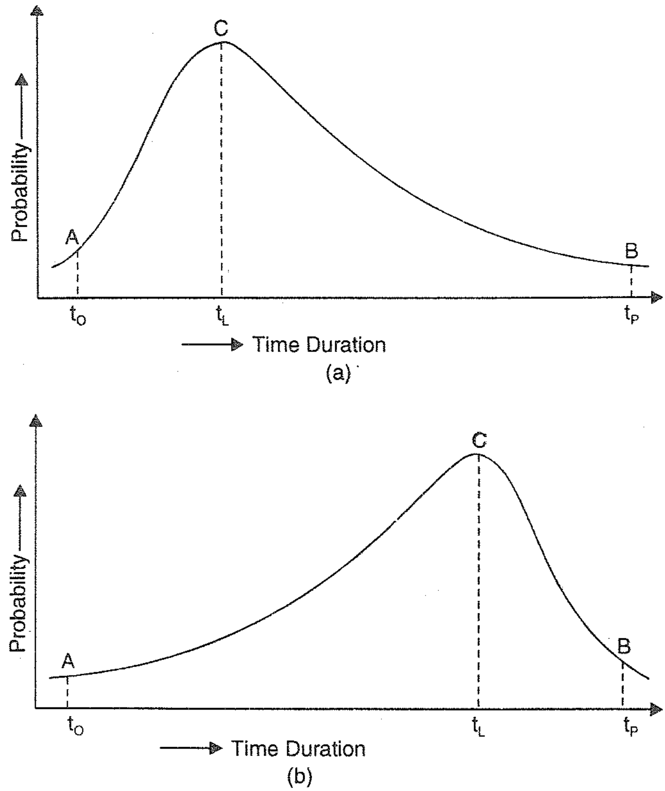


FIG. 5.6. BETA DISTRIBUTIONS.

4. The distribution should be such that the amount of uncertainty in the estimating can be measured easily.

The above mentioned four requirements are met with beta distribution. Hence this distribution is used in PERT analysis.

It can be shown that for Beta distribution. the standard deviation is given by

$$\sigma = \frac{t_P - t_O}{6}$$

The variance $\sigma^2 = \left(\frac{t_P - t_O}{6} \right)^2$.

We have already seen that *variance* is the measure of *uncertainty*. Greater the variance, greater will be the uncertainty.

5.8. EXPECTED TIME

The three time estimates t_O (optimistic time), t_P (pessimistic time) and t_L (most likely time) are identified on the Beta-distribution. The variance and standard deviation can be computed using t_O and t_P. However, one must combine the three time estimates into one single time—the average time taken for the completion of the activity or job. This average time or single workable time is commonly called the *expected time* and is denoted by t_E. If the exact shape of the probability distribution curve is known, the average time or expected time could be accurately calculated. However, since the precise curves are never available (specially for non-repetitive jobs) we must use *approximation*. This is done algebraically, using a weighted average derived by statisticians. In computing the expected time, a weightage of 1 is given to the optimistic time t_O, weightage of 4 to the most likely time (t_L) and weightage of 1 to the most pessimistic time (t_P).

Thus, $$t_E = \frac{t_O + 4t_L + t_P}{6} \qquad ...(5.7)$$

The above expression for t_E, based on weighted average method, is reasonable since the chance of completion of the job in t_O or t_P is much less than the most likely time (t_L).

Let us take examples of estimated times of completion of two jobs *A* and *B*, as under.

	t_O	t_L	t_P (days)
Job *A*	4	6	11
Job *B*	5	10	12

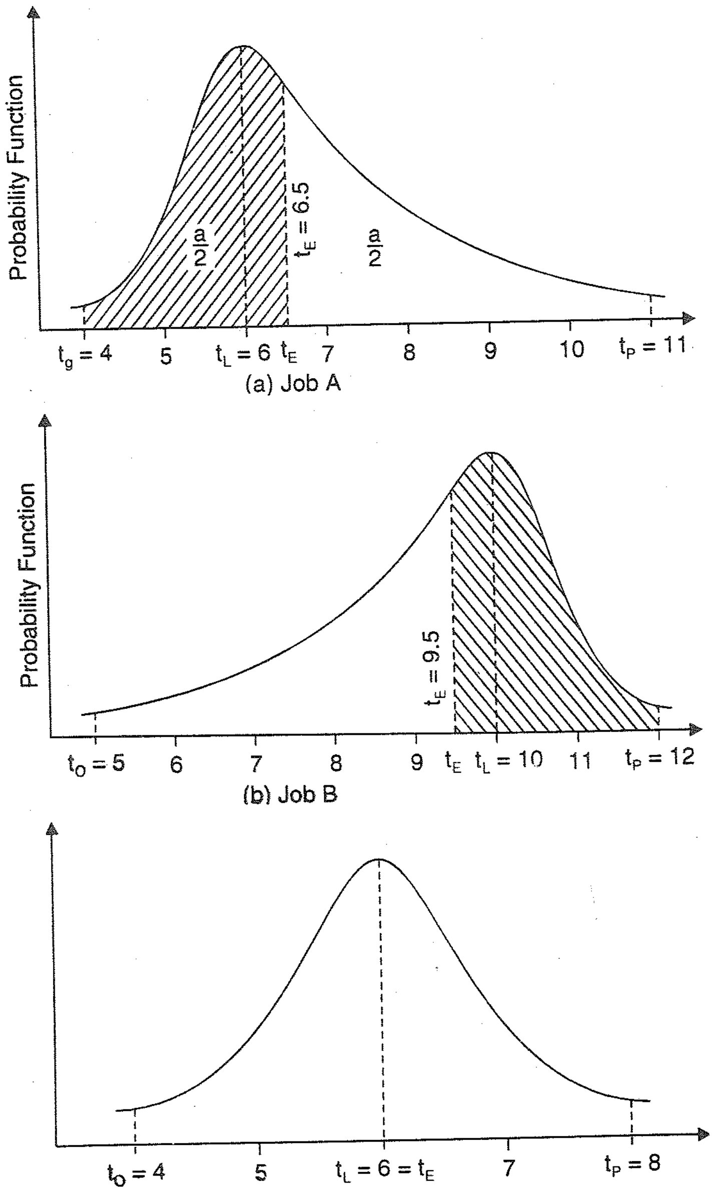
The expected time for these jobs are

$$(t_E)_A = \frac{t_O + 4t_L + t_P}{6} = \frac{4 + (4 \times 6) + 11}{6} = 6.5 \text{ days}$$

$$(t_E)_B = \frac{t_O + 4t_L + t_P}{6} = \frac{5 + (4 \times 10) + 12}{6} = 9.5 \text{ days}$$

Thus, for job *A*, the expected time falls to the right of the most likely time, though the curve has skew to the left [Fig. 5.7 (*a*)]. For job *B*, the expected time t_E falls to the left of the most likely time, though the curve has skew to the right.

One important point should be noted about the expected time t_E. The expected time t_E represents the *average value* while the most likely time t_L represents the *mode* of the β-distribution. The expected time represents a particular value on the distribution curve, that has both a 50-50 chance of being exceeded and a 50-50 chance of



(c)
FIG. 5.7

being met. There is the same chance of the actual time taken will be *greater* than t_E as there is it will be *less* than t_E. Naturally, the vertical ordinate through t_E will divide the probability curve into two equal areas, as shown in Fig. 5.7.

If the estimated time (t_O, t_L and t_P) are such that the expected time t_E computed with these comes out to be equal to t_L, the distribution curve will be symmetrical about the mode (t_L). For example, if $t_O = 4$, $t_L = 6$ and $t_P = 8$, for a job C, we have

$$t_E = \frac{t_O + 4t_L + t_P}{6}$$

$$= \frac{4 + 4 \times 6 + 8}{6} = 6$$

$$\therefore \qquad t_E = t_L$$

Such a situation is shown in Fig. 5.7 (c), in which the curve is symmetrical.

For the three jobs A, B and C mentioned before, the standard deviations and variance are as under :

For activity A,

Standard deviation $\sigma_A = \frac{t_P - t_O}{6} = \frac{11 - 4}{6} = 1.167$

Variance $\quad \sigma_A^2 = (1.167)^2 = 1.36$

For activity B,

$$\sigma_B = \frac{t_P - t_O}{6} = \frac{12 - 5}{6} = 1.167$$

Variance $\quad = (\sigma_B)^2 = (1.167)^2 = 1.36$

For activity C,

$$\sigma_C = \frac{t_P - t_O}{t} = \frac{8 - 4}{6} = 0.667$$

Variance $(\sigma_C)^2 = (0.667)^2 = 0.444.$

EXPECTED TIME FOR ACTIVITIES IN SERIES

When a number of activities are in series, the expected time for the path, along the activities, can be found by first finding the t_E for each activity, and then taking their sum. Alternatively, the optimistic time (t_O), the most likely time (t_L) and the pessimistic times (t_P) of the path can be calculated first by taking the sum of all t_O, t_L and t_P respectively and then t_E can be computed.

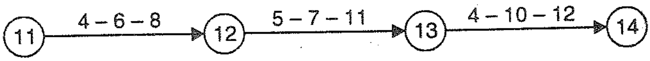


FIG. 5.8

For example, consider the three activities. 11—12, 12—13 and 13—14 shown in Fig. 5.8 with their individual time estimates (t_O, t_L and t_P) marked.

t_E for the path = Σt_E

Computations are arranged below (Table 5.4)

Table 5.4

Activity	t_O	t_L	t_P	t_E
11—12	4	6	8	6
12—13	5	7	11	7.333
13—14	4	10	12	9.333
				Σt_E = 22.666

Alternatively,

$$\Sigma t_O = 4 + 5 + 4 = 13$$

$$\Sigma t_L = 6 + 7 + 10 = 23$$

$$\Sigma t_P = 8 + 11 + 12 = 31$$

$$\therefore \quad \Sigma t_E = \frac{\Sigma t_O + 4\Sigma t_L + \Sigma t_P}{6}$$

$$= \frac{13 + 4 \times 23 + 31}{6} = 22.67$$

Thus, t_E for the series of activities computed by both the methods is the same.

The *standard deviation* for the last event (network ending event) in a series of activity, is given by

$$\sigma_{t_E} = \sqrt{\sigma_1^2 + \sigma_2^2 + ... + \sigma_n^2} \qquad ...(5.8)$$

where $\sigma_1, \sigma_2, ..., \sigma_n$ are the standard deviations of each of the activities.

σ_{t_E} = standard deviation of network ending event.

For the above example,

$$\sigma \text{ for } 11\text{—}12 = \frac{8 - 4}{6} = 0.667$$

$$\sigma \text{ for } 12\text{—}13 = \frac{11-5}{6} = 1$$

$$\sigma \text{ for } 13\text{—}14 = \frac{12-4}{6} = 1.337$$

Hence the standard deviation of event 14, symbolised by σ_{t_E} is

$$= \sqrt{(0.667)^2 + (1)^2 + (1.333)^2}$$
$$= \sqrt{3.222} = 1.795.$$

A similar approach can be made for a network consisting of several paths, each path with a number of activities in series. When t_E for path in a network is known, the *critical path* can be chosen easily. A *critical path* is the one which consumes maximum of time resources. This is illustrated in example 5.3.

Example 5.2. *For a particular activity of a project, time estimates received from two engineers X and Y are as follows :*

	Optimistic time	*Most likely time*	*Pessimistic time*
Engineer X	*4*	*6*	*8*
Engineer Y	*3*	*5*	*8*

State who is more certain about the time of completion of the job.

Solution.

The degree of uncertainty (or otherwise) is indicated by the variance of the time estimates.

The variance of time estimates given by engineer *X* is

$$\sigma_X^2 = \left(\frac{t_P - t_O}{6}\right)^2 = \left(\frac{8-4}{6}\right)^2 = 0.4356$$

The variance of time estimate of engineer *Y* is

$$\sigma_Y^2 = \left(\frac{t_P - t_O}{6}\right)^2 = \left(\frac{8-3}{6}\right)^2 = 0.69.$$

Thus, the variance of time estimates given by *Y* is more. Since greater the variance, greater will be uncertainty, engineer *X*'s time estimates have more certainty.

Example 5.3. *The network for a certain project is shown in Fig. 5.9. Determine the expected time for each of the path. Which path is critical ?*

Solution.

In the network, event 1 is the starting event while event 8 is the end event. There are following four paths from the starting event to the end event :

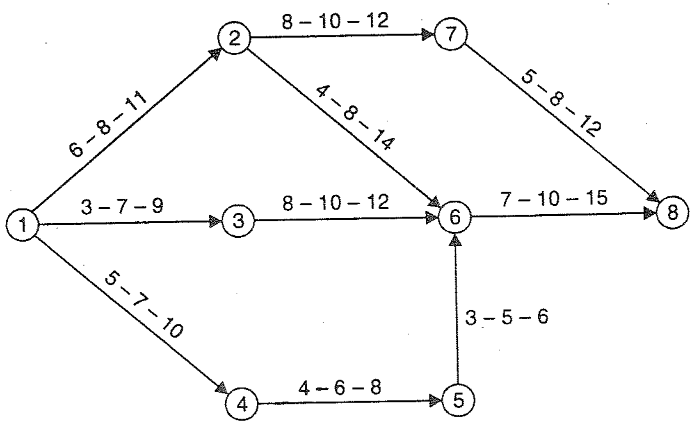


FIG. 5.9

Path A : 1—2—7—8

Path B : 1—2—6—8

Path C : 1—3—6—8

Path D : 1—4—5—6—8

The optimistic time for each of the path is equal to the sum of all t_O's of activity of that path. Similarly, the pessimistic time and most likely time of each of the paths can be found. These times are shown in Table 5.5.

Table 5.5

	Optimistic time	*Most likely time*	*Pessimistic time*
Path A 1—2—7—8	19	26	35
Path B 1—2—6—8	17	26	40
Path C 1—3—6—8	18	27	36
Path D 1—4—5—6—8	19	28	39

From the point of view of pessimistic time, path *B* is *critical* since it takes longest duration. From the point of view of optimistic time, paths A and D are equally critical. However, from the most likely time point of view, path D is the most critical.

In the PERT analysis, the *expected time* t_E is taken as the basis for finding the *critical path*. As indicated earlier, the expected time t_E is computed from the equation

$$t_E = \frac{t_O + 4t_L + t_P}{6}$$

Thus, expected time for each activity can be found. The expected time (t_E) for any path is equal to Σt_E of all activities. The computations are shown in Table 5.6.

Table 5.6

Path	*Activity*	t_O	t_L	t_P	t_E	Σt_E
	1—2	6	8	11	8.17	
A	2—7	8	10	12	10.00	26.34
	7—8	5	8	12	8.17	
	1—2	6	8	11	8.17	
B	2—6	4	8	14	8.33	26.83
	6—8	7	10	15	10.33	
	1—3	3	7	9	6.67	
C	3—6	8	10	12	10.00	27.00
	6—8	7	10	15	10.33	
	1—4	5	7	10	7.17	
D	4—5	4	6	8	6.00	28.33
	5—6	3	5	6	4.83	
	6—8	7	10	15	10.33	

From Table 5.6, we find that path D is *critical* since Σt_E for this path is the maximum.

PROBLEMS

1. Define 'optimistic time estimate', 'pessimistic time estimate' and 'most likely time estimate'.
2. Differentiate clearly between most likely time estimate (t_L), mean time (t_m) and expected time (t_E).

3. What do you understand by frequency distribution ? How do you determine (*i*) most likely time, (*ii*) variance and (*iii*) standard deviation from the frequency distribution ?
4. What is meant by probability distribution curve ? Differentiate clearly between normal probability distribution curve and beta distribution.
5. How do you use the normal probability curve for determining standard deviation ?
6. Explain how beta distribution is suitable for PERT analysis. Explain how do you determine the expected time and standard deviation.
7. The time estimates for three activities *A*, *B* and *C* are as follows :

	Optimistic time	*Most likely time*	*Pessimistic time*
A	10	12	14
B	6	8	12
C	5	10	12

Determine expected time and variance for each activity. Which activity has more reliable time estimates ?

8. A path of a certain network is shown in Fig. 5.10 with the time estimates for its activities as mentioned along each activity. Determine the expected time for the path. What is the standard deviation for the path ?

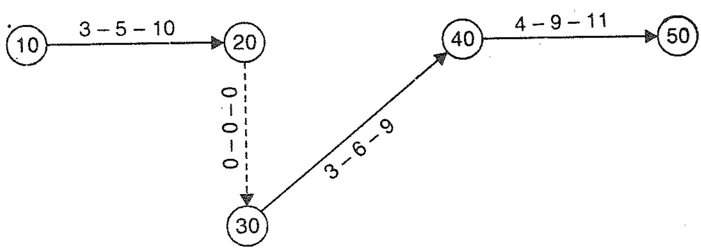


FIG. 5.10

9. The network for a certain project is shown in Fig. 5.11. Determine the expected time for each path. Which path is critical ?

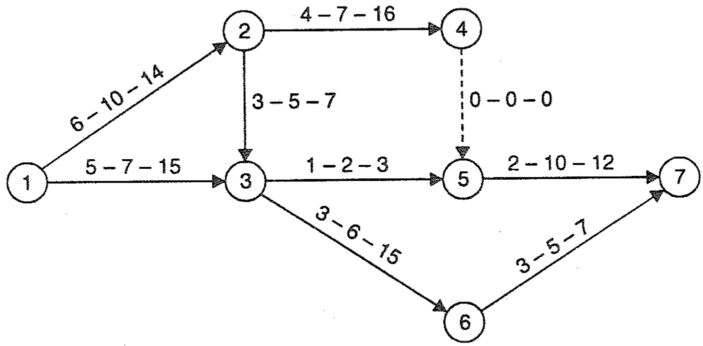


FIG. 5.11

6

PERT : Time Computations

6.1. INTRODUCTION

In the previous chapter, we have considered *time estimates* from which the *expected time* t_E was determined. All these times (*i.e.* optimistic time, pessimistic time, most likely time and expected time) refer to the completion of an activity. Let us now concentrate on the time of occurrence of an *event*. Though, for simple networks, the *expected time* or *average time* of completion of activities enables us to find the critical path, but in complex network, it is necessary to follow a systematic method for determining the critical path. This is achieved by first computing for each event and the following two time estimates are :

(*a*) Earliest expected time (T_E).

(*b*) Latest allowable occurrence time (T_L).

The three times estimates t_O, t_L and t_P, as well as the expected or average time t_E, which refer to an activity or job are designated by small t while the above two times (*i.e.* earliest expected time T_E and latest allowable occurrence time T_L) which refer to an event are symbolised by capital T.

6.2. EARLIEST EXPECTED TIME

The *earliest expected time* is the time when an event can be expected to occur. It is represented by symbol T_E and appear above or below the node (event circle) in a network.

The earliest expected time (T_E) is computed by adding the *expected times* (t_E) of all the *activities* along an *activities path* leading to that event. If more than one activity paths lead to that event, then the *maximum* of the *sum* of t_E's along the various paths will give the earliest expected time.

Let us first consider a simple network shown in Fig. 6.1 in which there is only one *activity path*. The three time estimates (t_O,

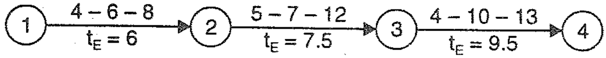


FIG. 6.1

t_L and t_P) of each activity are normally expressed above activity arrow while the activity expected time (t_E) is written below it.

Let us assume that event 1 is the initial event, which occurs at zero time. The activity 1—2, which connects events 1 and 2, has t_E = 6. Hence the earliest expected time for event 2 will be

$$= 0 + 6 = 6.$$

Event 3 is connected to event 2 by activity 2—3 which has t_E = 7.5. Hence T_E for 3 = T_E for 2 + t_E for activity (2—3)

$$= 6 + 7.5 = 13.5.$$

Similarly, T_E for the last event 4 = $(T_E)_3 + (t_E)_{3\text{-}4}$

$$= 13.5 + 9.5 = 23.$$

Hence we conclude that the earliest expected time for any event is the sum of the activities expected times (t_E) of the activity path leading to the event under consideration. The expected time T_E for each event is entered near the event circle (normally above the circle) as shown in Fig. 6.2.

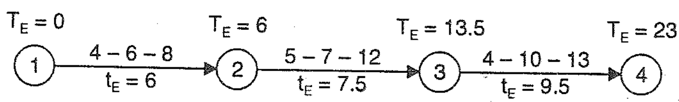


FIG. 6.2

Let us now consider a network shown in Fig. 6.3. The expected time (t_E) for each activity is shown on the activity arrow.

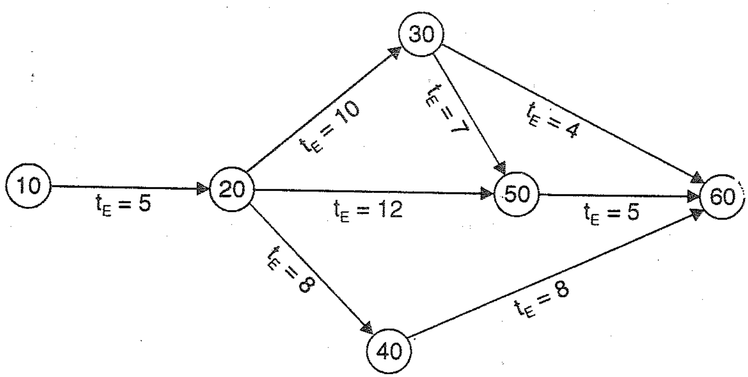


FIG. 6.3

Consider event 50. It has two activity paths : (*i*) path 10—20—30—50 and (*ii*) 10—20—50. In both these paths, events 10 and 20 are common. Since event 10 is the initial event, $T_E = 0$ for this. For event 20, $T_E = 0 + 5 = 5$ units. The time computations along each path leading to event 50 are as under :

Events		T_E
10		0
20		5
30		15
50	(*i*) Path 10—20—30—50	22
	(*ii*) Path 10—20—50	(17)

From the above, we note that T_E for event 50 are different along the two activity paths. However, *no event can be considered to have reached until all activities leading to the event are completed.* Hence event 50 cannot be considered to have occurred until all the activities along both the paths are complete. Thus, T_E will be the greater of the two values obtained from the two paths. Thus, T_E for 50 is 22 days.

Now let us come to the last event 60. It has four paths leading to it :

(*i*) Path 10—20—30—50—60.

(*ii*) Path 10—20—50—60.

(*iii*) Path 10—20—30—60.

(*iv*) Path 10—20—40—60.

Out of these, the first two paths pass through event 50. Since T_E for event 50 is 22, T_E for event 60 along both the paths will be

$$= 22 + 5 = 27.$$

For the third path 10—20—30—60,

$$T_E = 0 + 5 + 10 + 4 = 19.$$

Similarly, for the fourth path 10—20—40—60,

$$T_E = 0 + 5 + 8 + 8 = 21.$$

Since the earliest expected time for event 60 has to be the largest of the above, $T_E = 27$.

The following table gives T_E for each event of the network shown in Fig. 6.3. Fig. 6.4 shows the T_E's marked on the event circles.

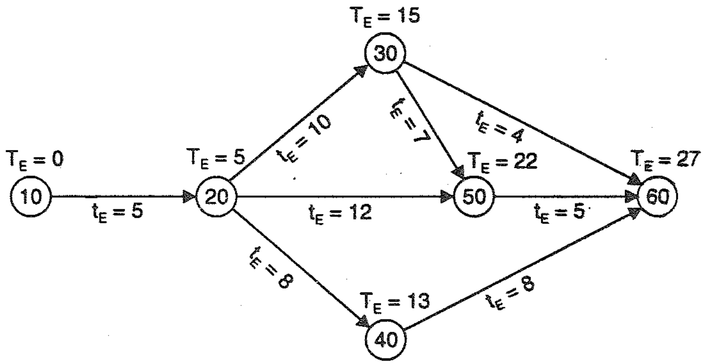


FIG. 6.4

Event	T_E	*Remarks*
10	0	
20	5	
30	15	
40	13	
50	22	Path 10—20—30—50
60	27	Path 10—20—30—50—60

6.3. FORMULATION FOR T_E

The method described above may be all right for small networks, but for large or complicated networks in which an event under consideration may have many predecessor events, it is better

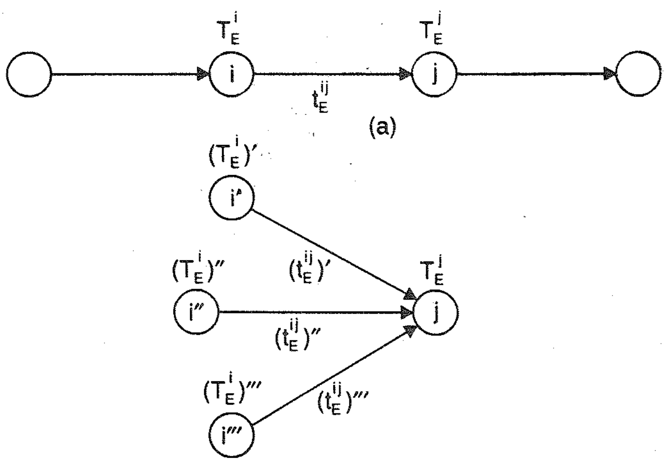


FIG. 6.5

to formulate a rule for computation of T_E so that frequent references to the network may not be necessary.

Let us represent an activity symbolically by ij where i is the predecessor event and j is the successor event, and i—j is the activity connecting the two events. Since T_E for a successor event is equal to T_E for the predecessor event plus the expected activity time (t_E), we have

T_E (successor event) = T_E (predecessor event) + t_E (activity).

Expressed symbolically,

$$T_E^j = T_E^i + t_E^{ij} \quad \text{...(6.1)}$$

The above formulation is true if there is only one predecessor event. If, however, there are more than one predecessor events to the successor event (*i.e.* event under consideration), the above rule needs modification since the event j cannot occur unless all activities leading to it are completed. Hence T_E for the event will be equal to *maximum* of $(T_E^i + t_E^{ij})$ along various activity paths. Hence

$$T_E^j = (T_E^i + t_{ij})_{max}. \quad \text{...(6.2)}$$

Thus, in Fig. 6.5, event j has three predecessor events i', i'' and i'''. With the three activities $(ij)'$, $(ij)''$ and $(ij)'''$ leading to it. The earliest expected time T_E^j for the event will be the maximum of $(T_E^i + t_E^{ij})'$, $(T_E^i + t_E^{ij})''$ and $(T_E^i + t_E^{ij})'''$.

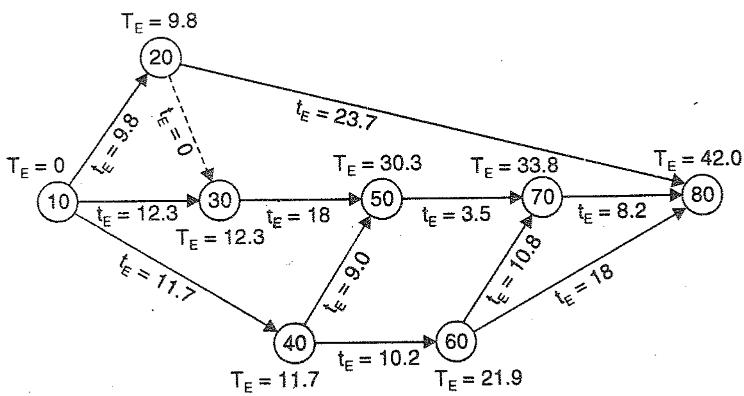


FIG. 6.6

Let us now apply the above formulation to the network shown in Fig. 6.6.

Event 10 has no predecessor event. Hence

$$T_E^{10} = 0$$

For event 20,

$$T_E^{20} = T_E^{10} + T_E^{10-20}$$

or $$T_E^{20} = 0 + 9.8 = 9.8$$

Event 30 has two predecessor events, 10 and 20. For each of these,

$$T_E^{30} = T_E^{10} + t_E^{10-30}$$
$$= 0 + 12.3 = 12.3$$

and $$T_E^{30} = T_E^{20} + t_E^{20-30}$$
$$= 9.8 + 0 = 9.8.$$

Out of these, the maximum value is 12.3. Hence $T_E^{30} = 12.3$. The maximum value is often *underscored* by dash (—). Similarly, the computations for other events are as under :

Event 40 : $T_E^{40} = T_E^{10} + t_E^{10-40} = 0 + 11.7 = 11.7$

Event 50 : (*i*) $T_E^{50} = T_E^{30} + t_E^{30-50} = 12.3 + 18 = \underline{30.3}$

(*ii*) $T_E^{50} = T_E^{40} + t_E^{40-50} = 11.7 + 9 = 20.7$

∴ $T^{50} = \underline{30.3}$

Event 60 : $T_E^{60} = T_E^{40} + t_E^{40-60} = 11.7 + 10.2 = 21.9$

Event 70 : (*i*) $T_E^{70} = T_E^{50} + t_E^{50-70} = 30.3 + 3.5 = 33.8$

(*ii*) $T_E^{70} = T_E^{60} + t_E^{60-70} = 21.9 + 10.8 = 32.7$

∴ $T_E^{70} = \underline{33.8}$

Event 80 : (*i*) $T_E^{80} = T_E^{20} + t_E^{20-80} = 9.8 + 23.7 = 33.5$

(*ii*) $T_E^{80} = T_E^{70} + t_E^{70-80} = 33.8 + 8.2 = \underline{42.0}$

(*iii*) $T_E^{80} = T_E^{60} + t_E^{60-80} = 21.9 + 18 = 39.9$

∴ $T_E^{80} = \underline{42.0}$

Computations are usually done in a tabular form. This is illustrated in Table 6.1, in which events are tabulated, starting with the end event, for convenience in computations. Thus, event 80 is entered first in the successor event column. Since it has three predecessor events (70, 60 and 20), these are entered next in their numerical order, with the high numbered event first. For all these three entries, the successor event is the same (*i.e.* 80). The *next lower event* 70 is then entered in the successor event column. It has two predecessor events 60 and 50 which are entered in the predecessor event column in the decreasing order of their numerical value. This procedure is followed till all the events are entered. The last successor event that is entered is 20 which has a predecessor event 10. The initial event 10 is not entered in the table since it does not have any predecessor event.

After having entered the successor and predecessor events, the expected time t_E^{ij} for each activity is entered. The column for *earliest expected time* t_E^j is then filled, starting from the bottom.

Table 6.1
Computation of Earliest Expected Time

Successor event j	*Predecessor event* i	*Activity* i—j	t_E^{ij}	t_E^j
80	70	70—80	8.2	<u>42.0</u>
	60	60—80	18.0	39.9
	20	20—80	23.7	33.5
70	60	60—70	10.8	32.7
	50	50—70	3.5	<u>33.8</u>
60	40	40—60	10.2	<u>21.9</u>
50	40	40—50	9.0	20.7
	30	30—50	18.0	<u>30.3</u>
40	10	10—40	11.7	<u>11.7</u>
30	20	20—30	0	9.8
	10	10—30	12.3	<u>12.3</u>
20	10	10—20	9.8	<u>9.8</u>

For example, ***for event 20***,

$$T_E^{20} = T_E^{10} + t_E^{10-20} = 0 + 9.8 = 9.8.$$

Since there is only one path to event 20, only one value of T_E^{20} has been obtained. This value is entered in the table and is underscored.

Next, ***for the activity 30***, we have

$$T_E^{30} = T_E^{10} + t_E^{10-30} = 0 + 12.3 = \underline{12.3}$$

also $$T_E^{30} = T_E^{20} + t_E^{20-30} = 9.8 + 0 = 9.8.$$

Here we obtain two values for T_E^{30} out of which the greater value (*i.e.* 12.3) is underscored in the table, to *identify* it, so that this value is used for further computations.

For event 40, $T_E^{40} = T_E^{10} + t_E^{10-40} = 0 + 11.7 = \underline{11.7}$

Since this is the only value, it is underscored.

For event 50, $T_E^{50} = T_E^{30} + t_E^{30-50} = 12.3 + 18.0 = 30.3$

It is to be noted that for T_E^{30}, greater of the two values (shown underscored) has been used.

Also $T_E^{50} = T_E^{40} + t_E^{40-50} = 11.7 + 9.0 = 20.7.$

Out of the two values (*i.e.* 30.3 and 20.7) of T_E^{50}, the greater value (*i.e.* 30.3) is underscored.

For event 60, $T_E^{60} = T_E^{40} + t_E^{40-60} = 11.7 + 10.2 = 21.9.$

For event 70, $T_E^{70} = T_E^{50} + t_E^{50-70} = 30.3 + 3.5 = 33.8.$

Here also the greater of the two values of T_E^{60} (*i.e.* 30.3) has been used.

Also, $T_E^{70} = T_E^{60} + t_E^{60-70} = 21.9 + 10.8 = 32.7$

Out of the two values of T_E^{70} so obtained, the greater value (*i.e.* 33.8) is underscored.

For event 80,

$$T_E^{80} = T_E^{20} + t_E^{20-80} = 9.8 + 23.7 = 33.5$$

$$T_E^{80} = T_E^{60} + t_E^{60-80} = 21.9 + 18.0 = 39.9$$

$$T_E^{80} = T_E^{70} + t_E^{70-80} = 33.8 + 8.2 = \underline{42.0}.$$

Here also, for T_E^{70}, the greater of the two values (*i.e.* 33.8) has been used. Thus, we obtain three values of T_E^{80}. Out of these, the maximum values (*i.e.* 42.2) is underscored.

In the computations of T_E^j for the above table, we proceeded from the *initial event* and ended with the end event. This is known as the *forward pass*, in which the network is traversed from initial event node to the final event node.

6.4. LATEST ALLOWABLE OCCURRENCE TIME

A planner is equally concerned with the completion of the project within the scheduled time. For each event, therefore, some time limit is allotted by which that event must occur. The latest time by which an event must occur, to keep the project on schedule is called the *latest allowable occurrence time*. It is denoted by symbol T_L. This is, therefore, another *event time*.

Whenever a project is taken in hand, decision is made regarding the completion time of the project and the accepted figure is called the *scheduled completion time* (or *the contractual obligation time*) and is denoted by symbol T_S. Naturally, T_S refers to the latest time of the last event (*i.e.* $T_S = T_L$).

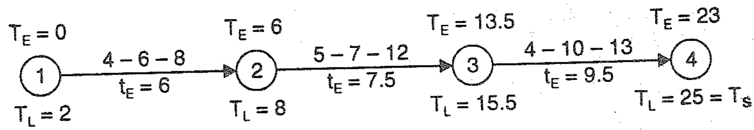


FIG. 6.7

To compute the latest allowable occurrence time for various events, let us again consider the simple network of Fig. 6.2. Let us assume that *scheduled completion time* T_S for the project is 25, meaning thereby that the end event 4 must occur, latest by 25 units of time after the project is initiated.

Thus, for the last event, $T_L^4 = T_S = 25$.

The activity 3—4 takes 9.5 units of time for its completion. Hence event 3 cannot occur later than $25 - 9.5 = 15.5$. Thus, $T_L^3 = 15.5$.

Similarly, for event 2, $T_L^2 = T_L^3 - t_E^{2-3} = 15.5 - 7.5 = 8.0$

and for the initial event.

$$T_E^1 = T_L^2 - t_E^{1-2} = 8.0 - 6 = 2.0.$$

These values of latest occurrence time for each event are indicated below the corresponding event circles.

Let us now consider the network of Fig. 6.4, reproduced in Fig. 6.8, where some events may have more than one successor events.

Let us presume that the *scheduled completion time* T_S for the project is 27 units of time. The latest occurrence time T_L for the last event is therefore 27.

For event 50, latest occurrence time is given by

$$T_L^{50} = T_L \text{ for } 60 - t_E \text{ for } (50\text{—}60)$$
$$= 27 - 5 = 22.$$

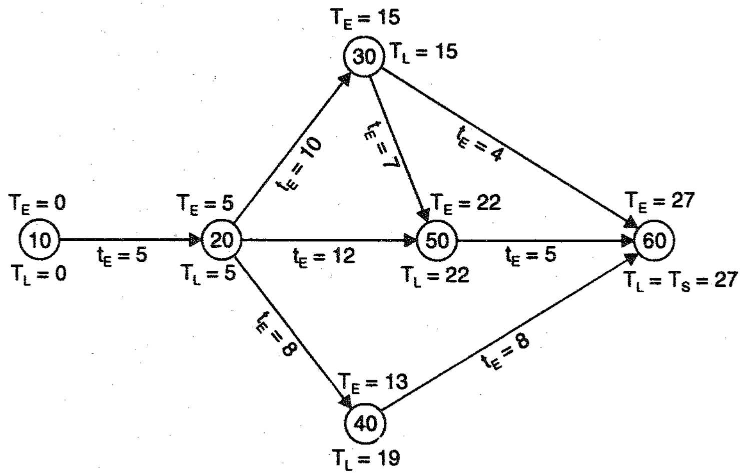


FIG. 6.8

For event 40,

$$T_L^{40} = T_L \text{ for } 60 - t_E \text{ for } (40\text{—}60)$$
$$= 27 - 8 = 19.$$

Event 30 has two successor events : event 40 and event 50. Hence two values of T_L are obtained as under :

$$T_L^{30} = T_L \text{ for } 60 - t_E \text{ for } (30\text{—}60)$$
$$= 27 - 4 = 23$$

and
$$T_L^{30} = T_L \text{ for } 50 - t_E \text{ for } (30\text{—}50)$$
$$= 22 - 7 = 15.$$

Out of this, the *minimum* value (*i.e.* 15) will be the appropriate value of T_L^{30}. This is because if event 50 cannot occur later than 22 units of time after the beginning of the project, event 30 cannot occur later than 15 units of time after the initiation of project since activity 30—50 takes 7 units of time for its completion. If a higher value of T_L^{30} (= 23) is permitted, T_L^{50} will be = 23 + 7 = 30 and the event 50 will be late by 8 units of time. Hence a minimum value, out of the various values, is to be selected.

Similarly, event 20 has three successor events : 50, 40 and 30 having T_L's as 22, 19 and 15 respectively. Therefore, we get three values of T_L^{20} as under :

(*i*) $T_L^{20} = T_L$ for $50 - t_E$ for (20—50)

$= 22 - 12 = 10$

(*ii*) $T_L^{20} = T_L$ for $40 - t_E$ for (20—40)

$= 19 - 8 = 11$

(*iii*) $T_L^{20} = T_L$ for $30 - t_E$ for (20—30)

$= 15 - 10 = 5.$

Out of above, the minimum value (*i.e.* 5) is the appropriate value of T_L^{20}.

For the initial event,

$T_L^{10} = T_L$ for $20 - t_E$ for (10—20)

$= 5 - 5 = 0.$

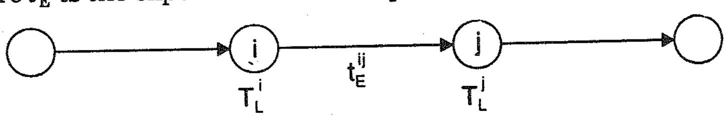
6.5. FORMULATION FOR T_L

Consider an activity *i j* [Fig. 6.9 (*a*)], in which *i* is the predecessor event and *j* is the successor event.

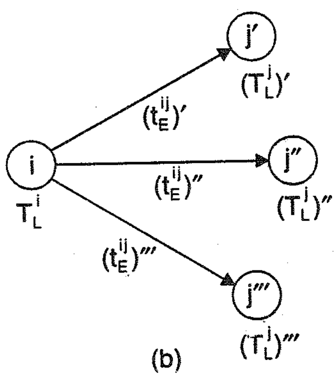
Let the latest occurrence time T_L^j be known. The latest occurrence time T_L^i for predecessor event is given by

$$T_L^i = T_L^j - t_E^{ij} \quad \text{...(6.3)}$$

where t_E^{ij} is the expected time of completion of activity *i j*.



(a)



(b)

FIG. 6.9

The above formulation is useful when the event i under consideration has only one successor event (j). If however, there are more than one successor events ($j', j'' j'''$ etc.), the *minimum* of $(T_L^j - t_E^{ij})$ will be the appropriate latest occurrence time T_L^i for event i. This is so because if the higher of the various values is taken, the latest occurrence time for the successor events will be also increased, suggesting a delay in the project completion.

Thus, $$T_L^i = (T_L^j - t_E^{ij})_{min} \quad ...(6.4)$$

Thus, in Fig. 6.9 (b), there are three successor events (j', j'' and j''') to event i. The latest occurrence time T_L^i for the event i will be the *minimum* of $(T_L^j - t_E^{ij})'$, $(T_L^j - t_E^{ij})''$ and $(T_L^j - t_E)'''$.

Let us now apply this formulation to the network of Fig. 6.6, reproduced in Fig. 6.10.

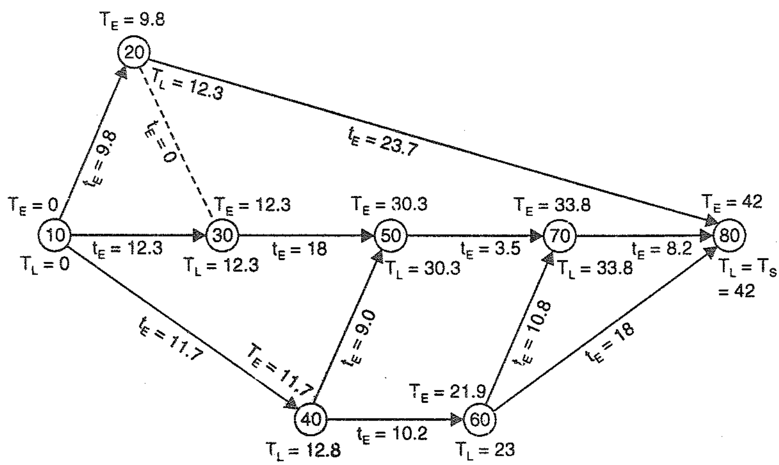


FIG. 6.10

Here, the *scheduled completion time* (T_S) is not given. In such a circumstance, T_S can be taken equal to T_E of end event.

Thus, $$T_S = T_E^{80} = 42$$

Hence $$T_L^{80} = T_S = 42$$

For computation of T_L for other events, we start backwards from the end event 80 and apply the rule

$$T_L^i = (T_L^j - t_E^{ij})_{min}.$$

This is done till the initial event is reached. The procedure is known as the *backward pass*, in contrast to the procedure of *forward pass* used for the computation of earliest expected time T_E.

(*i*) ***Event 70***

It has only one successor event (80).

$$\therefore \quad T_L^{70} = T_L^{80} - t_E^{70-80}$$
$$= 42 - 8.2 = \underline{33.8}.$$

This value is underscored for further use.

(*ii*) ***Event 60***

This has two successor events : 80 and 70

$$T_L^{60} = T_L^{80} - t_E^{60-80}$$
$$= 42 - 18 = 24$$

and
$$T_L^{60} = T_L^{70} - t_E^{60-70}$$
$$= 33.8 - 10.8 = \underline{23.}$$

Out of these, minimum value (*i.e.* 23) is the appropriate value of T_L^{60} and is underscored for further use.

(*iii*) ***Event 50***

It has only one successor event (70)

$$\therefore \quad T_L^{50} = T_L^{70} - t_E^{50-70}$$
$$= 33.8 - 3.5 = \underline{30.3.}$$

(*iv*) ***Event 40***

It has two successor event : 60 and 50

$$\therefore \quad T_L^{40} = T_L^{60} - t_E^{40-60} = 23 - 10.2 = \underline{12.8}$$

and
$$T_L^{40} = T_L^{50} - t_E^{40-50}$$
$$= 30.3 - 9 = 21.3$$

Out of these two, the lower value of 12.8 is adopted.

(*v*) ***Event 30***

It has also one successor event (50)

$$\therefore \quad T_L^{30} = T_L^{50} - t_E^{30-50}$$
$$= 30.3 - 18 = \underline{12.3}$$

(*vi*) ***Event 20***

It has two successor events : 80 and 30.

$$\therefore \quad T_L^{20} = T_L^{80} - t_E^{30-80}$$
$$= 42 - 23.7 = 18.3$$

and
$$T_L^{20} = T_L^{30} - t_E^{20-30}$$
$$= 12.3 - 0 = \underline{12.3}$$

Out of these, the minimum value (12.3) is adopted.

(*vii*) ***Event 10***

It has three successor events 40, 30 and 20.

$$\therefore \quad T_L^{10} = T_L^{40} - t_E^{10-40}$$
$$= 12.8 - 11.7$$
$$= 1.1$$
$$T_L^{10} = T_L^{30} - t_E^{10-30}$$
$$= 12.3 - 12.3$$
$$= \underline{0}$$

and
$$T_L^{10} = T_L^{20} - t_E^{10-20}$$
$$= 12.3 - 9.8$$
$$= 2.5.$$

Out of these, the minimum value is 0, and is adopted as the appropriate value of T_L for the initial event.

The computation of T_L is usually done in Tabular form. This is illustrated in Table 6.2, for the network shown in Fig. 6.10.

The first two columns are for predecessor events (i) and successor events (j) respectively. Here also, the tabulation of events is done with the highest numbered events, decreasing downwards in numerical order. Column 3 is for the activity i—j while column 4 is for the corresponding activity expected times t_E^{ij}.

The computation of T_L^i is started *in the backward* direction, starting with the highest numbered predecessor event. Thus, for event 70,

$$T_L^{70} = T_L^{80} - t_E^{70-80}$$
$$= 42 - 8.2$$
$$= \underline{33.8}.$$

This is the only value ; it is underscored for further use. For activity 60, there are two successor events, 80 and 70.

$$\therefore \quad T_E^{60} = T_E^{80} - t_E^{60-80}$$
$$= 42 - 18.0$$
$$= 24.0$$

and
$$T_E^{60} = T_E^{70} - t_E^{60-70}$$
$$= 33.8 - 10.8$$
$$= \underline{23.0}.$$

Table 6.2
Computation of Latest Allowable Occurrence Time

Predecessor event (i)	*Successor event (j)*	*Activity i–j*	t_E^{ij}	T_L^i
70	80	70—80	8.2	<u>33.8</u>
60	80	60—80	18.0	24.0
	70	60—70	10.8	<u>23.0</u>
50	70	50—70	3.5	<u>30.3</u>
40	60	40—60	10.2	<u>12.8</u>
	50	40—50	9.0	21.3
30	50	30—50	18.0	<u>12.3</u>
20	80	20—80	23.7	18.3
	30	20—30	0.0	<u>12.3</u>
10	40	10—40	11.7	1.1
	30	10—30	12.3	<u>0.0</u>
	20	10—20	9.8	2.5

The minimum value (*i.e.* 23.0) is underscored.

For activity 50, the successor event is 70 which has a T_L of 33.8.

$$\therefore \quad T_L^{50} = T_L^{70} - t_E^{50-70}$$
$$= 33.8 - 3.5 = 30.3.$$

The computations are thus continued till the first event is reached.

6.6. COMBINED TABULAR COMPUTATIONS FOR T_E and T_L

The computation of *earliest expected time* (T_E) is done by *forward pass*, starting from the *initial event*. For such computations, predecessor events (*i* events) form the base. Table 6.1 is used for such computations. On the other hand, the computation of *latest allowable occurrence time* (T_L) is done by *backward pass*, starting from the *end event*. For such computations, successor events (*j* events) form the base. Table 6.2 is used for such computations. However, for most of the networks, computations of both T_E and T_L

for each event is required. None of the two tables suggested earlier are convenient. A combined tabular form (Table 6.3) is therefore suggested, with the help of which both T_E and T_L can be computed.

For illustration, let us take the same example of network shown in Fig. 6.6.

Table 6.3
Computations for T_E and T_L

Event No.	Earliest expected time (↓)				Latest occurrence time (↑)			
	Predecessor event (i)	t_E^{ij}	T_E^j	T_E	Successor event (j)	t_E^{ij}	T_L^i	T_L
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
10	—	—	0	0	20	9.8	2.5	0
					30	12.3	0	
					40	11.7	1.1	
20	10	9.8	9.8	9.8	30	0	12.3	12.3
					80	23.7	18.3	
30	10	12.3	12.3	12.3	50	18.0	12.3	12.3
	20	0	9.8					
40	10	11.7	11.7	11.7	50	9.0	21.3	12.8
					60	10.2	12.8	
50	30	18.0	30.3	30.3	70	3.5	30.3	30.3
60	40	10.2	21.9	21.9	70	10.8	23.0	23.0
					80	18.0	24.0	
70	50	3.5	33.8	33.8	80	8.2	33.8	33.8
	60	10.8	32.7					
80	20	23.7	33.5	42.0	—	—	42.0	42.0
	60	18.0	39.9					
	70	8.2	42.0					

Column 1 of Table 6.3 gives the event number, starting with the initial event, and proceeding in the direction of increasing numbers of the events.

Column 2 gives the predecessor events while column 6 gives the successor events to the events of column 1. These columns are therefore completed first, using the network diagram. An event under consideration (column 1) may have one or more than one predecessor events (column 2), and one or more than one successor events (column 6). A horizontal line is drawn after entering all the predecessor events and successor events to every event of column 1.

Then, computations are done for the *earliest expected times of the event,* in columns (3), (4) and (5). Column 3 is for the activity times t_E^{ij} where j is the event under consideration (column 1) and i is the predecessor event (column 2). T_E^j is computed by using Eq. 6.1

$$T_E^j = T_E^i + t_E^{ij}$$

Where there are more than one predecessor events, several values of T_E^j are obtained, which are entered in column (4). The maximum value of $\underline{T_E^j}$ is underscored. This underscored value is the appropriate value of the *earliest expected time* for the event under consideration (column 1), and is entered as T_E in column 5. For the computations of T_E, thus, we use the *forward pass*, starting with the initial event, and proceeding in the downward direction (↓) in the Table. We observe that T_E for last event comes out to be 42.

Then, we compute the *latest occurrence time* of the event under consideration (column 1), in columns 7, 8 and 9. Column (7) is the activity time t_E^{ij}, where i is the event under consideration (column 1) and j is the successor event (column 6). T_L^i is computed by using Eq. 6.3

$$T_L^i = T_L^i - T_E^{ij}$$

Computations are done by *backward pass*, starting with the end event and proceeding upwards (↑) in the table. The scheduled completion time T_S is taken equal to T_E. Hence $T_L = T_S = T_E = 42$ for the last event.

When there are more than one successor events, several values of T_L^i are obtained, which are entered in column (8). The minimum value of T_L^i is underscored. This underscored value is the appropriate value of the *latest allowable occurrence time* for the event under consideration (column 1) and is entered as T_L in column 9.

Thus, for each of the activities of column 1, T_E is given in column 5 and T_L is given in column 9.

PROBLEMS

1. Explain the term 'earliest expected time'. Formulate an expression for determining the same.
2. What do you understand by the 'latest allowable occurrence time'? How do you determine it ?
3. Explain with the help of a tabular form, how do you determine the 'earliest expected time' and the 'latest allowable occurrence time' for a network. Differentiate clearly between the 'forward pass' and 'backward pass'.
4. A network is shown in Fig. 6.11, with the expected time of completion of each activity. Determine the earliest expected time and latest occurrence time for each event.

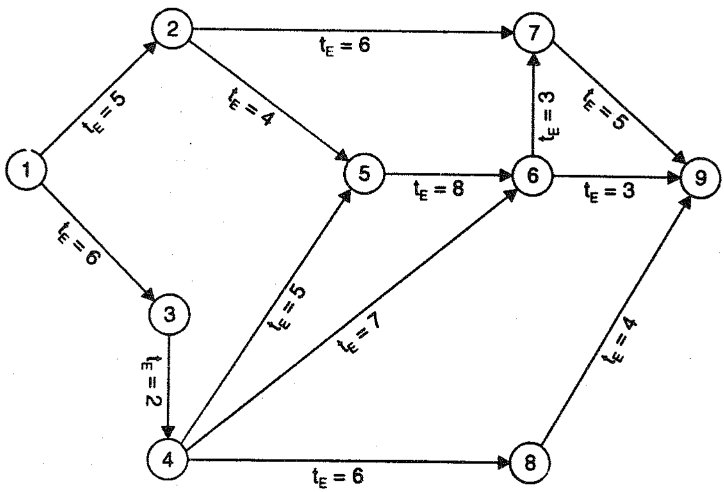


FIG. 6.11

5. The network for a construction project is shown in Fig. 6.12. The three time estimates for each activity are given along each activity arrow. Compute (*a*) expected time of completion of each activity, (*b*) earliest expected time for each event, (*c*) latest allowable occurrence time for each event.

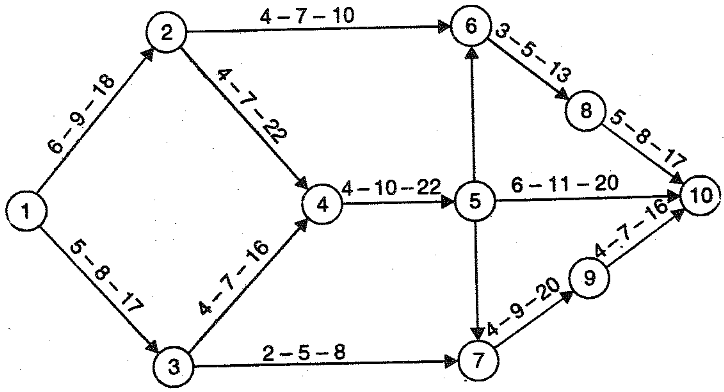

1
2
3
4
5
6
7
8
9
10
6 – 9 – 18
5 – 8 – 17
4 – 7 – 10
4 – 7 – 22
4 – 7 – 16
2 – 5 – 8
4 – 10 – 22
3 – 5 – 13
5 – 8 – 17
6 – 11 – 20
4 – 7 – 16
4 – 9 – 20

FIG. 6.12

7

PERT : Network Analysis

7.1. SLACK

In the previous chapter, we have computed two important times for the events : (*i*) the earliest expected time (T_E) and (*ii*) the latest allowable occurrence time T_L. The difference between the two times of an activity indicates the range between which the occurrence time of an event can vary. *Slack may be simply defined as the difference between the latest allowable time and the earliest expected time of an event.*

$$\therefore \qquad S = T_L - T_E \qquad ...(7.1)$$

where S is the slack for any event.

Let us take the case of a simple network of Fig. 6.3 (chapter 6), reproduced in Fig. 7.1, with T_E and T_L for each event marked.

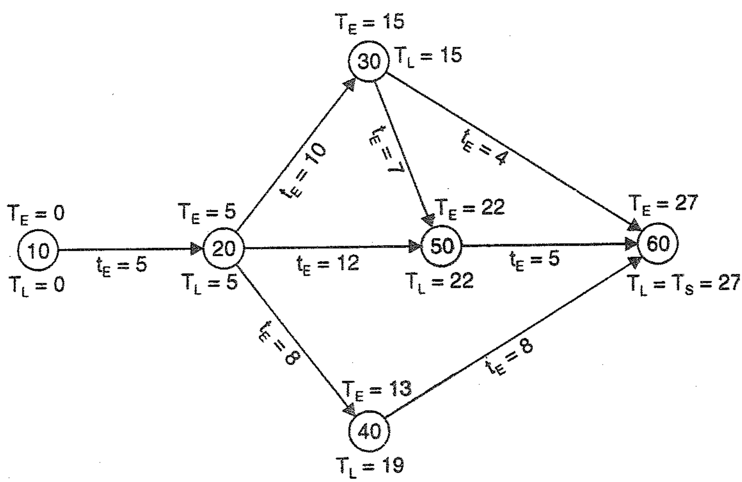


FIG. 7.1

The values of T_E, T_L and S for each event is tabulated in Table 7.1.

Table 7.1
Computation of Slack

Event No.	T_E	T_L	$S = T_L - T_E$
10	0	0	0
20	5	5	0
30	15	15	0
40	13	19	6
50	22	22	0
60	27	27	0

Here we observe that the slack for all events, except for event 40, is zero. The slack for event 40 is 6 units of time. This means that even if event 40 occurs 6 units of time *late*, the scheduled completion date of the project will not be affected. After the completion of event 20, event 40 can occur at 8 + (0 to 6) later. The other events 10, 20, 30, 50 and 60 do not have any slack time and hence their *occurrence is critical*.

Thus, we find that *slack* gives the idea of 'time to spare'. Slack means more time to work, less to worry about. It reveals about those areas which have an excess of resources from which trade-offs can be rearranged. It also spots those areas which are potential trouble areas, *i.e.* those areas of zero or minimum slack.

Slack can be *positive*, *zero* or *negative*, depending upon the relationship between T_L and T_E.

Positive slack. Positive slack is obtained when T_L is more than T_E for an event. It is an indication of an *ahead of schedule condition* (excess resources).

Zero slack. Zero slack is obtained when T_L is equal to T_E for an event. It is an indication of a *on schedule condition* (adequate resources).

Negative slack. Negative slack is obtained when the scheduled time of completion, T_S (and hence T_L) is less than the T_E. It is an indication of a *behind of schedule condition* (lack of resources).

Example 7.1. *Analyse with respect to resources the network shown in Fig. 7.2. Values are in days.*

Solution. The computations of T_E, T_L and S are arranged in Table 7.2. First T_E for all the events are computed by the *forward pass*, taking $T_E^1 = 0$. From the Table, we find $T_E^5 = 20$ for the last event. Since the scheduled time of completion (T_S) is not given, it is taken equal to $T_E^5 = 20$ days. Thus, $T_L^5 = T_S = 20$ days. Having found T_L for the last event, T_L for other events are computed by the *backward pass*.

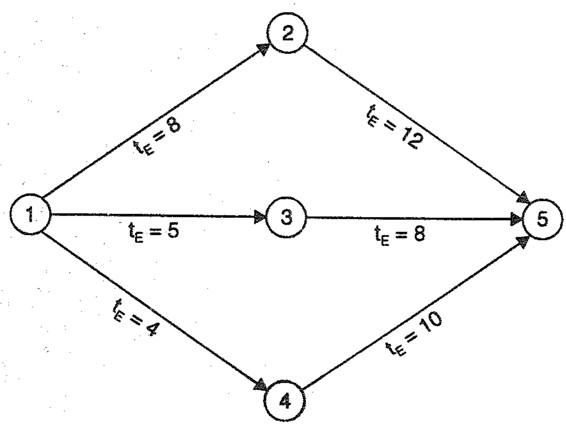


FIG. 7.2

Table 7.2

Event No.	*Earliest expected time* ↓				*Latest occurrence time* ↑				*Slack* $S = T_L - T_E$
	Predecessor event (i)	t_E^{ij}	T_E^i	T_E	*Successor event (j)*	t_E^{ij}	T_L^i	T_L	
1	—	—	<u>0</u>	0	2 3 4	8 5 4	<u>0</u> 7 6	0	0
2	1	8	<u>8</u>	8	5	12	<u>8</u>	8	0
3	1	5	<u>5</u>	5	5	8	<u>12</u>	12	7
4	1	4	<u>4</u>	4	5	10	<u>10</u>	10	6
5	2 3 4	12 8 10	<u>20</u> 13 14	20	—	—	20	20	0

The values of T_E and T_L, so obtained, for each event are marked on the network diagram, reproduced in Fig. 7.3.

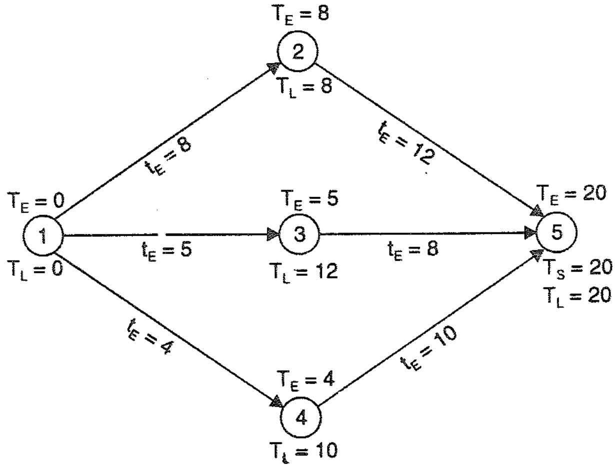


FIG. 7.3

The following conclusions are drawn :

1. Events 1, 2 and 5 have zero slack. Any delay in the activities connecting them would cause corresponding delay in the completion of the project. The occurrence of these events is *critical.*

2. Events 3 and 4 have slack of 7 and 6 days respectively. The activities connecting either of these events can be *delayed* by the slack value without affecting the scheduled completion time of the project.

3. Event 1 must start exactly on schedule. However, the above analysis suggests that the resources of activities 1—3 and 1—4 can be partly shifted to aid activities 1—2 and 2—5.

7.2. CRITICAL PATH

It is important to note that the value of slack, associated with an event, determine how critical that event is. The less the slack (more negative), the more critical an event is. A *critical path* is the one which connects the events having zero or minimum slack times. All the events along the *critical path* are considered to be *critical* in the sense that any delay in their occurrence will result in the delay in the scheduled completion of the project. *Eventually, a critical path is the longest path (time wise) connecting the initial and end event.*

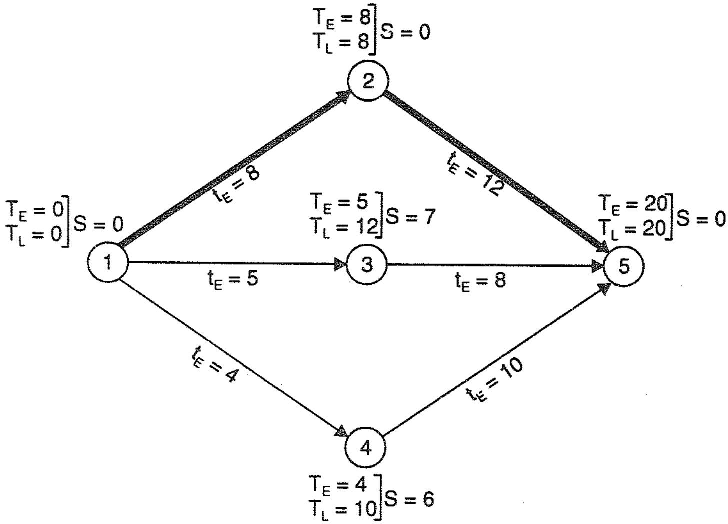


FIG. 7.4. THE CRITICAL PATH.

A critical path is distinctly marked in the network, usually by thick line.

For example, consider the network of Fig. 7.3 (Example 7.1). The path 1—2—5 is along the events having zero slack times. This path is therefore the critical path, shown by thick lines in Fig. 7.4. Note that Σt_E of all the activities along the critical path is equal to T_E of that last event. Along any other path, Σt_E is less than T_E of last event. *Thus, critical path is the longest path.*

7.3. ILLUSTRATIVE EXAMPLES

Example 7.2. *Determine the critical path for the network of Fig. 6.10.*

Solution. The earliest occurrence time (T_E) and latest allowable occurrence time (T_L) for each event are marked in Fig. 6.10. Slack for each event can be computed by taking the difference between T_L and T_E for each event, as indicated in Table 7.3.

Table 7.3

Event	T_E	T_L	*Slack* $S = T_L - T_E$
10	0	0	0
20	9.8	12.3	2.5
30	12.3	12.3	0
40	11.7	12.8	1.1
50	30.3	30.3	0
60	21.9	23.0	1.1
70	33.8	33.8	0
80	42.0	42.0	0

From the above table, we find that events 10, 30, 50, 60 and 80 have zero slack. Hence the path joining the events is the *critical path*. This is shown by thick lines in Fig. 7.5. Note that the critical path is the longest path, timewise. Also, note that Σt_E of all activities along the critical path is equal to T_E of last event.

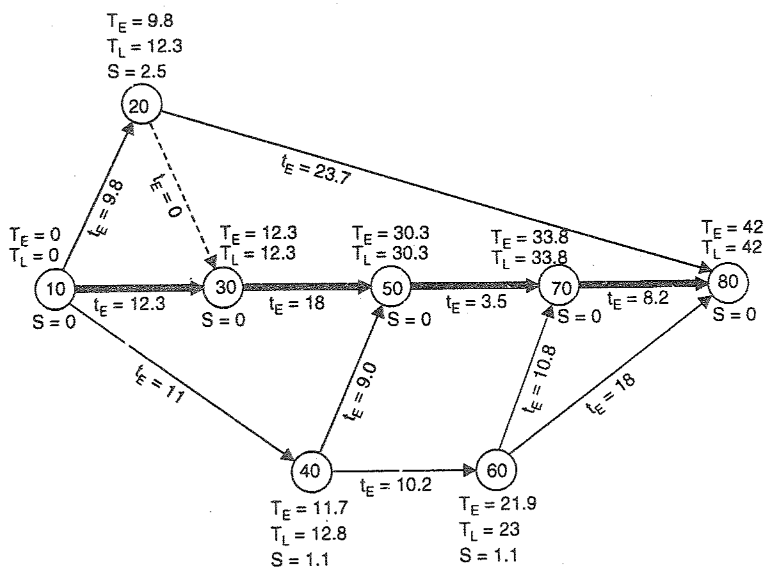


FIG. 7.5

Example 7.3. *Determine the critical path for the network shown in Fig. 7.6. Numbers indicate time in weeks.*

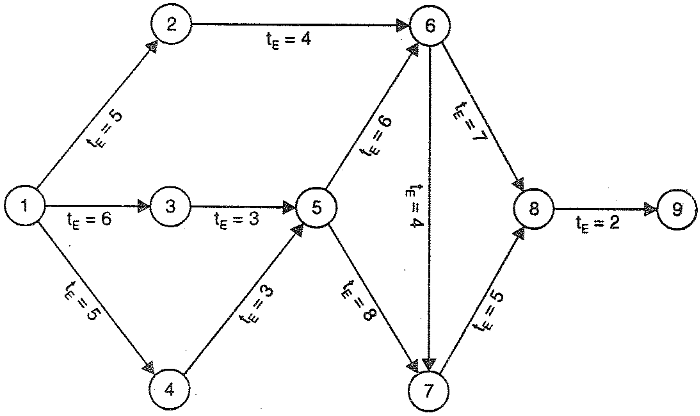


FIG. 7.6

Solution. For the determination of the critical path, it is essential to first determine T_E, T_L and S for each event. This is conveniently done in a tabular form. In Table 7.4, the earliest expected time for each event is first computed using forward pass and column 5 is obtained. This way T_E for the last event (9) is obtained. Since the scheduled time of completion of the project is not given, it is taken equal to T_E^9 = 26 weeks. Thus

$$T_L^9 = T_S = T_E^9 = 26 \text{ weeks.}$$

Knowing T_L^9, we calculate T_L for other events, using, backward pass, till the initial event is reached. Column 9 gives the latest allowable completion time for each event. Knowing T_E and T_L for each event, slack time S is found by taking the difference between T_L and T_S. This is tabulated in column 10.

From Table 7.4, we find that slack $S = 0$ at events 1, 3, 5, 6, 7, 8 and 9. Hence *critical path* is along 1—3—5—6—7—8—9, as shown by thick lines in Fig. 7.7. *Note that Σt_E of all activities along the critical path is equal to T_E of the last event.* Along any other path Σt_E is *less* than T_E of last event. Thus, the critical path is the longest path.

Table 7.4

Event No.	*Earliest expected time* ↓				*Latest occurrence time* ↑				Slack $S = T_L - T_E$
	Predecessor event (i)	t_E^{ij}	T_L^i	T_E	*Successor event (j)*	t_E^{ij}	T_L^i	T_L	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1	—	—	0	0	2 3 4	5 6 5	6 0 1	0	0
2	1	5	5	5	6	4	11	11	6
3	1	6	6	6	5	3	6	6	0
4	1	5	5	5	5	3	6	6	1
5	3 4	3 3	9 8	9	6 7	6 8	9 11	9	0
6	2 5	4 6	9 15	15	7 8	4 7	15 17	15	0
7	5 6	8 4	17 19	19	8	5	19	19	0
8	6 7	7 5	22 24	24	9	2	24	24	0
9	8	2	26	26	—	—	26	26	0

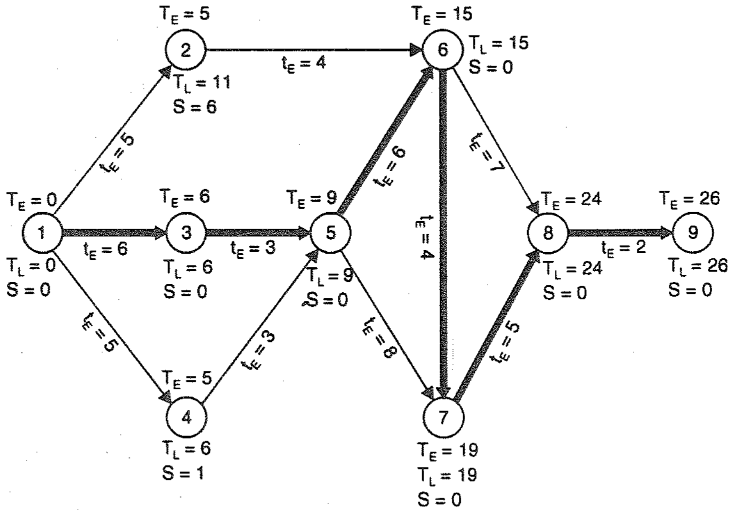


FIG. 7.7

Example 7.4. *The expected time of completion (in days) for each activity of a network is shown in Fig. 7.8. Determine the critical path. It is given that the scheduled completion time is 21 days.*

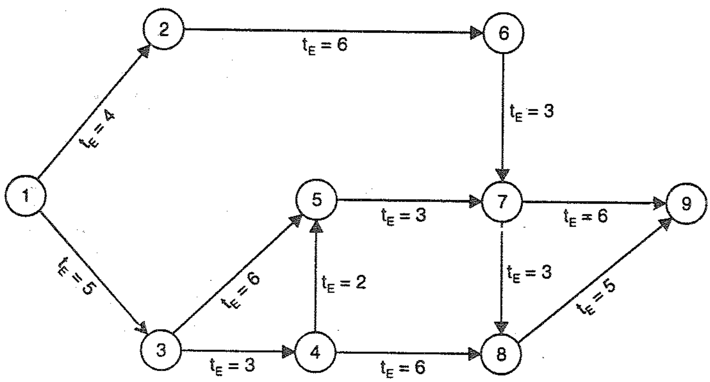


FIG. 7.8

Solution. The computations are arranged in the tabular form (Table 7.5). First, T_E for each activity is computed, as shown in column 5, from which we find that T_E for the last event comes out to be 22 days. The scheduled completion time (T_S) is 21 days only. This means that the project will have to be completed 1 day earlier than what it would normally take for its completion. This, of course, will have to be done by allocation of extra resources. The latest allowable occurrence time for the last event (T_L^9) will, therefore, be equal to T_L = 21 days. With this value, T_L is computed for all other events, by using *backward pass*. Column 9 gives the T_L values for all the events.

The slack for each event is determined by finding algebraic difference between T_L and T_E. This is tabulated in column 10 of Table 7.5. From this column, we find that the minimum value of slack is –1 day, and this occurs at events 1, 3, 5, 7, 8 and 9. Hence *critical path* is along 1—3—5—7—8—9, as shown by thick lines in Fig. 7.9. Note that Σt_E of the activities along the critical path is equal to T_E of the last event. This is not so along any other path.

Table 7.5

Event No.	*Earliest expected time* ↓				*Latest occurrence time* ↑				$S = T_L - T_E$
	Predecessor event (i)	t_E^{ij}	T_E^j	T_E	*Successor event (j)*	t_E^{ij}	T_L^i	T_L	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1	—	—	0	0	2 3	4 5	0 – 1	– 1	– 1
2	1	4	4	4	6	6	4	4	0
3	1	5	5	5	4 5	3 6	5 4	4	– 1
4	3	3	8	8	5 8	2 6	8 10	8	0
5	3 4	6 2	11 10	11	7	3	10	10	– 1
6	2	6	10	10	7	3	10	10	0
7	5 6	3 3	14 13	14	8 9	3 6	13 15	13	– 1
8	4 7	6 3	14 17	17	9	5	16	16	– 1
9	7 8	6 5	20	22	—	—	21	21	– 1

From column (10), we observe that next minimum value of slack is 0 which occurs at events 2, 4 and 6. Therefore, the path connecting events 2 and 6, *i.e.* path 1—2—6—7 is the *sub-critical* or *semi-critical path*. Similarly, the path connecting event 4, *i.e.* path 3—4—5 is also sub-critical or semi-critical path. These two sub-critical paths are shown by dotted lines drawn adjacent to the corresponding activity arrows, in Fig. 7.9.

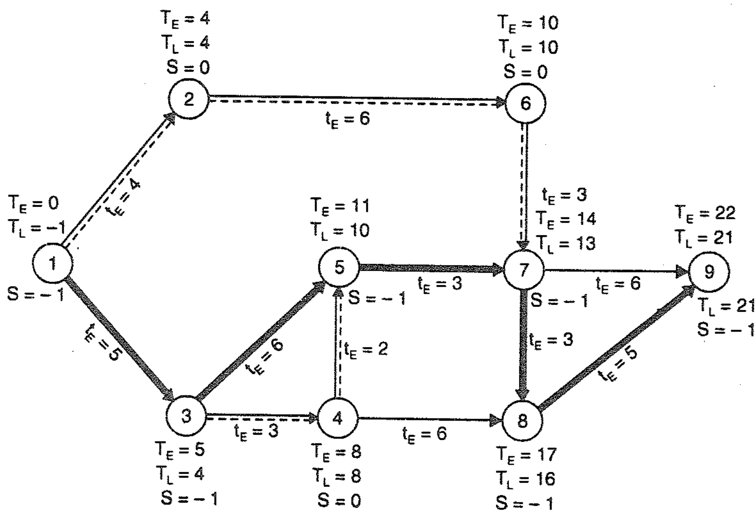


FIG. 7.9

7.4. PROBABILITY OF MEETING SCHEDULED DATE

After having calculated, the latest allowable occurrence time with the help of the assumed or given *scheduled completion time* of the project, and after having determined the critical path, the next question that remains to be answered is 'what is the probability of meeting the scheduled time ?' The answer to this question is sought by applying the probability theory to the network analysis.

We know that *critical path* is timewise the *longest path*. The critical path is along a number to activities of the path. In chapter 5, we considered three time estimates for any activity : (*i*) the optimistic time estimate (t_O), (*ii*) the most likely time estimate (t_L) and (*iii*) the pessimistic time estimate (t_P). We have also seen that these times have beta distribution (Fig. 5.6). Assuming the beta distribution, the *expected time* (t_E) for the completion of any activity is computed from Eq. 5.7.

$$t_E = \frac{t_O + 4t_L + t_P}{6} \quad ...(5.7)$$

The expected time (t_E) is such that there is fifty-fifty chance of completion of the activity in this time. Fig. 7.10 shows the beta distribution, with t_O, t_L, t_E and t_P marked. If we draw a vertical line through t_E, the shaded area to its left is *half* of the total area. In other words, this vertical line through t_E divides the beta distribution curve in two-halves. Hence the activity *ij*, for which the above probability distribution curve refers, has fifty per cent probability of its completion within time t_E.

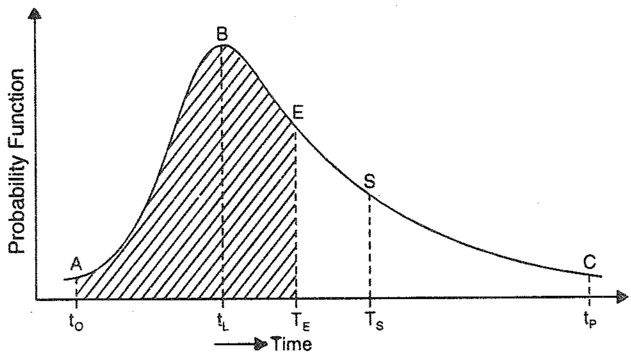


FIG. 7.10. BETA DISTRIBUTION.

Again the probability of completion of the activity within some other time t_S will be equal to the area under the curve upto vertical line through t_S divided by the total area of the curve :

$$\text{Probability} = \frac{\text{area under } ABS}{\text{area under } ABC}$$

We have seen in the previous article that Σt_E of all activities along the critical path is equal to T_E of the last event. Though t_E of individual activities has random probability distribution (assumed as beta distribution), the variation of T_E for the project as a whole has fortunately a *normal distribution* for all practical purposes. This assumption of normal distribution is based on the *central limit theorem* as given below.

Central Limit Theorem

This theorem states that if there are n activities, each having its own β-distribution with means $\mu_1, \mu_2, \mu_3, ..., \mu_n$ and standard deviations $\sigma_1, \sigma_2, \sigma_3, ..., \sigma_n$ respectively, then the distribution of time for the *project as a whole* will approximately be a *normal distribution curve*. The normal distribution curve will have a mean μ and variance σ^2 given by

$$\mu = \mu_1 + \mu_2 + \mu_3 + ... + \mu_n \quad ...(7.2)$$

and

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + ... + \sigma_n^2 \quad ...(7.3)$$

Normal Distribution Curve

Fig. 7.11 shows the normal distribution curve, which is symmetrical about its apex. Since $T_E = \Sigma t_E$, and since each t_E has

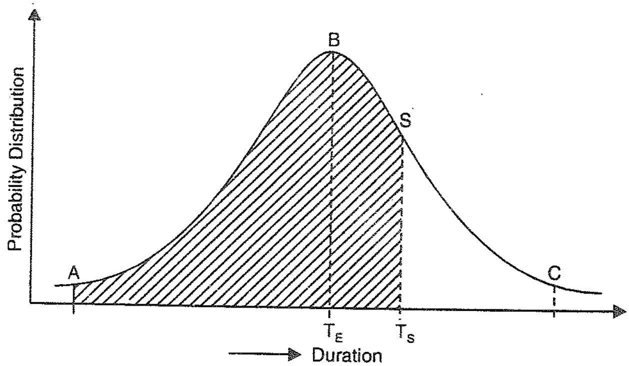


FIG. 7.11. NORMAL DISTRIBUTION.

fifty-fifty probability of completion of the activity, T_E should also have fifty-fifty probability. Hence we have to make the T_E value for the end event coincide with the modal value of the normal distribution curve. When once this is achieved, the random curve derived from a particular network will be reduced to the *normal form*.

Now, if T_S is the scheduled completion time of the project, the probability of completion of the project within the time T_S is given by

$$\text{Probability} = \frac{\text{Area under } ABS}{\text{Area under } ABC} \quad ...(7.4)$$

In order to find the shaded area, we take the help of the important property of the normal distribution curve discussed in § 5.6. Refer Fig. 5.5 and 7.12.

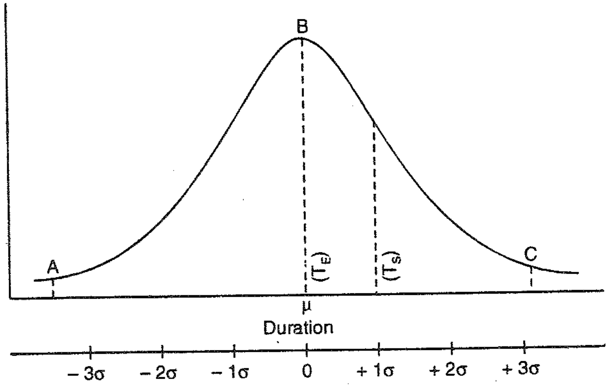


FIG. 7.12

If mean of the normal distribution is denoted by μ, it can be proved that

(*a*) Approximately 68 per cent of the value of normal distribution lie within ± σ from the average, where σ is the standard deviation. This means that the area of the curve between $x = \mu - \sigma$ to $x = \mu + \sigma$ is 68% of the total area (Fig. 5.5).

(*b*) Approximately 95 per cent of all the values lie within ± 2σ from the average. This means that area of the curve between $x = \mu - 2\sigma$ to $x = \mu + 2\sigma$ is 95% of the total area.

(*c*) Approximately 99.7 per cent of all the values lie within ± 3σ from the average. This means that the area of the curve between $x = \mu - 3\sigma$ to $x = \mu + 3\sigma$ is 99.7%.

Table 7.6 gives the values of probability corresponding the various values of the *normal deviate Z* (*i.e.* distance from the mean expressed in terms of σ).

Procedure for finding the probability of meeting the scheduled time of completion

The above properties of the normal distribution curve, and the % probability given in Table 7.6 can be used only if the random curve obtained from a particular network is reduced to the *normalized form*. This is obtained by making T_E value for the end event coincide with the modal value of the normal distribution curve. The following procedure is adopted for determining the probability of meeting the scheduled time of completion :

Table 7.6
Standard Normal Distribution Function

$Z(+)$	*Probability* (P_r) (%)	$Z(-)$	*Probability* (P_r) (%)
0	50.0	0	50.0
+ 0.1	53.98	– 0.1	46.02
+ 0.2	57.93	– 0.2	42.07
+ 0.3	61.79	– 0.3	38.21
+ 0.4	65.54	– 0.4	34.46
+ 0.5	69.15	– 0.5	30.85
+ 0.6	72.57	– 0.6	27.43
+ 0.7	75.80	– 0.7	24.20
+ 0.8	78.81	– 0.8	21.19
+ 0.9	81.59	– 0.9	18.41
+ 1.0	84.13	– 1.0	15.87
+ 1.1	86.43	– 1.1	13.57
+ 1.2	88.49	– 1.2	11.51
+ 1.3	90.32	– 1.3	9.68
+ 1.4	91.92	– 1.4	8.08
+ 1.5	93.32	– 1.5	6.68
+ 1.6	94.52	– 1.6	5.48
+ 1.7	95.54	– 1.7	4.46
+ 1.8	96.41	– 1.8	3.59
+ 1.9	97.13	– 1.9	2.87
+ 2.0	97.72	– 2.0	2.28
+ 2.1	98.21	– 2.1	1.79
+ 2.2	98.61	– 2.2	1.39
+ 2.3	98.93	– 2.3	1.07
+ 2.4	99.18	– 2.4	0.82
+ 2.5	99.38	– 2.5	0.62
+ 2.6	99.53	– 2.6	0.47
+ 2.7	99.65	– 2.7	0.35
+ 2.8	99.74	– 2.8	0.26
+ 2.9	99.81	– 2.9	0.19
+ 3.0	99.87	– 3.0	0.13

Step 1. Determine the standard deviation (σ) appropriate to the critical path, for the network, using the relation.

$$\sigma = \sqrt{\text{sum of variances along critical path}}$$

or $$\sigma = \sqrt{\Sigma \sigma_{ij}^2} \quad ...(7.5)$$

where σ_{ij}^2 = variance for the activity i—j along the critical path

$$= \left(\frac{t_P^{ij} - t_O^{ij}}{6} \right)^2 \quad ...(7.6)$$

Step 2. Knowing the scheduled time of completion (t_S) and earliest expected time of completion (T_E), find the time distance $T_S - T_E$ and express it in terms of *probability factor Z by the relation :*

$$Z = \frac{T_S - T_E}{\sigma} \quad ...(7.7)$$

or $$Z = \frac{T_S - T_E}{\sqrt{\Sigma \sigma_{ij}^2}} \quad ...[7.7\ (a)]$$

The probability factor (Z) is the same as *normal deviate* of Table 7.6.

The probability factor (Z) can be positive, zero or negative.

When Z is *positive* (*i.e.* T_S to the right of T_E), the chances of completing the project in time are *more* than 50%.

When Z is *zero* (*i.e.* T_S coinciding with T_E), the chances of completing the project in time is fifty-fifty.

When Z is *negative* (*i.e.* T_S to the left of T_E), the chances of completing the project in time is *less* than 50%.

Step 3. Find % probability with respect to the normal deviate Z from Table 7.6.

Example 7.5. *A project is expected to take 15 months along the critical path, having a standard deviation of 3 months. What is the probability of completing the project within (a) 15 months, (b) 18 months and (c) 12 months ?*

Solution. The probability factor is given by

$$Z = \frac{T_S - T_E}{\sigma} \quad ...(7.7)$$

where T_E = 15 months and σ = 3 months (*a*) Given : $T_S = 15$ months

$$\therefore \quad Z = \frac{15 - 15}{3} = 0$$

From Table 7.7, for $Z = 0$, probability = **50%**

(*b*) Given : $T_S = 18$ months

$$\therefore \quad Z = \frac{18 - 15}{3} = 1$$

From Table 7.7, for $Z = 1$, probability = **84.13%**

(*c*) Given : $T_S = 12$ months

$$Z = \frac{12 - 15}{3} = -1.$$

From Table 7.6, for $Z = -1$, probability = **15.87%**

Example 7.6. *PERT calculations yield a project length of 50 weeks, with a variance of 16. Within how many weeks would you expect the project to be completed with probability of (a) 95% (b) 75% (c) 40% ?*

Solution. Standard deviation $\sigma = \sqrt{16} = 4$.

(*a*) For 95% probability $Z \approx 1.65$ (Table 7.6)

Now $\dfrac{T_S - T_E}{\sigma} = Z$

$$\therefore \quad T_S = \sigma Z + T_E$$
$$= 4 \times 1.65 + 50$$
$$= \mathbf{56.6} \approx 57 \text{ weeks.}$$

(*b*) For 75% probability, $Z = 0.69$

$$\therefore \quad T_S = \sigma Z + T_E$$
$$= 4 \times 0.69 + 50$$
$$= \mathbf{52.76} \approx 53 \text{ weeks.}$$

(*c*) For 40% probability, $Z \approx -0.25$

$$\therefore \quad T_S = \sigma Z + T_E$$
$$= .4\,(-0.25) + 50$$
$$= \mathbf{49 \text{ weeks.}}$$

Example 7.7. *For the network shown in Fig. 7.13, the time estimates (in days) each for activity are mentioned. Determine the probability of completing the project in 35 days.*

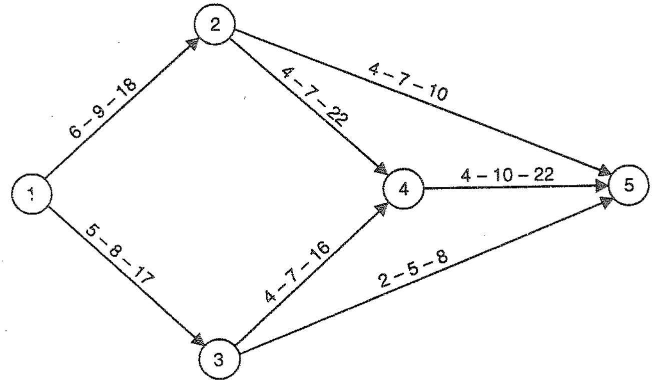


FIG. 7.13

Solution. Let us first determine the expected time of completion (t_E) and variance σ^2 for each activity using the relations

$$t_E = \frac{t_O + 4t_L + t_P}{6}$$

and
$$\sigma^2 = \left(\frac{t_P - t_O}{6}\right)^2$$

The computation are arranged in Table 7.7.

Table 7.7

Activity	t_O	t_L	t_P	t_E	σ^2
1—2	6	9	18	10	4
1—3	5	8	17	9	4
2—4	4	7	22	9	9
3—4	4	7	16	8	4
4—5	4	10	22	11	9
2—5	4	7	10	7	1
3—5	2	5	8	5	1

The computation for T_E and T_L and slack for various events are arranged in Table 7.8. For the last event, T_L is taken equal to its T_E. From Table 7.8, we find that slack is minimum at events 1, 2, 4 and 5. Hence path 1—2—4—5 is the critical path. The critical path is marked by thick lines on Fig. 7.14.

Table 7.8

Event No.	*Earliest expected time* ↓				*Latest occurrence time* ↑				Slack S
	Predecessor event (i)	t_E^{ij}	T_E^j	T_E	Successor event (j)	t_E^{ij}	T_L^i	T_L	
1	—	—	0	0	2 3	10 9	<u>0</u> 2	0	0
2	1	10	<u>10</u>	10	4 5	9 7	<u>10</u> 23	10	0
3	1	9	<u>9</u>	9	4 5	8 5	<u>11</u> 25	11	2
4	2 3	9 8	<u>19</u> 17	19	5	11	<u>19</u>	19	0
5	2 3	7 5	17 14	30			30	30	0

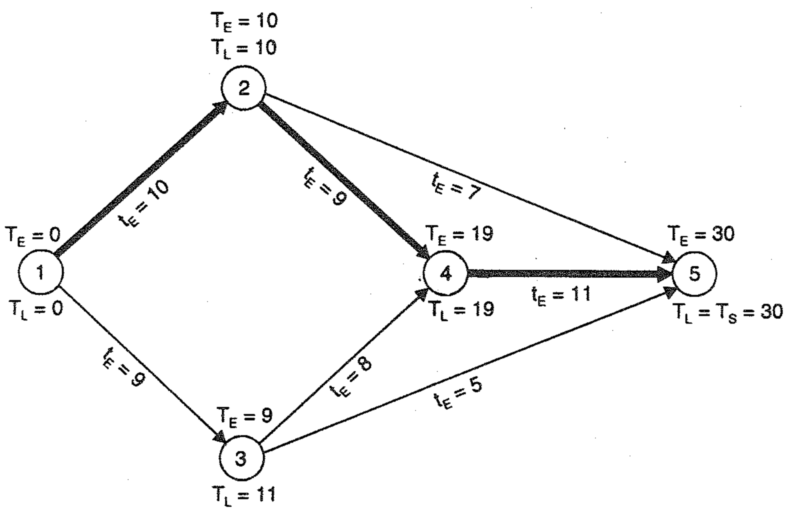


FIG. 7.14

Now deviation along the critical path is given by

$$\sigma = \sqrt{\Sigma\sigma_{ij}^2}$$

where $\Sigma\sigma_{ij}^2$ = sum of variances along critical path

$$= \sigma^2_{1-2} + \sigma^2_{2-4} + \sigma^2_{4-5}$$

$$= 4 + 9 + 9 = 22$$

$$\therefore \quad \sigma = \sqrt{22} = 4.69$$

$$\therefore \quad Z = \frac{T_S - T_E}{\sigma}, \text{ where } T_S = 35 \text{ days (given)}$$

$$= \frac{35 - 30}{4.69} = 1.066$$

From Table 7.6, for $Z = 1.066$, $P_r \approx$ **85.7%**.

Hence there is 85.7% probability of completion of the project in 35 days.

Example 7.8. *Fig. 7.15 shows the network for a construction project, with the three time estimates of each activity marked. Determine :*

(a) Critical path and its standard deviation.

(b) Probability of completion of project in 40 days.

(c) Time duration that will provide 95% probability of its completion in time.

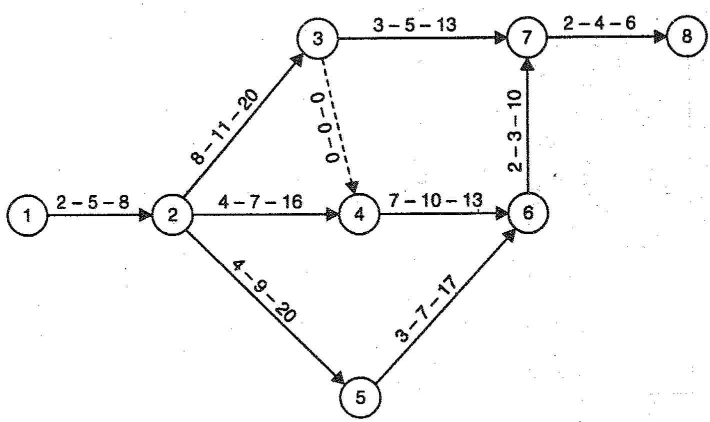


FIG. 7.15

Solution. The computations of t_E and σ^2 are arranged in Table 7.9. The computation for T_E, T_L and S for each event are given in Table 7.10. For the last event, T_L is taken equal to its T_E. From Table 7.10, we find that slack is minimum at events 1, 2, 3, 4, 6, 7 and 8. Hence critical path is along 1—2—3—4—6—7—8. The value of T_E and T_L for each event, along with the critical path is marked on Fig. 7.16.

Table 7.9

Activity	t_O	t_L	t_P	t_E	σ^2
1—2	2	5	8	5	1
2—3	8	11	20	12	4
3—4	0	0	0	0	0
2—4	4	7	16	8	4
2—5	4	9	20	10	7.11
4—6	7	10	13	10	1
5—6	3	7	17	8	5.44
3—7	3	5	13	6	2.78
6—7	2	3	10	4	1.77
7—8	2	4	6	4	0.44

Table 7.10

Event No.	*Earliest expected time* ↓				*Latest occurrence time* ↑				*S*
	Predecessor event (i)	t_E^{ij}	T_E^j	T_E	*Successor event (j)*	t_E^{ij}	T_L^i	T_L	
1	—	—	0	0	2	5	0	0	0
2	1	5	5	5	3	12	5	5	0
					4	8	9		
					5	10	9		
3	2	12	17	17	7	6	25	17	0
					4	0	17		
4	2	8	13	17	6	10	17	17	0
	3	0	17						
5	2	10	15	15	6	8	19	19	4
6	4	10	27	27	7	4	27	27	0
	5	8	23						
7	3	6	23	31	8	4	31	31	0
	6	4	31						
8	7	4	35	35	—	—	35	35	0

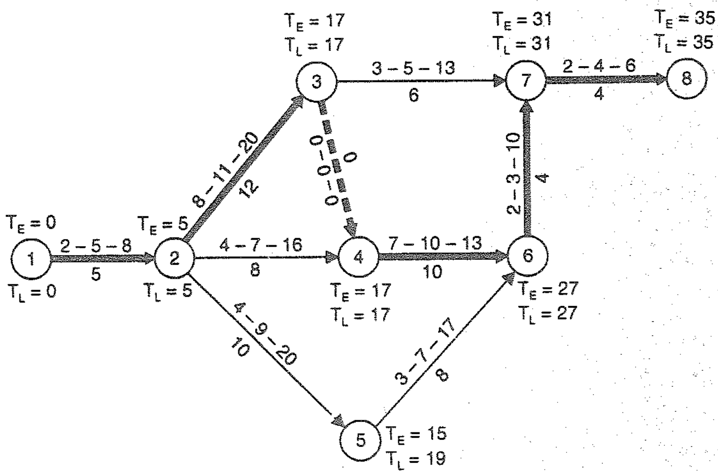


FIG. 7.16

Standard deviation along critical path

$$= \sqrt{\Sigma(\sigma_{ij})^2}$$

where $\Sigma(\sigma_{ij})^2 = 1 + 4 + 0 + 7.11 + 1.77 + 0.44 = 14.32$

$\therefore \quad \sigma = \sqrt{14.32} = \mathbf{3.78}.$

Now $\quad Z = \dfrac{T_S - T_E}{\sigma}$

(*a*) When $T_S = 40$ days

$$Z = \frac{40 - 36}{3.75} = 1.07.$$

Hence from Table 7.6, Probability = **85.7%.**

(*b*) For $P_r = 95\%$, we have $Z \approx 1.65$.

$\therefore \quad T_S = \sigma Z + T_E$

$= (3.78 \times 1.65) + 35 = \mathbf{41.2}$ days.

PROBLEMS

1. Explain the following terms : (*i*) Latest allowable occurrence time, (*ii*) earliest expected time, (*iii*) slack, (*iv*) critical path. What does a negative stack indicate ?
2. For the network shown in Fig. 7.17 (on next page), determine the slack for various events, if the scheduled date of completion of the project is 36 days. Present the computations in tabular form.

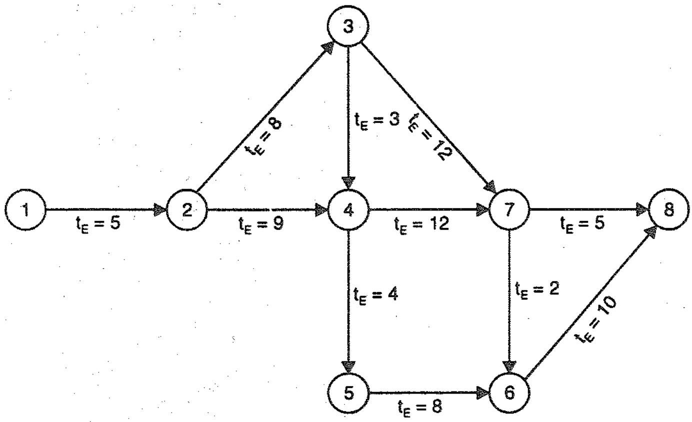


FIG. 7.17

3. For the network of problem 2, determine the critical path.
4. Explain how do you determine the probability of meeting the scheduled date of completion of a project.
5. If the expected time along the critical path of a project is 27 weeks and the standard deviation along it is 6 weeks, determine the probability of completing the project within (*a*) 21 weeks, (*b*) 24 weeks, (*c*) 36 weeks.
6. On a network, PERT calculations yield a project length of 60 days with a variance of 9 days. Estimate the number of days required to complete the project with a probability of 98%.
7. A construction company has an opportunity to submit a bid for the construction of a new apartment building. From the specification provided by the developer, the PERT network along with the three time estimate (in week) for each activity are shown in Fig. 7.18.

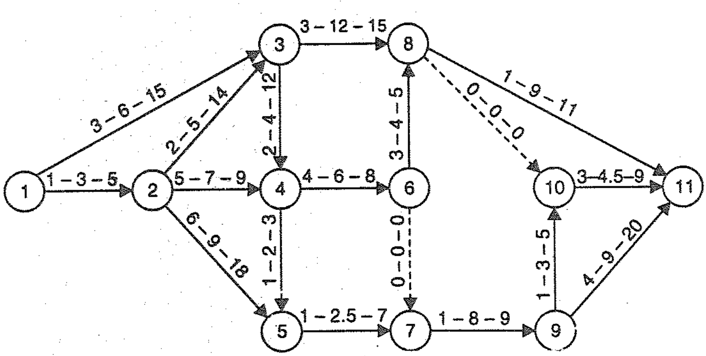


FIG. 7.18

Determine :

(*a*) Critical path and its standard deviation.

(*b*) Probability of completing the work in 38 weeks.

(*c*) Completion time duration for which the company should bid to provide 95% probability of completing the project in time.