UNIT 11 ENERGY METHODS AND APPLICATIONS

Structure

11.1 INTRODUCTION

Energy methods are extensively used for the determination of force or any internal stress resultant (for example, bending moment etc.) and displacements (linear and angular, both) of structures. It is particularly useful in the analysis of indeterminate structures. The energy theorems are applicable in elementary analysis as well as in advanced analysis and also in finite element methods. They are very convenient and general in their applications.

Objectives

After studying this unit, you should be able to

- calculate the strain energy stored by determinate as well as indeterminate structures,
- describe the concept of virtual work – due to virtual displacements and virtual forces,
- discuss the applications of Castigliano's Theorems I and II and Minimum Energy Principles,
- explain the applications of Maxwell's Reciprocal and Betti's Theorems,
- analyse redundant beams, frames and trusses, and
- calculate displacements at different coordinates of indeterminate structures.

11.2 STRAIN ENERGY IN LINEAR ELASTIC SYSTEMS

This section is a brief summary of the topic which you have already studied in Unit 10 on Strain Energy under "Strength of Materials" course and which may also be referred.

A structural member obeying Hook's law of elasticity may be subjected to axial forces, shear forces, bending moments and twisting moments. The strain energy is calculated in the following way :

Strain energy stored by a member = Amount of the work done by the external forces to produce the deformation

11.2.1 Strain Energy Due to Axial Forces

Let strain energy stored by an elemental member ds be dU subject to the axial force F.

$\therefore$ *dU = Average load × axial displacement of element ds due to the force F.*

$$dU = \frac{F}{2} \times \left(\frac{F ds}{AE}\right) = \frac{F^2\, ds}{2AE}$$

where, $A =$ Area of cross-section of the member, and

$E =$ Modulus of elasticity.

$\therefore$ Strain energy for the entire length of the member

$$U = \int dU = \int \frac{F^2 ds}{2AE} \qquad (11.1)$$

11.2.2 Strain Energy Due to Shear Forces

Let strain energy stored by an elemental member ds is dU subject to the shear force Q.

$\therefore$ *dU = Average shear force × Shear displacement (deformation) of element ds due to the shear force Q*

$$dU = \frac{Q}{2} \times \left(\frac{Q ds}{A_r G}\right) = \frac{Q^2 ds}{2A_r G}$$

where, $A_r =$ Reduced area of cross section

$G =$ Shear modulus of elasticity.

$\therefore$ Total strain energy for the entire length of the member

$$U = \int dU = \int \frac{Q^2\, ds}{2A_r G} \qquad (11.2)$$

Now, if we consider strain energy in the xz plane is U_{xz} and corresponding reduced area of cross section A_{rx} and those in the yz plane are U_{yz} and A_{ry} respectively,

$$U_{xz} = \int \frac{Q_x^2\, ds}{2A_{rx} G} \text{ and } U_{yz} = \int \frac{Q_y^2\, dS}{2A_{ry} G} \qquad (11.2a)$$

where, Q_x and Q_y = Biaxial shear forces (in the x and y directions respectively)

11.2.3 Strain Energy Due to Bending Moment

Let strain energy stored by an elemental member ds be dU, subject to the bending moment M.

$\therefore$ dU = *Average bending moment × bending displacement (Angular rotation) of element ds due to the bending moment M.*

$$dU = \frac{M}{2} \times \left(\frac{M\, ds}{EI}\right) = \frac{M^2\, ds}{2EI}$$

where, I = Moment of inertia of the cross-section of the member with respect to the neutral axis.

$\therefore$ Total strain energy for the entire length of the member

$$U = \int dU = \int \frac{M^2\, dS}{2EI} \qquad (11.3)$$

Similarly, strain energy in the xz and yz planes are as follows :

$$U_{xz} = \int \frac{M_x^2\, ds}{2EI_y}, \text{ and}$$

$$U_{yz} = \int \frac{M_y^2\, ds}{2EI_x} \qquad (11.3a)$$

It is due to bending moments M_x and M_y in the xz and yz planes respectively.

11.2.4 Strain Energy Due to Twisting Moment (Torsion)

Let strain energy stored by an elemental member ds be dU, subject to the twisting moment or torsion T.

$\therefore\ dU =$ *Average torsion × Torsional displacement (angle of twist) of element ds due to the torsional moment T.*

$$dU = \frac{T}{2} \times \left(\frac{T\, ds}{GK}\right) = \frac{T^2\, ds}{2GK}$$

where, K = a constant of the twisted member based on shape of the section. (For a circular section it is equal to the polar moment of inertia J)

$\therefore$ Total strain energy for the entire length of the member

$$U = \int dU = \int \frac{T^2\, ds}{2GK} \qquad (11.4)$$

11.2.5 General Equation of Strain Energy

The general equation of strain energy is the sum of energy due to six internal force components comprising the axial force S, the biaxial shear forces Q_x and Q_y, the biaxial bending moments M_x and M_y and torsion T.

Therefore,

$$U = \int \frac{S^2\, ds}{2AE} + \int \frac{Q_x^2\, ds}{2A_{rx}G} + \int \frac{Q_y^2\, ds}{2A_{ry}G} + \int \frac{M_x^2\, ds}{2EI_x} + \int \frac{M_y^2\, ds}{2EI_y} + \int \frac{T^2\, ds}{2GK} \qquad (11.5)$$

In the case of pin-jointed frames or trusses, axial forces of the members are dominant.

$$U = \sum \frac{S^2 L}{2AE} \qquad (11.6)$$

In the case of plane rigid-jointed frames where twisting moments are absent, the other three components, namely axial forces, shear forces and bending moments are dominant.

Thus,

$$U = \int \frac{S^2\, ds}{2AE} + \int \frac{Q^2\, ds}{2A_r G} + \int \frac{M^2\, ds}{2EI} \qquad (11.7)$$

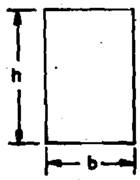
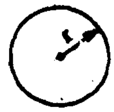
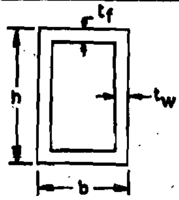
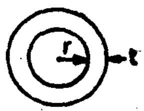
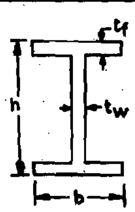
But generally axial forces and shear forces are very small in comparison to bending moment then energy due to the small components may be neglected and we can use the equation as follows :

$$U = \int \frac{M^2\, ds}{2EI} \qquad (11.7a)$$

Now, we give a few values of A_r and K due to shear and torsion for different cross-sectional areas in the Table 11.1.

The calculation of strain energy is very important for the determination of deformation of determinate and indeterminate structures. We shall discuss more elaborately the strain energy method in Block 4.

Table 11.1

Section	A_r	K
	$\frac{A}{1.2}$	$\approx hb^2\left[\frac{1}{3} - 0.21\frac{b}{h}\left(1 - \frac{b^4}{12h^4}\right)\right]$
	$0.9A$	$\frac{\pi r^4}{2}$
	$2ht_w$	$\approx 2b^2h^2 \frac{t_f t_w}{bt_w + ht_f}$
	$\frac{A}{2}$	$\approx 2\pi r^2 t$
	ht_w	$\approx \frac{1}{3}\left(ht_w^2 + 2bt_f^2\right)$

Example 11.1

Determine the total strain energy of the L-shaped member which is subjected to 1000 N load as shown. The cross-sectional area of the member is 6 cm × 12 cm.

Assume $E = 2 \times 10^7$ N/cm^2 and $G = 0.8 \times 10^7$ N/cm^2.

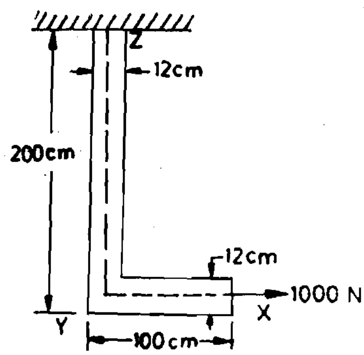


Figure 11.1

Solution

Member XY: Axial force in the member XY = 1000 N (tensile)

SF, BM and twisting moment in the member XY are zero.

Member YZ: Axial force and twisting moment of the member YZ are zero.

SF at $Y = 1000$ N, SF at $Z = 1000$ N, constant throughout YZ.

BM at $Y = 0$ and BM at $Z = 1000 \times 200$ N cm $= 2 \times 10^5$ N cm, linearly varying from zero at Y to 2×10^5 N cm at Z.

BM at x from Y in the YZ portion = $1000\,x$.

Area, $(A) = 6 \times 12 \text{ cm}^2$; Reduced Area, $(A_r) = \dfrac{6 \times 12}{1.2} \text{ cm}^2$

Moment of inertia, $I = \dfrac{6 \times 12^3}{12} \text{ cm}^4$

Now, strain energy in the member XY

$$U_{xy} = \frac{S^2 L}{2AE} + 0 = \frac{1000^2 \times 100}{2 \times 6 \times 12 \times 2 \times 10^7} = 0.0347 \text{ N cm}$$

and strain energy in the member YZ

$$U_{yz} = \int_Y^Z \frac{Q^2\, ds}{2A_r G} + \int_Y^Z \frac{M^2\, ds}{2EI}$$

$$= \int_0^{200} \frac{1000^2\, dx}{2 \times \frac{6 \times 12}{1.2} \times 0.8 \times 10^7} + \int_0^{200} \frac{(1000x)^2 dx}{2 \times 2 \times 10^7 \times \frac{6 \times 12^3}{12}}$$

$$= \frac{1000^2 \times 200}{2 \times \frac{6 \times 12}{1.2} \times 0.8 \times 10^7} + \frac{1000^2 \times (200)^3}{3 \times 2 \times 2 \times 10^7 \times \frac{6 \times 12^3}{12}}$$

$$= (0.2083 + 77.1605) \text{ N cm}$$

$$= 77.3688 \text{ N cm}$$

$\therefore$ Total strain energy $= U_{xy} + U_{yz} = (0.0347 + 77.3688) \text{ N cm} = 77.4035 \text{ N cm}$

11.3 VIRTUAL WORK

Work is done when the point of application of a force is moved and is given by the product of *force* $\times$ *displacement*. The word virtual indicates imaginary, so the *virtual work* is the hypothetical work consisting of *real* forces with *virtual* displacements or *virtual* forces with *real* displacements. The *principle of virtual work* was postulated by Aristotle in the 4th century BC. In fact, all the energy methods can be developed from the principle of virtual work. The principle of virtual work is based on the physical principle of conservation of energy and is applicable to both linear and non-linear elastic systems of determinate and indeterminate structures.

11.3.1 Principle of Virtual Displacements (Rigid Bodies)

The total work done by a rigid body held in equilibrium by a system of forces and reactions during a small virtual displacement is zero.

This principle is useful in determining forces and influence lines. Unit displacement method is developed based on this concept.

Interested students may see the proof of this principle in any standard book of Theory of Structures suggested in the section, "Further Reading".

11.3.2 Principle of Virtual Forces

The total work done by a rigid body subjected to a deformation compatible with the support conditions, held in equilibrium, by virtual forces and reactions on the body is equal to zero.

This principle is useful in computing displacements in a structure. Unit load (for trusses) unit moment (for beams) and unit torsion (for shafts) have been developed based on this concept for determination of deformation of various structures.

11.4 CASTIGLIANO'S THEOREMS

Real energy calculation being very tedious, Castigliano in 1876 developed two theorems to calculate the forces and deformation in a structure based on the concept of strain energy.

11.4.1 Theorem I

The partial derivative of the strain energy of a linearly elastic structure (represented in terms of displacements) with respect to any displacement Δ_j at coordinate j is equal to the force P_j at coordinate j.

Mathematically, $$\frac{\partial U}{\partial \Delta_j} = P_j \qquad (11.8)$$

Proof

We assume a set of forces $P_1, P_2, \ldots, P_j, \ldots, P_n$ acting on a structure at coordinates $1, 2, \ldots, j, \ldots, n$ creating displacements $\Delta_1, \Delta_2, \ldots, \Delta_j, \ldots, \Delta_n$. Now, we impose a small increment $\delta\Delta_j$ to the displacement at coordinate j. Keeping the displacements at all other coordinates unchanged. As a result, the increments in the forces are $\delta P_1, \delta P_2, \ldots, \delta P_j, \ldots, \delta P_n$. The increment in displacement at coordinate j and the consequent increment in loads series is considered as the second set. We are showing the two sets of forces and corresponding displacements in Table 11.2.

Table 11.2

Set ↓	Coordinate →	1	2	--	j	--	n
I	P	P_1	P_2	--	P_j	--	P_n
	Δ	Δ_1	Δ_2	--	Δ_j	--	Δ_n
II	P'	δP_1	δP_2	--	δP_j	--	δP_n
	Δ'	0	0	0	$\delta\Delta_j$	0	0

$\therefore$ The work done at the coordinate j during these displacements will be

$$P_j\delta\Delta_j = \Delta_1\delta P_1 + \Delta_2\delta P_2 + \ldots + \Delta_j\delta P_j + \ldots + \Delta_n\delta P_n$$

or, $$P_j\delta\Delta_j = \delta U$$

or, $$\frac{\delta U}{\delta \Delta_j} = P_j$$

Using limit $\delta\Delta_j \to 0$,

$$\frac{\partial U}{\partial \Delta_j} = P_j$$

This theorem is also applicable to the system of moments and the resulting angular deformations, thus $\frac{\partial U}{\partial M_j} = \theta_j$.

This principle is widely used in analysis of structures.

11.4.2 Theorem II

The partial derivative of the strain energy of a linearly elastic structure (represented in terms of forces) with respect to any force P_j at coordinate j is equal to the displacement Δ_j at coordinate j.

Mathematically, $$\frac{\partial U}{\partial P_j} = \Delta_j \qquad (11.9)$$

Proof

We assume a set of forces $P_1, P_2, \ldots, P_j, \ldots, P_n$ acting on a structure at coordinates $1, 2, \ldots, j, \ldots, n$, creating displacements $\Delta_1, \Delta_2, \ldots, \Delta_j, \ldots, \Delta_n$. Now, we impose a small increment δP_j to the load at coordinate j keeping the forces at all other coordinates unchanged. As a result, the increment in the displacements are $\delta\Delta_1, \delta\Delta_2, \ldots, \delta\Delta_j, \ldots, \delta\Delta_n$. The increment in load at coordinate j and the consequent increments in displacements at all the coordinates is considered as the second set. We are showing the two sets of forces and corresponding displacements in Table 11.3.

Table 11.3

Set ↓	Coordinate →	1	2	--	j	--	n
I	P	P_1	P_2	--	P_j	--	P_n
	Δ	Δ_1	Δ_2	--	Δ_j	--	Δ_v
II	P'	0	0	0	δP_j	0	0
	Δ'	$\delta\Delta_1$	$\delta\Delta_2$	--	$\delta\Delta_j$	-	$\delta\Delta_n$

∴ The work done at the coordinate j during these displacements will be

$$\delta P_j \Delta_j = P_1 \delta\Delta_1 + P_2 \delta\Delta_2 + \ldots + P_j \delta\Delta_j + \ldots + P_n \delta\Delta_n$$

or, $$\delta P_j \Delta_j = \delta U$$

or, $$\frac{\delta U}{\delta P_j} = \Delta_j$$

Using limit $\delta\Delta_j \rightarrow 0$,

$$\frac{\partial U}{\partial P_j} = \Delta_j$$

This theorem is extensively used for determination of displacement in a structure of both the determinate and indeterminate types.

In fact, it is a powerful tool for the analysis of the structure.

11.4.3 Statically Determinate Structures

In the case of determinate structures, Castigliano's theorems may be applied in the following ways :

(a) In case of *trusses* where axial forces (S) are predominant,

$$\Delta_j = \frac{\partial U}{\partial P_j} = \int \frac{S}{AE}\left(\frac{\partial S}{\partial P_j}\right) ds \quad (11.10a)$$

(b) In case of *beams* and *plane jointed frames* where bending moments (M) are predominant,

$$\Delta_j = \frac{\partial U}{\partial P_j} = \int \frac{M}{EI}\left(\frac{\partial M}{\partial P_j}\right) ds \quad (11.10b)$$

(c) In case of *shafts* where torsion (T) is predominant,

$$\Delta_j = \int \frac{T}{GJ}\left(\frac{\partial T}{\partial P_j}\right) ds \quad (11.10c)$$

If there is no load at the coordinate j, we assume an imaginary or dummy load acting at that particular coordinate for finding out the displacement equation and ultimately we put the value of the dummy load as zero which is known as dummy load method. An elegant way to analyse the displacement of structures considering the dummy load as unit force is popularly known as unit load method.

Then, the above expressions become as follows :

$$\Delta_j = \int \frac{Su\, ds}{AE} \quad \text{For trusses} \quad (11.11a)$$

$$\Delta_j = \int \frac{Mm\, ds}{EI} \quad \text{For beams and frames} \quad (11.11b)$$

$$\Delta_j = \int \frac{Tt\, ds}{GJ} \quad \text{For shafts} \quad (11.11c)$$

where $u = \left(\frac{\partial S}{\partial P_j}\right)$ = force in the members due to unit force at coordinate j

$m = \left(\frac{\partial M}{\partial P_j}\right)$ = bending moment in the members due to unit force at coordinate j

$t = \left(\frac{\partial T}{\partial P_j}\right)$ = torsion in the members due to unit force at coordinate j

Interested students may see the problems on determinate structures applying these theorems from any standard book of Theory of Structures suggested in section, "Further Reading".

Before concentrating the application of the above theorem in the statically indeterminate structures, we just explain the Minimum Energy Theorem which is closely linked to Castigliano's second theorem.

11.5 MINIMUM ENERGY THEOREM

In any and every case of statically indeterminate structure, where an indefinite number of different values of the redundant forces and displacements satisfy the condition of statical equilibrium, their actual values are those that render the total strain energy stored to a minimum.

Therefore, $$\frac{\partial U}{\partial X} = 0 \text{ and } \frac{\partial^2 U}{\partial X^2} \text{ is positive} \quad (11.12)$$

where X = redundant force.

In general, the strain energy stored by a structure subjected to bending and/or axial loading is given by

$$U = \int \frac{M^2\, ds}{2EI} + \int \frac{S^2\, ds}{2AE}$$

We consider a continuous beam with reaction X at A which is treated as redundant as shown below :

Here, V_b and V_c = reactions at B and C.

W_1, W_2, W_3 = External loading

According to Castigliano's second theorem, displacement at A :

$$\Delta_A = \frac{\partial U}{\partial X} \text{ where } U = \text{strain energy stored by the beam.}$$

Due to no displacement of the support A, we have, $\Delta_A = 0$.

Thus, $\frac{\partial U}{\partial X} = 0$, which satisfies the first condition of the Minimum Energy Theorem. It is also applicable to all types of forces such as axial force, twisting moment, bending moment or a combination thereof.

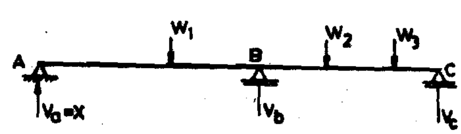


Figure 11.2

Now to test the condition for maximum or minimum value of U

$$\frac{\partial U}{\partial X} = \int \frac{M\,ds}{EI} \cdot \frac{\partial M}{\partial X} + \int \frac{S\,ds}{AE}\left(\frac{\partial S}{\partial X}\right) = 0$$

Differentiating again we get,

$$\frac{\partial^2 U}{\partial X^2} = \int \frac{ds}{EI}\left[M\frac{\partial^2 M}{\partial X^2} + \left(\frac{\partial M}{\partial X}\right)^2\right] + \int \frac{ds}{AE}\left[S\frac{\partial^2 S}{\partial X^2} + \left(\frac{\partial S}{\partial X}\right)^2\right]$$

The bending moment M at any section or the force S in any member is a linear function of X,

$$\therefore \quad \frac{\partial M}{\partial X} \text{ and } \frac{\partial S}{\partial X} = \text{Constants, and } \left(\frac{\partial M}{\partial X}\right)^2 \text{ and } \left(\frac{\partial S}{\partial X}\right)^2 = \text{positive value}$$

$$\text{and} \quad \frac{\partial^2 M}{\partial X^2} = \frac{\partial^2 S}{\partial X^2} = 0$$

$$\therefore \quad \frac{\partial^2 U}{\partial X^2} = \text{positive value which indicates that the stored strain energy is minimum.}$$

Example 11.2

Find the reaction at the prop of a propped cantilever beam loaded as shown in Figure 11.3.

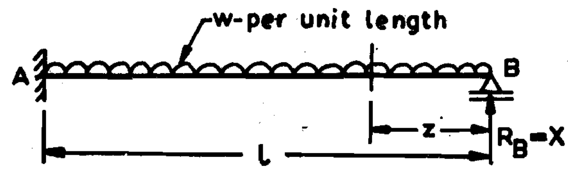


Figure 11.3

Solution

Let X be the reaction at the prop (considered as the redundant reaction)

$$\therefore \text{ B.M. at any section distant } Z \text{ from } B, \; M = Xz - \frac{wz^2}{2}$$

$$\therefore \text{ Strain energy stored by the beam} = U = \int \frac{M^2 dz}{2EI} = \int_o^l \left(Xz - \frac{wz^2}{2}\right)^2 \frac{dz}{2EI}$$

By the Minimum Energy Principle $\frac{\partial U}{\partial X} = 0$

We get, $$\int_o^l 2\left[Xz - \frac{wz^2}{2}\right] z \frac{dz}{2EI} = 0$$

or, $$\frac{X}{EI}\int_o^l z^2 dz - \frac{w}{2EI}\int_o^l z^3 dz = 0$$

or, $$\frac{Xl^3}{3EI} - \frac{wl^4}{\Delta 8EI} = 0$$

or, $$X = \frac{3}{8}wl$$

Knowing $R_B = X = \frac{3}{8}wl$, we can find out all the reactions at the support A.

Example 11.3

Determine the reactions at the supports of the continuous beam loaded as shown in Figure 11.4 by the principle of least work.

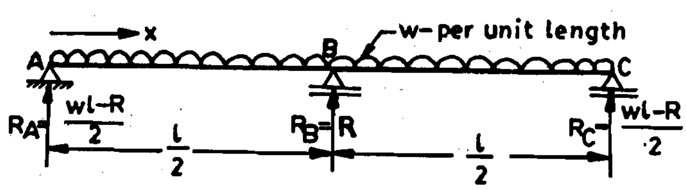


Figure 11.4

Solution

Let the reaction at B, $R_B = R$ (considered as redundant reaction).

From symmetry, we get reaction at each end.

We get the reaction,

$$R_A \text{ or } R_C = \frac{wl - R}{2}$$

B.M. at any section in AB at a distance x from A

$$M = \left(\frac{wl - R}{2}\right)x - \frac{wx^2}{2}$$

Total strain energy

$$U = \int \frac{M^2 dx}{2EI} = 2\int_0^{\frac{l}{2}} \left[\frac{wl - R}{2}x - \frac{wx^2}{2}\right]^2 \frac{dx}{2EI}$$

$$= \frac{1}{EI}\int_0^{\frac{l}{2}} \left[\frac{wl - R}{2}x - \frac{wx^2}{2}\right]^2 dx$$

By the principle of least work $\frac{\partial U}{\partial R} = 0$

We get, $$\frac{1}{EI}\int_0^{\frac{l}{2}} 2\left[\frac{wl - R}{2}x - \frac{wx^2}{2}\right]\left(\frac{x}{2}\right)dx = 0$$

or, $$\frac{1}{EI}\int_0^{\frac{l}{2}} \left[\frac{wx^3}{2} - \frac{wl - R}{2}x^2\right]dx = 0$$

or, $$\frac{1}{EI}\left[\left(\frac{1}{2}\times\frac{1}{4}\times\frac{wl^4}{16}\right) - \left(\frac{wl - R}{2}\times\frac{1}{3}\times\frac{l^3}{8}\right)\right] = 0$$

or, $$\frac{wl^4}{128} - \frac{wl^4}{48} + \frac{Rl^3}{48} = 0$$

or, $$R = \frac{5}{8}wl = \text{reaction at } B$$

and reaction at A or C = $\dfrac{wl - \frac{5}{8}wl}{2} = \dfrac{3}{16}wl$ each

Example 11.4

Determine the forces in the members of the truss loaded as shown in Figure 11.5 (a). The sectional area of vertical member = 3000 mm^2; horizontal member = 4000 mm^2 and diagonal members = 5000 mm^2 each. The members are of same material.

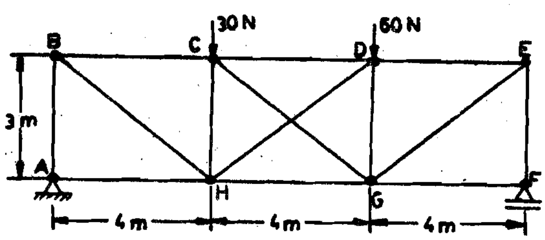


Figure 11.5 (a)

Solution

Degree of redundancy of the truss, $D = m + r - 2j$

$$= 14 + 3 - 2 \times 8 = 1$$

D_e = Degree of external redundancy = $r - 3 = 3 - 3 = 0$

$\therefore$ Degree of internal redundancy $= D_i = D - D_e = 1 - 0 = 1$

Let, DH member be redundant. Axial force in the member $DH = X$ (say). So, we can analyse the given structure by the following two equivalent structures.

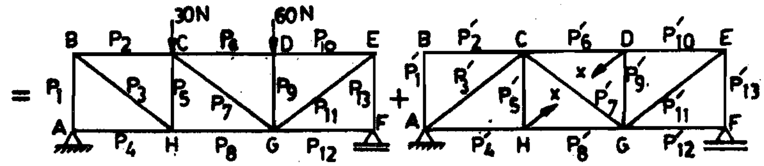


Figure 11.5 (b)

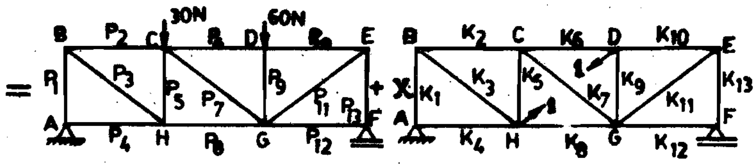


Figure 11.5 (c)

Here, $S_1 = P_1 + P_1', S_2 = P_2 + P_2'$ and so on [Figure 11.5 (b)].

Here, $S_1 = P_1 + XK_1,\ S_2 = P_2 + XK_2$ and so on [Figure 11.5 (c)].

$\therefore$ Total strain energy stored by the frame $= U = \sum \frac{S_1^2 l_1}{2A_1E}$

$$= \sum (P_1 + XK_1)^2 \frac{l_1}{2A_1E}$$

According to least work principle $\frac{\partial U}{\partial X} = 0$

$$\Rightarrow \sum 2\,(P_1 + XK_1)\frac{K_1 l_1}{2A_1E} = 0$$

$$\text{or, } \sum \frac{P_1K_1l_1}{A_1E} + X\sum\frac{K_1^2l_1}{A_1E} = 0$$

$$\text{or, } X = -\frac{\sum\frac{P_1K_1l_1}{A_1E}}{\sum\frac{K_1^2l_1}{A_1E}}$$

Now, we solve the problem stepwise.

Step 1 : Evaluation of P_1 etc.

We remove the member DH

$$\Sigma M_A = 0 \Rightarrow V_f \times 12 - 30\times 4 - 60\times 8$$

or $\quad V_f = 50\text{ N}$ and $V_a = (30+60) - 50 = 40\text{ N}$

From the joints A and F, $P_{ah} = P_{fg} = 0$

We get, $\tan\theta = \frac{3}{4}$, $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$

At Joint A

$P_{ab} = 40$ N (compressive)

At Joint B

$P_{bh}\sin\theta = 40 \Rightarrow P_{bh} = \frac{200}{3}$ N (tensile)

$P_{bc} = \frac{200}{3}\cos\theta = \frac{160}{3}$ N (compressive)

At Joint H

$$P_{hc} = \frac{200}{3} \sin \theta = 40 \text{ N (compressive)}$$

$$P_{hg} = \frac{200}{3} \cos \theta = \frac{160}{3} \text{ N (tensile)}$$

At Joint C

$$P_{cg} \sin \theta = 40 - 30 = 10 \Rightarrow P_{cg} = \frac{50}{3} \text{ N (tensile)}$$

$$P_{cd} = \frac{160}{3} + \frac{50}{3} \cos \theta = \frac{200}{3} \text{ N (compressive)}$$

At Joint D

$$P_{de} = \frac{200}{3} \text{ N (compressive)}$$

$$P_{dg} = 60 \text{ N (compressive)}$$

At Joint E

$$P_{eg} \cos \theta = \frac{200}{3} \Rightarrow P_{eg} = \frac{250}{3} \text{ (tensile)}$$

$$P_{ef} = \frac{250}{3} \sin \theta = 50 \text{ N (compressive)}$$

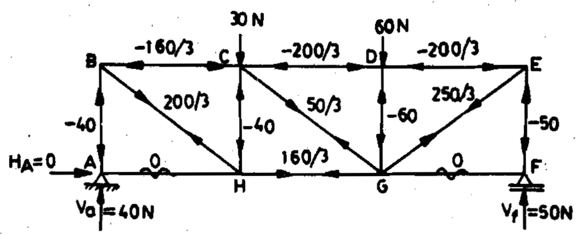


Figure 11.5 (d) : P-forces in Truss Members

Step 2 : Evaluation of K_1 etc.

We remove the external loading and impose a pair of unit loads (tensile force in member DH) at D and H in place of the member DH.

Here, $V_a = H_a = V_f = 0$ and

$$K_{ab} = K_{bc} = K_{bh} = K_{ah} = K_{ed} = K_{ef} = K_{fg} = K_{eg} = 0$$

At Joint H

$$K_{ng} = 1 \times \cos \theta = \frac{4}{5} \text{ N (tensile)}$$

$$K_{nc} = 1 \times \sin \theta = \frac{3}{5} \text{ N (tensile)}$$

At Joint D

$$K_{dg} = 1 \times \sin \theta = \frac{3}{5} \text{ N (tensile)}$$

$$K_{dc} = 1 \times \cos\theta = \frac{4}{5} \text{ N (tensile)}$$

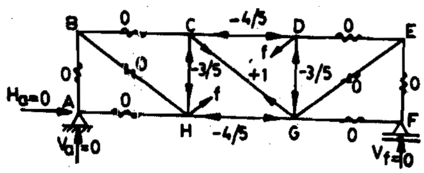


Figure 11.5 (e) : K-forces in Truss Members

At Joint C

$$K_{cg} \sin\theta = \frac{3}{5} \Rightarrow K_{cg} = 1 \text{ N (compressive)}$$

Step 3 : Table

We assume the compressive force as negative and the tensile force as positive. Now we fill up the above results in Table 11.4.

Table 11.4

Member	P (N)	K (N)	L (mm)	A (mm^2)	$\frac{PKL}{A}$	$\frac{K^2L}{A}$	$F = P + KX$ $F = P - \frac{131}{6}K$
AB	− 40	0	3000	3000	0	0	− 40 (C)
BC	$-\frac{160}{3}$	0	4000	4000	0	0	− 53.33 (C)
CD	$-\frac{200}{3}$	$-\frac{4}{5}$	4000	4000	$+\frac{160}{3}$	$\frac{16}{25}$	− 49.2 (C)
DE	$-\frac{200}{3}$	0	4000	4000	0	0	− 66.67 (C)
EF	− 50	0	3000	3000	0	0	− 50 (C)
FG	0	0	4000	4000	0	0	0
GH	$+\frac{160}{3}$	$-\frac{4}{5}$	4000	4000	$-\frac{128}{3}$	$\frac{16}{25}$	70.8)T)
HA	0	0	4000	4000	0	0	0
BH	$+\frac{200}{3}$	0	5000	5000	0	0	66.67 (T)
HC	− 40	$-\frac{3}{5}$	3000	3000	+ 24	$\frac{9}{25}$	− 12.90 (C)
CG	$+\frac{50}{3}$	1	5000	5000	$+\frac{50}{3}$	1	− 5.17 (C)
GD	− 60	$-\frac{3}{5}$	3000	3000	+ 36	$\frac{9}{25}$	− 46.9 (C)
GE	$+\frac{250}{3}$	0	5000	5000	0	0	83.33 (T)
DH	0	1	5000	5000	0	1	− 21.83 (C)
				$\sum =$	$+\frac{262}{3}$	4	

Step 4 : Correcting Factor X

$$X = -\frac{\sum \frac{P_1 K_1 l_1}{A_1 E}}{\sum \frac{K_1^2 l_1}{A_1 E}} = -\frac{\left(\frac{262}{3}\right)}{4} = -\frac{131}{6}$$

Step 5 : Force in the members

Force in the member $S = P + XK$

$$S_{ab} = -40 + 0 = -40 \text{ N (compressive)}$$

$$S_{bc} = -\frac{160}{3} + 0 = -53.33 \text{ N (compressive)}$$

$$S_{cd} = -\frac{200}{3} - \frac{4}{5} \times \left(-\frac{131}{6}\right) = -49.2 \text{ N (compressive)}$$

$$S_{de} = -\frac{200}{3} + 0 = -66.67 \text{ N (compressive)}$$

$$S_{ef} = -50 + 0 = -50 \text{ N (compressive)}$$

$$S_{fg} = 0 + 0 = 0$$

$$S_{gh} = \frac{160}{3} + \left(-\frac{4}{5}\right)\left(-\frac{131}{6}\right) = 70.8 \text{ N (tensile)}$$

$$S_{ha} = 0 + 0 = 0$$

$$S_{bh} = -\frac{200}{3} + 0 = 66.67 \text{ N (tensile)}$$

$$S_{hc} = -40 + \left(-\frac{3}{5}\right)\left(-\frac{131}{6}\right) = -26.9 \text{ N (compressive)}$$

$$S_{cg} = +\frac{50}{3} + 1 \times \left(-\frac{131}{6}\right) = -5.17 \text{ N (tensile)}$$

$$S_{gd} = -60 + \left(-\frac{3}{5}\right)\left(-\frac{131}{6}\right) = -46.9 \text{ N (compressive)}$$

$$S_{ge} = \frac{250}{3} + 0 = 83.33 \text{ N (tensile)}$$

$$S_{dh} = 0 + \left(-\frac{131}{6}\right) = -21.83 \text{ N (compressive)}$$

Figure 11.5 (f) shows the final forces in the members.

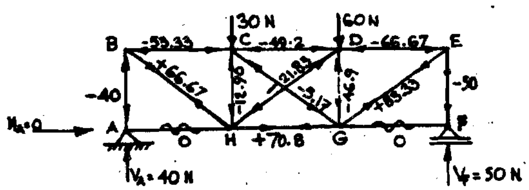


Figure 11.5 (f) : Final Member Forces in the Truss

Example 11.5

Determine the forces in all the members of the pin jointed frame [Figure 11.6 (a)] if the member AC is 1 mm short of the required length and the last member to be fitted. Assume area of each diagonal members = 1000 mm^2, area of each remaining members = 2000 mm^2 and E = 200 kN/mm^2.

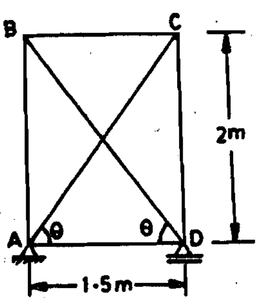


Figure 11.6 (a)

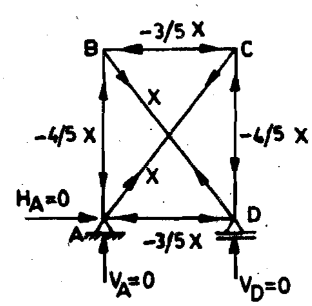


Figure 11.6 (b)

Solution

Total degree of redundancy of the frame = $m + r - 2j = 6 + 3 - 2 \times 4 = 1$

Degree of external redundancy = $D_e = r - 3 = 3 - 3 = 0$

$\therefore$ Degree of internal redundancy = $D_i = 1 - 0 = 1$

Let the member AC be redundant and unknown force (tension) in it when it is fitted into position = X

We assume $\angle CAD = \angle BDA = \theta$

$$\therefore \ \tan\theta = \frac{2}{1.5} = \frac{4}{3}$$

$$\therefore \ \sin\theta = \frac{4}{5}; \text{ and } \cos\theta = \frac{3}{5}$$

At Joint C

$$S_{cd} = X\sin\theta = \frac{4}{5}X \text{ (compressive)}$$

$$S_{cb} = X\cos\theta = \frac{3}{5}X \text{ (compressive)}$$

At Joint D

$$S_{db}\sin\theta = \frac{4}{5}X \Rightarrow S_{db} = X \text{ (tensile)}$$

$$S_{da} = X\cos\theta = \frac{3}{5}X \text{ (compressive)}$$

At Joint A

$$S_{ab} = X\sin\theta = \frac{4}{5}X \text{ (compressive).}$$

The forces are shown in Figure 11.6 (b).

$\therefore$ Total strain energy

$$U = \sum\frac{S^2 l}{2AE}$$

$$= 2\times\left(\frac{X^2\times 2500}{2\times 1000E}\right) + 2\times\left[\left(\frac{4}{5}X\right)^2\frac{2000}{2\times 2000E}\right] + 2\times\left[\left(\frac{3}{5}X\right)^2\frac{1500}{2\times 2000E}\right]$$

$$= \frac{5X^2}{2E} + \frac{16X^2}{25E} + \frac{27X^2}{100E} = \frac{341X^2}{100E}$$

According to Castigliano's second theorem,

Displacement of C with respect to $A = \Delta = \dfrac{\partial U}{\partial X}$

$$\Rightarrow \qquad \Delta = \frac{2\times 341X}{100E}$$

But we know that $\Delta = 1$ mm (positive, since short)

Therefore, $$1 = \frac{2\times 341X}{100E}$$

Thus, $$X = \frac{1\times 100\times 200}{2\times 341} = 29.32 \text{ kN}$$

$\therefore$ Tension in the diagonal member = 29.32 kN each.

Compression in the vertical member = $\dfrac{4}{5}\times 29.32 = 23.456$ kN each

Compression in the horizontal member $\dfrac{3}{5}\times 29.32 = 17.592$ kN each.

Note :

If the member AC is *little longer* than the required length, compression will develop in this member, therefore, Δ will be negative and we can analyse the frame due to lack of fit in the same manner.

Example 11.6

Find the tensions in the wires AD, BD and CD having the same cross-sectional area and of the same material supporting a load W at D as shown in the Figure 11.7 (a).

Prove that the horizontal displacement of D is equal to $\frac{1}{7}$th of extension of BD.

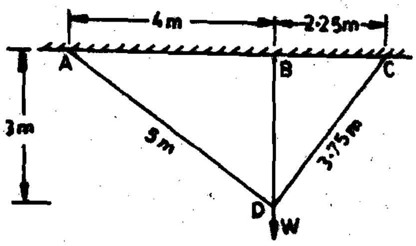


Figure 11.7 (a)

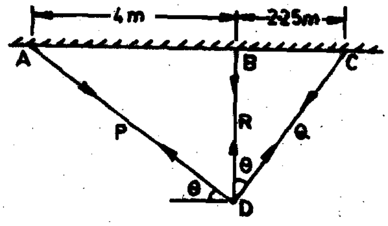


Figure 11.7(b)

Solution

First Part

Total degree of redundancy of the frame, $= m + r - 2j$

$$= 3 + (3 \times 2) - (4 \times 2) = 1$$

Degree of external redundancy $= 6 - 3 - 2 = 1$

Therefore, degree of internal redundancy $= 1 - 1 = 0$

Let the tension in DA, DC and DB are P, Q and R respectively.

We assume R, the reaction at B which is vertical as redundant, since the tension in the member $DB = R$.

Again we assume $\angle BDC = \theta$

$\therefore\ \angle EDA = \theta$, since ΔADC is a right angled triangle.

Here, $\tan\theta = \frac{3}{4}$, $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$

At Joint D

$$P\cos\theta = Q\sin\theta \Rightarrow P = Q\tan\theta$$

$$P = \frac{3}{4}Q \qquad \text{(i)}$$

and $\quad R + P\sin\theta + Q\cos\theta = W$

Putting the values from Eq. (i),

$$R + \left(\frac{3}{4} \times Q \times \frac{3}{5}\right) + \left(Q \times \frac{4}{5}\right) = W$$

or, $\quad R + \frac{5}{4}Q = W$

$$Q = \frac{4}{5}(W - R) \qquad \text{(ii)}$$

From Eq. (i) and Eq. (ii), we get, $P = \frac{3}{5}(W - R)$

Total strain energy stored by the frame

$$U = \frac{P^2 AD}{2AE} + \frac{Q^2 DC}{2AE} + \frac{R^2 BD}{2AE}$$

$$U = \frac{1}{2AE}[5P^2 + 3.75Q^2 + 3R^2]$$

According to Minimum Energy principle, i.e. $\frac{\partial U}{\partial R} = 0$

$$\Rightarrow \quad \frac{1}{2AE}\left[10P\frac{\partial P}{\partial R} + 7.5\frac{\partial Q}{\partial R} + 6R\right] = 0$$

or, $\quad 10P\left(-\frac{3}{5}\right) + 7.5Q\left(-\frac{4}{5}\right) + 6R = 0$

Putting the values, we get

$$-\left[\frac{10\times 3}{5}\times\frac{3}{5}(W-R)\right]-\left[\frac{7.5\times 4\times 4}{5\times 5}(W-R)\right]+6R = 0$$

$$-\frac{18}{5}(W-R)-\frac{24}{5}(W-R)+6R = 0$$

Thus, we get, $\qquad R=\frac{7}{12}W$

Now, putting the value of R to get values of P and Q,

$$P = \frac{3}{5}(W-R) = \frac{3}{5}\left(W-\frac{7}{12}W\right) = \frac{W}{4}$$

$$Q = \frac{4}{5}(W-R) = \frac{4}{5}\left(W-\frac{7}{12}W\right) = \frac{W}{3}$$

Second Part

Horizontal component of the extension of DA,

$$\frac{P}{AE}\times 5\cos\theta = \frac{W}{4AE}\times 5\times\frac{4}{5} = \frac{W}{AE}\text{ (right)}$$

Horizontal component of the extension of DC

$$\frac{Q}{AE}\times 3.75\sin\theta = \frac{W}{3AE}\times 3.75\times\frac{3}{5} = \frac{0.75W}{AE}\text{ (left)}$$

Horizontal displacement of D = Algebrical summation of the horizontal component of the extension of DA and DC.

$$\frac{W}{AE}-\frac{0.75W}{AE} = \frac{W}{4AE}\text{ (right)}$$

$$\text{Extension of } BD = \frac{R\times 3}{AE} = \frac{7W\times 3}{12AE} = \frac{7W}{4AE}$$

Thus, horizontal displacement of $D=\frac{1}{7}$ of the extension of BD.

Example 11.7

Analyse the portal frame, having the members of same moment of inertia and loaded as shown in Figure 11.8 (a). Draw the bending moment diagram.

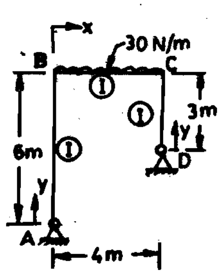


Figure 11.8 (a)

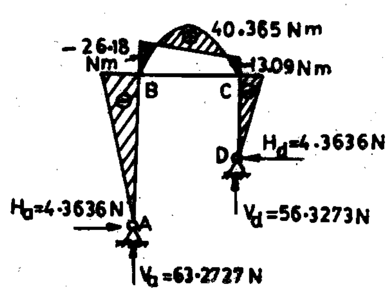


Figure 11.8 (b)

Solution

Total degree of redundancy $= (3\times\text{No. of loops})-\text{No. of releases at the supports}$

$= (3\times 1)-2 = 1$

We assume the horizontal thrust a at support A as redundant and vertical reactions at A and D are V_a and V_d respectively.

$\therefore\quad \Sigma H\equiv 0 \Rightarrow H_a=-H_d = H$ (say)

$\therefore\quad \Sigma M_A \equiv 0$

Taking moments about A, we get, $(V_d \times 4) + (H \times 3) = 30 \times 4 \times 2$

Thus, $$V_d = 60 - \frac{3}{4} H \text{ and } V_a = 60 + \frac{3}{4} H$$

Here, $U = \int \frac{M^2 ds}{2EI}$

According to Minimum Energy Principle, $\frac{\partial U}{\partial H} = 0$, thus,

$$\int \frac{M}{EI}\left(\frac{\partial M}{\partial H}\right) ds = 0$$

For this frame of uniform flexural rigidity, EI = constant

Hence, $$\sum \int M\left(\frac{\partial M}{\partial H}\right) ds = 0$$

Now, we prepare the following Table 11.5.

Table 12.5

Member	M	$\frac{\partial M}{\partial H}$	Limits of integration
AB	$-Hy$	$-y$	0 to 6
DC	$-Hy$	$-y$	0 to 3
BC	$\left(60 + \frac{3H}{4}\right)x - 15x^2 - 6H$	$\left(\frac{3x}{4} - 6\right)$	0 to 4

Assuming the bending moment producing concavity outside the frame as positive bending moment and that producing convexity outside the frame as negative bending moment.

$$\int_0^6 Hy^2\, dy + \int_0^3 Hy^2\, dy + \int_0^4 \left[\left(60 + \frac{3H}{4}\right)x - 15x^2 - 6H\right]\left(\frac{3x}{4} - 6\right) dx = 0$$

$$\int_0^6 Hy^2\, dy + \int_0^3 Hy^2\, dy + \int_0^4 \left(135x^2 + \frac{9}{16}Hx^2 - 9Hx - 11.25x^3 - 360x + 36H\right) dx = 0$$

On further simplifying, we get

$$H \times \frac{6^3}{3} + H \times \frac{3^3}{3} + 135 \times \frac{4^3}{3} + \frac{9}{16} H \times \frac{4^3}{3} - 9H \times \frac{4^2}{2} - 11.25 \times \frac{4^4}{4} - 360 \times \frac{4^2}{2} + 36H \times 4 = 0$$

or, $$\frac{495}{3} H = 720 \Rightarrow \text{giving } H = 4.3636 \text{ N}$$

Thus, $V_a = 60 + \frac{3}{4}(4.3636) = 63.2727$ N

$V_d = 120 - 63.2727 = 56.7273$ N

Now, we put the values of H, V_a and V_d in Figure 11.8 (b) and can calculate the BM at different points as given below :

BM at $A = 0$

BM at $B = -4.3636 \times 6 = -26.18$ N m

BM at $C = -4.3636 \times 3 = -13.09$ N m

BM at $D = 0$

BM at mid of BC due to external loading $= \frac{30 \times 4^2}{8} = +60$ N m

Net BM at mid of $BC = +60 - \left[\frac{26.18 - 13.09}{4} \times 27.09\right] = 40.365$ N m

The bending moment diagram is shown in Figure 11.8 (b).

Example 11.8

Analyse the frame shown in Figure 11.9 (a) made of the members of similar flexural rigidity.

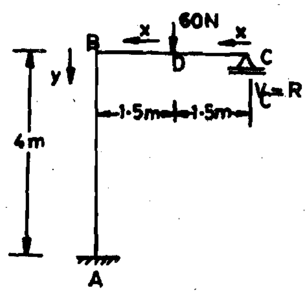


Figure 11.9 (a)

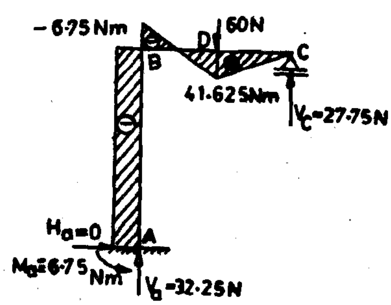


Figure 11.9 (b)

Solution

Total degree of redundancy of the frame = $3m + r - 3j = (3 \times 2) + 4 - (3 \times 3) = 1$

Degree of external redundancy = $r - 3 = 4 - 3 = 1$

Degree of internal redundancy = $1 - 1 = 0$

Let the reaction at $C = R$ as redundant.

Total strain energy of the frame, $U = \sum \int \frac{M^2 \, ds}{2EI}$

$$U = \int_0^{1.5} (Rx)^2 \frac{dx}{2EI} + \int_0^{1.5} [R(x + 1.5) - 60x]^2 \frac{dx}{2EI} + \int_0^4 (3R - 60 \times 1.5)^2 \frac{dy}{2EI}$$

Assuming the bending moment producing concavity outside the frame as positive bending moment and that producing convexity outside the frame as negative bending moment.

According to the Principle of Minimum Energy, $\frac{\partial U}{\partial R} = 0$

Thus, we get,

$$\int_0^{1.5} \frac{(2Rx \times x)\, dx}{2EI} + \int_0^{1.5} 2[R(x + 1.5) - 60x] \frac{(x + 1.5)\, dx}{2EI} + \int_0^4 2(3R - 90) \frac{3}{2EI} dy = 0$$

or, $\frac{R}{3}(1.5)^3 + \frac{R}{3}\left[(x + 1.5)^3\right]_0^{1.5} - \frac{60}{3}(1.5)^3 - \frac{90}{2}(1.5)^2 + 9(R - 30)\,4 = 0$

or, $R = 27.75 \text{ N} = V_c$

$\therefore \; V_a = (60 - 27.75) \text{ N} = 32.25 \text{ N}$

Now we can calculate the BM at different points as given below :

BM at $C = 0$

BM at $D = +27.75 \times 1.5 = +41.625$ N m

BM at B = BM at $A = (27.75 \times 3) - (60 \times 1.5) = -6.75$ N m

Now, we put the reaction and moment values in Figure 11.9 (b). Let the point of contraflexure be at a distance x from D,

Then, $27.75\,(x + 1.5) - 60x = 0$ giving $x = 1.29$ m

The bending moment diagram (BMD) is shown in Figure 11.9 (b).

Example 11.9

A semi circular arch of uniform flexural rigidity, having one end hinged and other end placed on roller subjected to a horizontal force P as shown in Figure 11.10 (a). Find the horizontal displacement of the roller end.

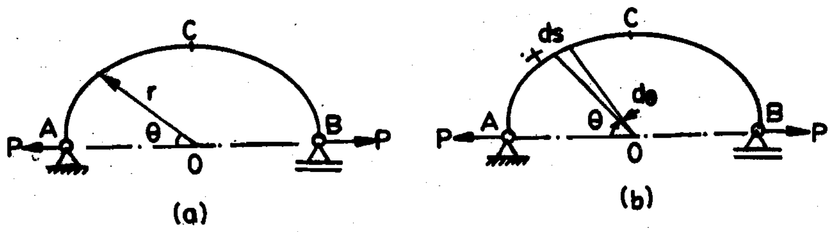


Figure 11.10

Solution

Due to equilibrium, $H_a = \underset{\leftarrow}{P}$

BM at a section X making an angle θ with the horizontal, $M = P\, r \sin\theta$

Here, the elemental length of the arch, $ds = r\, d\theta$

$$\therefore \text{ Total strain energy } U = \int \frac{M^2\, ds}{2EI} = 2\int_0^{\pi/2} \frac{P^2\, r^2 \sin^2\theta\, r\, d\theta}{2EI} = \frac{P^2 r^3}{EI} \times \frac{\pi}{4}$$

Let the horizontal movement of the roller end be Δ.

Thus, the external work done by $P = \frac{1}{2} P\Delta$

By equating total strain energy to external work done, we get,

$$\frac{1}{2} P\Delta = \frac{\pi}{4} \times \frac{P^2 r^3}{EI}$$

It gives the displacement of the roller end, $\Delta = \dfrac{\pi P r^3}{2EI}$

Example 11.10

Determine the various reactions of a thin semicircular ring lying in a horizontal plane having both ends clamped subjected to a central vertical force P perpendicular to its plane as shown in Figure 11.11 (a).

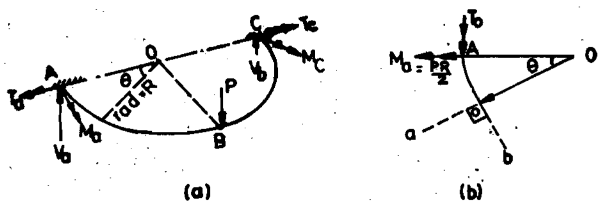


Figure 11.11

Solution

As the plane of loading is not in the plane of the structure, there will be twisting moments in addition to bending moment and vertical/horizontal reactions at supports.

From symmetry, we get, $V_a = V_c = \dfrac{P}{2}$

$$\text{and } (BM)_A = (BM)_C = \frac{P}{2} \times R = \frac{PR}{2}$$

Let the other reaction that is twisting moment acting at each of the support be T_o.

Now, we consider an arbitrary point o of the segment of the ring making an angle θ at the centre (where $0 < \theta < \frac{\pi}{2}$).

BM at o about oa axis, $M = \dfrac{P}{2} R \sin\theta - \dfrac{PR}{2} \cos\theta - T_0 \sin\theta$

Twisting moment at o about ob axis, $T = \frac{P}{2}(R - R\cos\theta) - \frac{PR}{2}\sin\theta + T_0\cos\theta$

Total strain energy of the ring

$$U = \int \frac{M^2\,ds}{2EI} + \int \frac{T^2\,ds}{2GJ}$$

Since angular rotation both at A and C = 0

According to Castigliano's second theorem, $\frac{\partial U}{\partial T_0} = 0,$

$$\frac{\partial U}{\partial T_0} = \int \frac{M\frac{\partial M}{\partial T_0}\,ds}{EI} + \int \frac{T\frac{\partial T}{\partial T_0}\,ds}{GJ} = 0$$

For this frame, being symmetrical, we have of AB and AC portion and $ds = R\,d\theta$

$$2\int_0^{\pi/2} \frac{M\frac{\partial M}{\partial T_0}}{EI} R\,d\theta + 2\int_0^{\pi/2} \frac{T\frac{\partial T}{\partial T_0}}{GJ} R\,d\theta = 0$$

Substituting the values of M and T, we get

$$\int_0^{\pi/2} \frac{\left[\frac{PR}{2}\sin\theta - \frac{PR}{2}\cos\theta - T_0\sin\theta\right]}{EI}(-\sin\theta)\,R\,d\theta +$$

$$\int_0^{\pi/2} \frac{\left[\frac{P}{2}(R - R\cos\theta) - \frac{PR}{2}\sin\theta + T_0\cos\theta\right]}{GJ}(\cos\theta)\,R\,d\theta = 0$$

Finally, we get,

$$T_0 = \frac{\frac{PR}{2}\left(\frac{2-\pi}{EI} + \frac{2-\pi}{GJ}\right)}{\left(\frac{\pi}{EI} - \frac{\pi}{GJ}\right)}$$

Example 11.11

A two-hinged symmetrical parabolic arch has a span of 30 m and rise of 7.5 m. The moment of inertia of arch section is proportional to sec θ, where θ is the slope of the arch axis at any point with the horizontal. Determine the horizontal thrust caused in the arch due to rise of temperature by 25°C.

Given $E = 2 \times 10^6$ kg/cm^2; Coefficient of thermal expansion, $\alpha = 6 \times 10^{-6}$ per °C; and Moment of inertia at the crown, $I_0 = 125 \times 10^4$ cm^4.

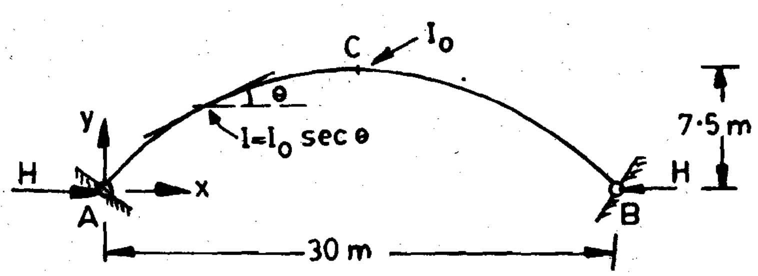


Figure 11.12

Solution

Total degree of redundancy of the arch = $3m + r - 3j = (3 \times 1) + 4 - (3 \times 2) = 1$

Degree of external redundancy = $4 - 3 = 1$

∴ Degree of internal redundancy = $1 - 1 = 0$

Let the horizontal thrust H developed in the parabolic arch due to rise of temperature be treated as redundant.

In the problem, span, $l = 30 \text{ m} = 3000 \text{ cm}$

Central rise, $y_c = 7.5 \text{ m} = 750 \text{ cm}$

Rise of temperature, $t = 25°\text{C}$

Modulus of elasticity, $E = 2.0 \times 10^6 \text{ kg/cm}^2$

Coefficient of linear expansion, $\alpha = 6 \times 10^{-6}$ per °C

Moment of inertia at the crown, $I_0 = 125 \times 10^4 \text{ cm}^4$

The horizontal expansion prevented by the hinges = αlt

BM on any element of the arch at a height y above the support, $M = Hy$

Total strain energy of the arch, $U = \int \frac{M^2 ds}{2EI}$

According to Castigliano's second theorem, $\frac{\partial U}{\partial H} = l\alpha t$

$$\int_0^{\text{total length of arch}} \frac{M}{EI}\left(\frac{\partial M}{\partial H}\right) ds = \alpha lt$$

or,

$$\int_0^{\text{total length of arch}} \frac{Hy \times y ds}{EI} = \alpha lt$$

or,

$$\frac{H}{E}\int_0^l \frac{y^2 \sec\theta \, dx}{I_0 \sec\theta} = \alpha lt \quad [\text{since, } I = I_0 \sec\theta, \text{ and } ds = dx \sec\theta]$$

or,

$$H = \frac{EI_0 \alpha lt}{\int_0^l y^2 \, dx}$$

For parabolic arch, $\int_0^l y^2 \, dx = \frac{8}{15} y_c^2 \, l$; equation of the arch being $y = \frac{4y_c}{l^2} x(l - x)$

Thus, we get,

$$H = \frac{15EI_0 \alpha lt}{8y_c^2 \, l}$$

On substituting the values, we get

$$H = \frac{15 \times (2 \times 10^6) \times (125 \times 10^4) \times (6 \times 10^{-6}) \times 25}{8 \times (750 \times 750)} = 1250 \text{ kg}$$

Example 11.12

Find the forces in the members FD and DH of the frame shown in Figure 11.13 (a) having the ratio of length to the cross sectional area of all the members as same.

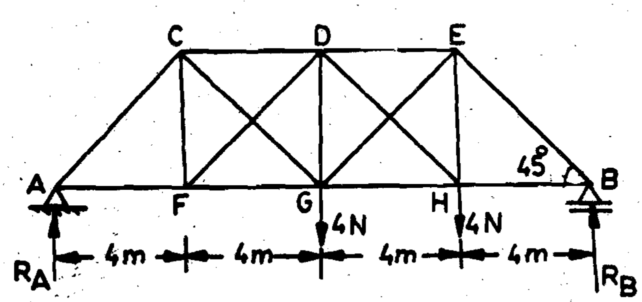


Figure 11.13 (a)

Solution

Total degree of redundancy

$$= m + r - 2j = 15 + 3 - (2 \times 8) = 2$$

Degree of external redundancy $= r - 3 = 3 - 3 = 0$

Degree of internal redundancy $= 2 - 0 = 2$

Here, we assume the axial force in the members FD and DH = X and Y respectively as redundant members

Now, we remove the redundant members FD and DH and analyse the truss by graphical method [Refer Figure 11.13 (b)].

From the vector diagram, we calculate the forces in the different members [Refer Figure 11.13 (c)].

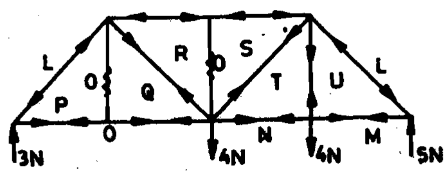


Figure 11.13 (b)

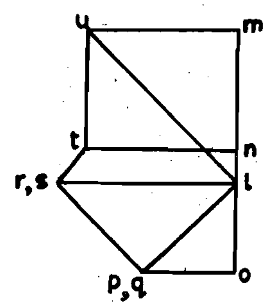


Figure 11.13 (c)

Now, we impose a unit tensile load in the member FD [Figure 11.14 (a)].

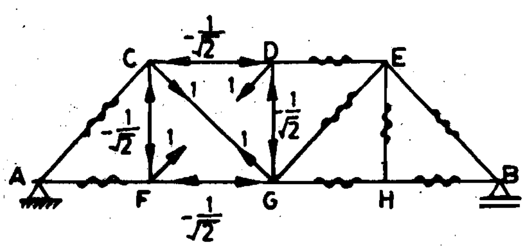


Figure 11.14 (a)

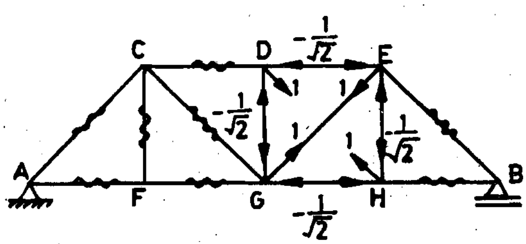


Figure 11.14 (b)

At Joint D

$$K_{DC} = \frac{1}{\sqrt{2}} \text{ N (compression)} \qquad K_{DG} = \frac{1}{\sqrt{2}} \text{ N (compression)}$$

At Joint F

$$K_{FG} = \frac{1}{\sqrt{2}} \text{ N (compression)} \qquad K_{FC} = \frac{1}{\sqrt{2}} \text{ N (compression)}$$

At Joint C

$$K_{CG} = 1 \text{ N (tension)}$$

Similarly, we impose unit tensile load in the member DH [Figure 11.14 (b)].

At Joint D

$$K_{DE}' = \frac{1}{\sqrt{2}} \text{ N (compression)} \qquad K_{DG}' = \frac{1}{\sqrt{2}} \text{ N (compression)}$$

At Joint H

$$K_{HG}' = \frac{1}{\sqrt{2}} \text{ N (compression)} \qquad K_{HE}' = \frac{1}{\sqrt{2}} \text{ N (compression)}$$

At Joint E

$$K_{EG}' = 1 \text{ N (tension)}$$

Now, we put the value of the forces in the different members as determinate truss, imposing unit load in FD and imposing unit load in DH in the tabular form as shown in Table 11.6, assuming tensile force as positive and compressive force as negative.

Table 11.6

Member	P	K	K'	PK	PK'	K^2	K'^2	KK'
AC	$-3\sqrt{2}$	0	0	0	0	0	0	0
CD	-6.0	$-\frac{1}{\sqrt{2}}$	0	$+\frac{6}{\sqrt{2}}$	0	$+\frac{1}{2}$	0	0
DE	-6.0	0	$-\frac{1}{\sqrt{2}}$	0	$+\frac{6}{\sqrt{2}}$	0	$+\frac{1}{2}$	0
EB	$-5\sqrt{2}$	0	0	0	0	0	0	0
AF	$+3.0$	0	0	0	0	0	0	0
FG	$+3.0$	$-\frac{1}{\sqrt{2}}$	0	$-\frac{3}{\sqrt{2}}$	0	$+\frac{1}{2}$	0	0
GH	$+5.0$	0	$-\frac{1}{\sqrt{2}}$	0	$-\frac{5}{\sqrt{2}}$	0	$+\frac{1}{2}$	0
HB	$+5.0$	0	0	0	0	0	0	0
CF	0	$-\frac{1}{\sqrt{2}}$	0	0	0	$+\frac{1}{2}$	0	0
FD	0	$+1$	0	0	0	$+1$	0	0
CG	$+3\sqrt{2}$	$+1.0$	0	$+3\sqrt{2}$	0	$+1.0$	0	0
DG	0	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0	0	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$
DH	0	0	$+1$	0	0	0	$+1$	0
EG	$+\sqrt{2}$	0	$+1.0$	0	$+\sqrt{2}$	0	$+1.0$	0
EH	$+4.0$	0	$-\frac{1}{\sqrt{2}}$	0	$-\frac{4}{\sqrt{2}}$	0	$+\frac{1}{2}$	0
			$\sum =$	$+\frac{9}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$+4.0$	$+4.0$	$+\frac{1}{2}$

Since we assume the force in each member $= P + KX + K'Y$

Total strain energy of the truss, $U = \sum \frac{(P + KX + K'Y)^2 C}{2AE}$

By minimum energy principle the redundant force X and Y should be such that $\frac{\partial U}{\partial X} = 0$ and $\frac{\partial U}{\partial Y} = 0$.

Thus, we get,

$$\sum \frac{PKl}{AE} + X \sum \frac{K^2 l}{AE} + Y \sum \frac{KK'l}{AE} = 0$$

and

$$\sum \frac{PK'l}{AE} + Y \sum \frac{K'^2 l}{AE} + X \sum \frac{KK'l}{AE} = 0$$

For this problem, since l/A and E are constant for all the members

Therefore,

$$\sum PK + X \sum K^2 + Y \sum KK' = 0$$

or, $+\frac{9}{\sqrt{2}} + 4X + \frac{1}{2}Y = 0$ (substituting the $\sum$ values from Table 11.6)

or, $4X + \frac{1}{2}Y = -\frac{9}{\sqrt{2}}$ (i)

Again, $\sum PK' + Y\sum K'^2 + X\sum KK' = 0$

or, $-\frac{1}{\sqrt{2}} + 4Y + \frac{1}{2}X = 0$ (substituting the $\sum$ values from Table 11.6)

or, $\frac{1}{2}X + 4Y = \frac{1}{\sqrt{2}}$

or, $4X + 32Y = \frac{8}{\sqrt{2}}$ (ii)

On solving Eqs. (i) and (ii), we get,

$$X = (-)\frac{73}{63}\sqrt{2}\text{ N (compression), and } Y = (+)\frac{17}{63}\sqrt{2}\text{ N (tension)}$$

Figure 11.15 shows the final member forces.

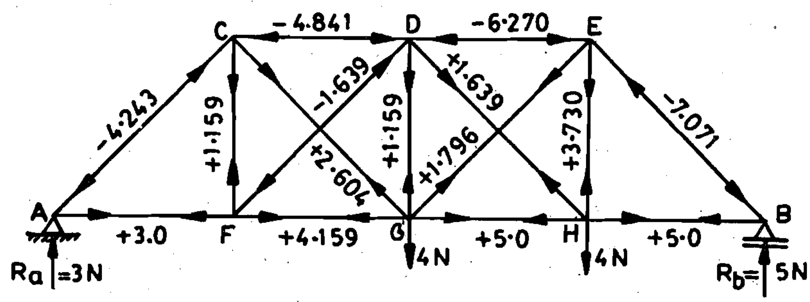


Figure 11.15 : Final Member Forces

Example 11.13

Analyse the portal frame shown in Figure 11.16 (a) using unit load method. Moment of inertia and length of the members are mentioned near the member in the figure.

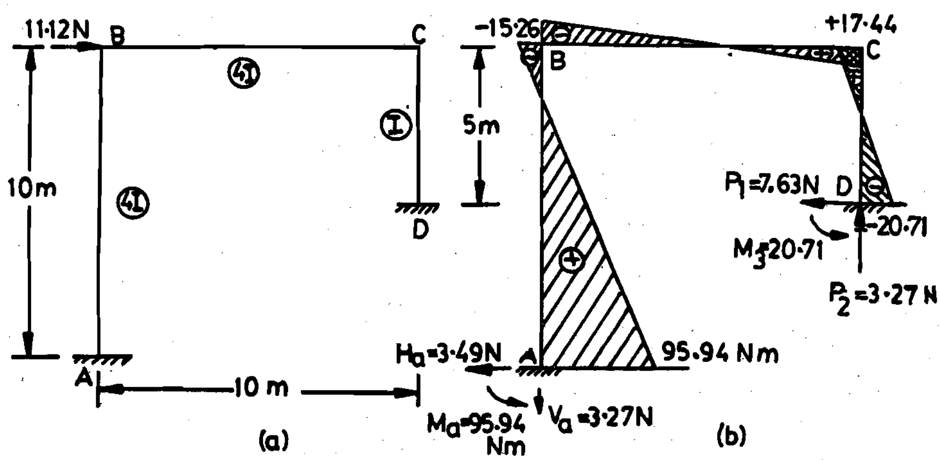


Figure 11.16

Solution

Total degree of redundancy of the frame = 3 × Number of loops

Released component at the support = $(3 \times 1) - 0 = 3$

Degree of external redundancy = $r - 3 = 6 - 3 = 3$

Therefore, the degree of internal redundancy = $3 - 3 = 0$

Let the horizontal reaction at $D = P_1$ →(right) and the vertical reaction at $D = P_2$ ↑ be taken as the redundant reactions.

And also let the moment at $D = M_3$ (clockwise) (upward).

Since the end D is fixed, linear or angular displacement at D due to each redundant reaction is zero.

According to Castigliano's Theorem II, we know

$$\frac{\partial U}{\partial M} = \Delta \quad => \quad \int \frac{M\left(\frac{\partial U}{\partial M}\right)ds}{EI} = \Delta$$

Using unit load method, this equation may be written as follows :

$$\Delta_1 = \int \frac{M}{EI}\left(\frac{\partial M}{\partial P_1}\right)dx = 0$$

$$\Delta_2 = \int \frac{M}{EI}\left(\frac{\partial M}{\partial P_2}\right)dx = 0$$

and

$$\Delta_3 = \int \frac{M}{EI}\left(\frac{\partial M}{\partial P_3}\right)dx = 0$$

We fill up the table (Table 11.7) as shown below :

$$\Delta_1 = \int_0^5 \frac{(P_1x - M_3)x}{EI}dx + \int_0^{10} \frac{(5P_1 + P_2x - M_3)(5)}{4EI}dx +$$

$$\int_0^{10} \frac{[10P_2 - M_3 + P_1(5-x) - 11.12x](5-x)}{4EI}dx = 0$$

or $\quad 150P_1 + 75P_2 - 30M_3 + 278 = 0 \qquad$ (i)

$$\Delta_2 = \int_0^{10} \frac{(5P_1 + P_2x - M_3)(x)}{4EI}dx + \int_0^{10} \frac{[10P_2 - M_3 + P_1(5-x) - 11.12x](10)}{4EI}dx = 0$$

or, $\quad 75P_1 + 400P_2 - 45M_3 - 1668 = 0 \qquad$ (ii)

$$\Delta_3 = \int_0^5 \frac{(P_1x - M_3)(-1)}{EI}dx + \int_0^{10} \frac{(5P_1 + P_2x - M_3)(-1)}{4EI}dx +$$

$$\int_0^{10} \frac{[10P_2 - M_3 + P_1(5-x) - 11.12x](-1)}{4EI}dx = 0$$

or, $\quad 25P_1 + 37.5P_2 - 10M_3 - 139 = 0 \qquad$ (iii)

Table 11.7

Portion of the Frame	**DC**	**CB**	**BA**
Moment of Inertia	I	4I	4I
Origin	D	C	B
Limits	0 to 5	0 to 10	0 to 10
M	$(P_1x - M_3)$	$(5P_1 + P_2x - M_3)$	$[10P_2 - M_3 + P_1(5-x) - 11.12x]$
$\frac{\partial M}{\partial P_1}$	x	5	$(5-x)$
$\frac{\partial M}{\partial P_2}$	0	x	10
$\frac{\partial M}{\partial M_3}$	− 1	− 1	− 1

Solving the Eqs. (i), (ii) and (iii), we get,

$$P_1 = -7.63 \text{ N};$$

$$P_2 = 3.27 \text{ N; and}$$

$$M_3 = -20.71 \text{ N m}$$

Now, we can determine the BM ordinates as follows :

$$\text{BM at } D = M_D = -20.71 \text{ N m}$$

$$\text{BM at } C = M_C = -20.71 + (7.63 \times 5) = +17.44 \text{ N m}$$

$$\text{BM at } B = M_B = 17.44 - (3.27 \times 10) = -15.26 \text{ N m}$$

$$\text{BM at } A = M_A = -15.26 + (11.12 \times 10) = +95.94 \text{ N m}$$

Assuming the bending moment producing tension at the outer face as positive and drawn on the compression side of the frame, the bending moment diagram is shown in Figure 11.16 (b).

SAQ 1

Find the prop reaction R of the beam loaded as shown in Figure 11.17.

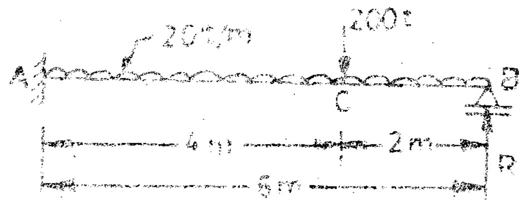


Figure 11.17

SAQ 2

Three wires AD, BD and CD of same cross sectional area and same materials carrying a load W at D. Find the tension in the wires as shown in Figure 11.18.

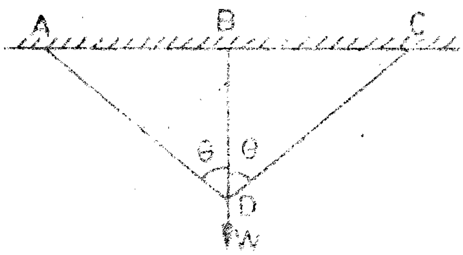


Figure 11.18

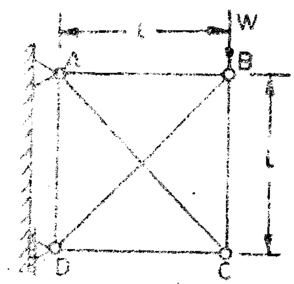


Figure 11.19

SAQ 3

Find the forces in the members of the frame having same area and of same material for all the members and carrying a load W at B as shown in Figure 11.19.

SAQ 4

Find the forces in all the members of the frame loaded as shown in Figure 11.20. The members have the same cross-sectional area and the same material.

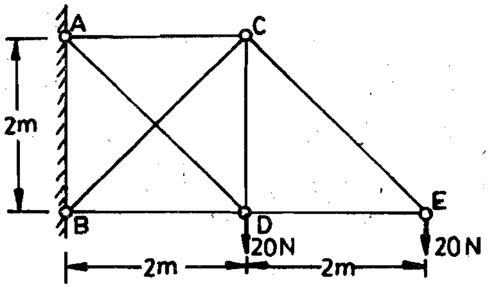


Figure 11.20

SAQ 5

(a) Analyse the two hinged portal frame loaded as shown in Figure 11.21.

(b) Find the maximum positive and negative moments in a ring of radius R having same cross sectional area and of same material throughout. The ring is subjected to the action of two equal and opposite vertical forces at the extremities of the vertical diameter shown in Figure 11.22.

Find also the contraction in the length of the vertical diameter and expansion in the length of the horizontal diameter. Assume strain energy stored in the ring is due to bending only.

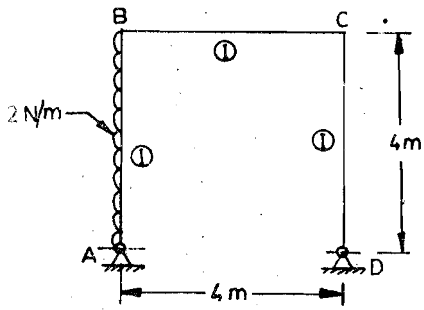


Figure 11.21

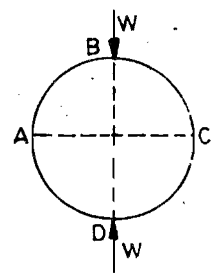


Figure 11.22

SAQ 6

Analyse the two hinged frame having the members of uniform flexural rigidity as shown in Figure 11.23. The joint C is rigid. Draw the bending moment diagram.

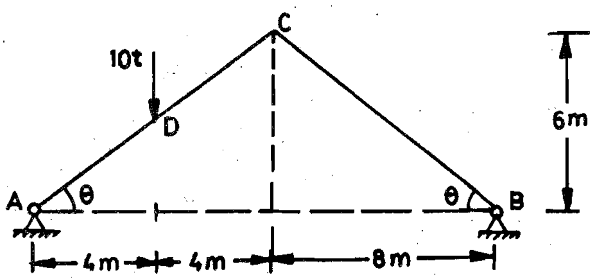


Figure 11.23

11.6 MAXWELL'S RECIPROCAL THEOREM

Clerk Maxwell developed a fundamental theorem based directly on the principle of conservation of energy and the principle of superposition which is applicable to both determinate and indeterminate structures.

Statement

In a linearly elastic structure in static equilibrium, the displacement at coordinate i due to a unit force acting at coordinate j is equal to the displacement at coordinate j due to unit force at coordinate i.

Mathematically,

$$K_{ij} = K_{ji}$$

Proof

We impose a displacement Δ_i at coordinate i without any displacement at coordinate j in the structure shown in Figure 11.24.

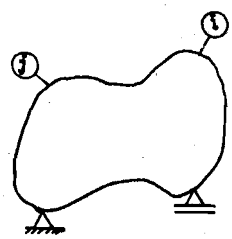


Figure 11.24

Here, the force at $i = K_{ii}\,\Delta_i$ and force at $j = K_{ji}\,\Delta_i$

Work done due to displacement, $\Delta_i = \frac{1}{2} \times K_{ii}\,\Delta_i \times \Delta_i = \frac{1}{2} \times K_{ii}\,\Delta_i^2$ (11.13)

Next, we provide a displacement Δ_j at coordinate j without any displacement at coordinate i.

Thus, additional force at $i = K_{ij}\,\Delta_j$ and that at $j = K_{jj}\,\Delta_j$.

Since the displacement given only at coordinate j here, we get

Work done due to displacement Δ_j at coordinate $i = (K_{ji}\Delta_i)\,\Delta_j$ and (11.14)

Additional work done due to displacement Δ_j a coordinate j

$$= \frac{1}{2} \times (K_{jj}\,\Delta_j)\,\Delta_j = \frac{1}{2} \times K_{jj}\,\Delta_j^2 \quad (11.15)$$

$\therefore$ Total work done due to displacements Δ_i and Δ_j

= sum of Eqs. (11.13), (11.14) and (11.15)

$$= \frac{1}{2} \times K_{ii}\,\Delta_i^2 + \frac{1}{2} \times K_{jj}\,\Delta_j^2 + K_{ji}\,\Delta_i\,\Delta_j \ \ldots$$

= Total strain energy stored by the structure $= U$ (11.16)

Now, we provide the displacement in this structure in the reverse order, i.e. at first we provide Δ_j and then Δ_i.

Work done due to first displacement Δ_j at coordinate $j = \frac{1}{2} \times K_{jj}\,\Delta_j^2$ (11.17)

Work done due to second displacement Δ_i at coordinate $i = (K_{ij}\,\Delta_j)\,\Delta_i$ (11.18)

Additional work done due to second displacement Δ_i at coordinate j

$$= \frac{1}{2} \times K_{ii}\,\Delta_i^2 \quad (11.19)$$

$\therefore$ Total strain energy due to displacement Δ_j and Δ_i

$$U = \frac{1}{2} \times K_{jj} \Delta_j^2 + K_{ij} \Delta_i \Delta_j + \frac{1}{2} \times K_{ii} \Delta_i^2 \qquad (11.20)$$

Since the strain energy stored U does not depend on the order of displacement provided, so the value of Eq. (11.16) and Eq. (11.20) should be equal.

Comparing the two values, we get

$$K_{ij} = K_{ji}$$

In this theorem, the word "displacement" means linear deflection as well as angular rotation. Similarly, the word "force" means both load or couple.

Example 11.14

Find the displacement at coordinate ② of the two-hinged arch as shown Figure 11.25 due to a 15 N load acting at coordinate ①. Given, a couple of 50 N m at coordinate ② creates a displacement of 0.005 m at coordinate ①.

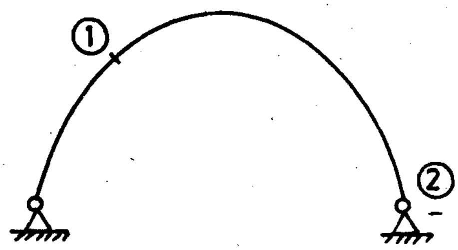


Figure 11.25

Solution

Here, in this problem, we have

Displacement at coordinate ① due to a couple of 50 N m at coordinate ② $\Delta = 0.005$ m

Displacement at coordinate ① due to a unit couple at coordinate ② $= \frac{0.005}{50}$ m

Therefore, $\quad K_{12} = 0.0001$ m

According to Maxwell's Reciprocal Theorem, we have

$$K_{12} = K_{21}$$

Thus, $\quad K_{21} = 0.0001$ radian which is a rotation.

Here, the rotation at coordinate ② due to a load of 15 N acting at coordinate ①

$$= \Delta_2 = K_{21} \times \text{Force at coordinate ①.}$$

$$= 0.0001 \times 15 = 0.0015 \text{ radian}$$

11.7 GENERALISED RECIPROCAL THEOREM OR BETTI'S THEOREM

This theorem is also based on the principle of energy conservation and the principle of superposition applied to both the determinate and indeterminate structures subjected to the action of several forces and displacements.

Statement

In a linearly elastic structure in static equilibrium subjected to two system of forces, the virtual work done by the first system of forces during the displacement caused by the second system of forces is equal to the virtual work done by the second system of forces during the displacements caused by the first system of forces.

Mathematically,

$$U = \sum P\Delta' = \sum P'\Delta$$

Proof

Let the first system of forces be P_1, P_2 and P_3 and the corresponding displacements at coordinates 1, 2 and 3 are Δ_1, Δ_2 and Δ_3 respectively.

Let the second system of forces be P_1', P_2' and P_3' and the corresponding displacements at coordinates 1, 2 and 3 are Δ'_1, Δ'_2 and Δ'_3 respectively.

The virtual work done by the first system of forces in undergoing the displacements caused by the second system of forces,

$$U = P_1\Delta'_1 + P_2\Delta'_2 + P_3\Delta'_3$$

where,

$$\Delta'_1 = \delta_{11}P_1' + \delta_{12}P_2' + \delta_{13}P_3'$$

$$\Delta'_2 = \delta_{21}P_1' + \delta_{22}P_2' + \delta_{33}P_3'$$

$$\Delta'_3 = \delta_{31}P_1' + \delta_{32}P_2' + \delta_{33}P_3'$$

Thus, we get

$$U = P_1(\delta_{11}P_1' + \delta_{12}P_2' + \delta_{13}P_3') + P_2(\delta_{21}P_1' + \delta_{22}P_2' + \delta_{23}P_3') + P_3(\delta_{31}P_1' + \delta_{32}P_2' + \delta_{33}P_3')$$

In the same manner, we get the virtual work done by the second system of forces in undergoing the displacements caused by the first system of forces,

$$U = P_1'\Delta_1 + P_2'\Delta_2 + P_3'\Delta_3$$

Similarly, we get,

$$U = P_1'(\delta_{11}P_1 + \delta_{12}P_2 + \delta_{13}P_3) + P_2'(\delta_{21}P_1 + \delta_{22}P_2 + \delta_{23}P_3) + P_3'(\delta_{31}P_1 + \delta_{32}P_2 + \delta_{33}P_3)$$

From reciprocal theorem we get, $\delta_{ij} = \delta_{ji}$

Thus, we get the virtual work done by the two conditions are same, i.e.

$$\sum P\Delta' = \sum P'\Delta$$

Example 11.15

A continuous beam is subjected to two systems of forces and displacements as shown in Figure 11.26. Find the upward deflection at the coordinate where 9 N is acting in the system II.

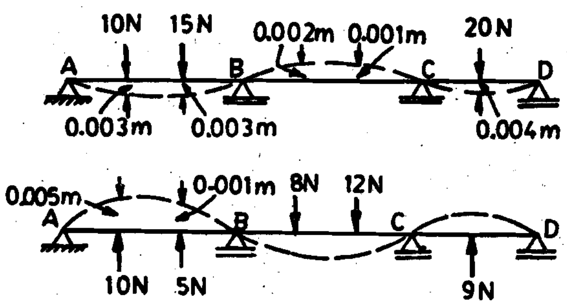


Figure 11.26

Solution

We give the numbers (1), (2), (3), (4) and (5) to the coordinate where forces are acting and displacements are shown or to be found out (Figure 11.27).

Now, we tabulate the values of forces and corresponding displacements in the two systems as shown in Table 11.8.

Taking downward forces and displacements as positive and upward forces and displacements as negative, we get

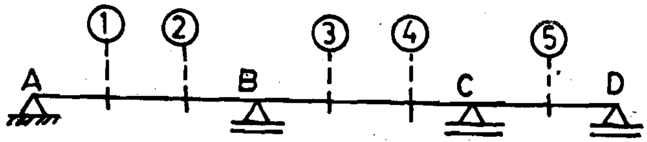


Figure 11.27

$$\sum P\Delta' = 10 \times (-0.005) + 15 \times (-0.001) + 0 + 20 \times (\Delta'_5)$$

$$= -0.065 + 20\Delta'_5$$

$$\sum P'\Delta = -10\,(0.003) - 5\,(0.003) + 8\,(-0.002) + 12\,(-0.001) - 9\,(0.004)$$

$$= -0.109$$

Table 11.8

System ↓	Coordinates →	1	2	3	4	5
I	P	10	15	0	0	20
	Δ	0.003	0.003	– 0.002	– 0.001	0.004
II	P′	– 10	– 5	8	12	– 9
	Δ′	– 0.005	– 0.001	–	–	?

According to Betti's law, we have

$$\sum P\Delta' = \sum P'\Delta$$

On putting the values, we get

$$-0.065 + 20\Delta'_5 = -0.109$$

or $$\Delta'_5 = -0.0022 \text{ m}$$

SAQ 7

Table 11.9 shows the forces and corresponding displacements at nine coordinates due to two systems of force of a portal frame (Figure 11.28).

Find the displacement Δ'_9 due to the second system of forces.

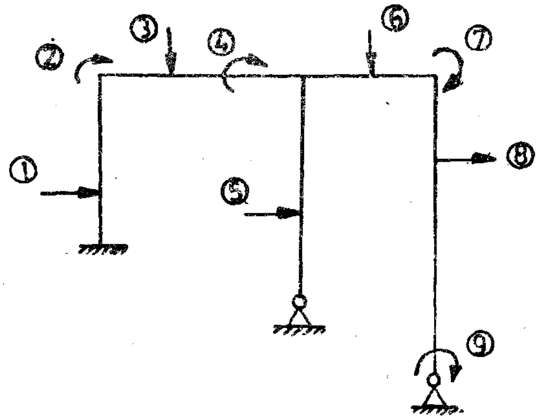


Figure 11.28

Table 11.9 (a) : System 1

Coordinates →	1	2	3	4	5	6	7	8	9
P	5 N	0	3 N	0	0	1 N	0	- 4 N	2 N m
Δ	–	0.001 rad	0.002 m	0.002 rad	0.001 m	–	0.002 rad	–	–

Table 11.9 (b) : System II

Coordinates →	1	2	3	4	5	6	7	8	9
P'	0	10 N m	4 N	5 N m	2 N	0	1 N m	0	0
Δ'	0.003 m	–	0.001 m	–	–	0.002 m	–	- 0.001 m	?

11.7.1 Müller-Breslau's Principle

This principle is used for obtaining "Influence Line Diagrams (ILD)" for any external reaction or internal stress resultant, e.g. BM, SF, axial force etc. in a structure.

Statement

The influence line diagram for any function (i.e. reaction/internal stress resultant) of a structure is given by the deflected shape of the line along which the unit load is moving; the deflected shape being obtained by removing the external/ internal constraint of the function and then applying a unit displacement in the direction of the removed constraint.

The proof of this theorem is obtained directly by using Betti's Theorem. This can be seen in any standard text book of Theory of Structures.

This principle is useful in finding out the "Influence Line Diagrams" for statically indeterminate structures as by removing the constraint of the function, we are reducing the indeterminacy by "one". Thus, a statically indeterminate structure of first order becomes statically determinate. In general, a statically indeterminate structure of order n is reduced to $n - 1$ for finding the deflected shape.

This is made clear by the following illustrations :

Case 1

The influence line diagram for reaction at $C(R_C)$ for the two span continuous beam shown in Figure 11.29 is obtained by removing the support C which is the constraint for the reaction R_C. After removal of the support the structure becomes a statically determinate one [Figure 11.29 (b)] which is an overhanging beam.

Now, if a unit deformation $\delta_C = 1$ is applied at C' the deflected shape gives influence line diagram for R_C.

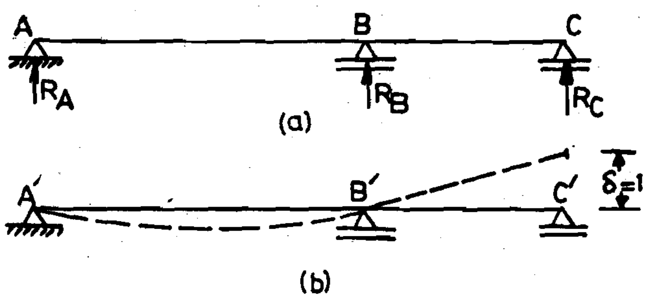


Figure 11.29

Case 2

The influence line diagram for the bending moment at point C in the propped cantilever shown in Figure 11.30 (a) is similarly obtained.

The bending moment restraint at C can be removed by introducing a hinge at C. The statically indeterminate structure in Figure 11.30 (a) is now reduced to a determinate one in Figure 11.30 (b). Next, introduce unit "rotation" at the hinge at C. Since the required force is a "moment" (i.e. couple at C), the corresponding deformation will be a "rotation". The deflected shape is given by the dotted line in Figure 11.30 (c). This will give the "influence line diagram" to the scale $\frac{1}{\theta}$ where θ is the total change in slope of the two parts of the beam at the hinge C'. Hence, $\frac{1}{\theta}$ will be called the "scale factor" of the diagram.

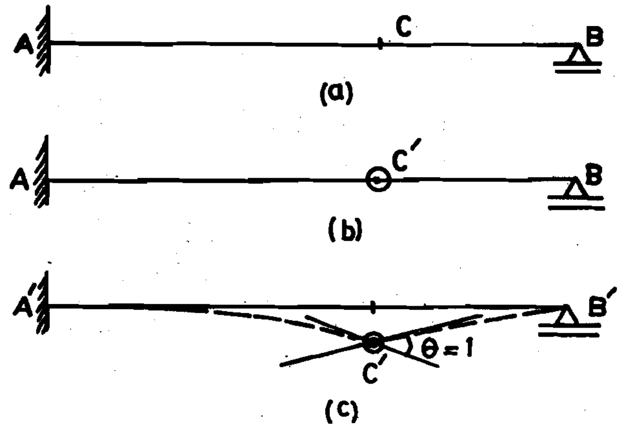


Figure 11.30

Case 3

The influence line diagram for the force in member HB is to be obtained for the pin-jointed truss shown in Figure 11.31 (a). Obviously, the pin-jointed truss is statically indeterminate (degree of redundancy = 1). By removing the member HB the truss becomes a statically determinate structure. Now, introduce unit forces in place of member HB at joints H and B as shown in Figure 11.31 (b). If the unit load rolls along the bottom chord *ABCDE*, the deflection of this chord due to this unit load system shown as *AB'C'D'E* gives the ILD for force in HB to a certain scale. The scale of the diagram is obtained by dividing all the ordinates of the deflection curve by the amount Δ where Δ is the deformation between the points H and B ($\Delta = HB - H'B'$). Thus, $\frac{1}{\Delta}$ is called the scale factor of the influence line diagram.

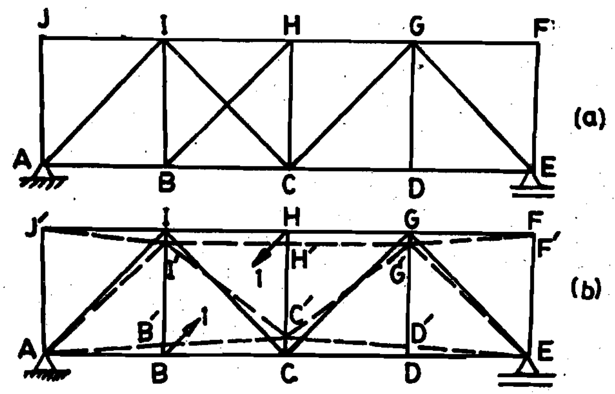


Figure 11.31

Example 11.15

Draw the influence line diagram for the bending moment at section *C* of the propped cantilever shown in Figure 11.32 (a).

Solution

Here the Figure 11.32 (b) shows the structure with the moment restraint at *C* removed by introducing a hinge there. A unit bending moment is introduced at the hinge *C* with the subsequent support reactions shown in Figure 11.32 (b) and the bending moment diagram in Figure 11.32 (c) which can be verified. For finding the deflection diagram of this beam, we use the conjugate beam method.

The conjugate beam with the $\frac{M}{EI}$ loading and the corresponding reactions are shown in Figure 11.32 (d). The bending moment diagram (BMD) of this loading is shown in Figure 11.32 (f) which is the deflection (δ) diagram, and the ordinates are calculated at every 1 m interval (which can be verified). According to Müller-Breslau's theorem, this deflected diagram gives the influence line diagram for bending moment at C, to a certain scale. The scale of the diagram is obtained by dividing the ordinates by $\theta_1 + \theta_2$ where θ_1 and θ_2 are slopes on either side of the hinge C. The value of θ_1 and θ_2 are the values of shear force of the $\frac{M}{EI}$ loading on conjugate beam and is equal to $\frac{9}{8}\downarrow$ and $\frac{71}{48}\uparrow$ respectively [Figure 11.32 (e)]. Total value of $\theta_1 + \theta_2$ is $\frac{125}{48\,EI}$. So all the ordinates of the deflection diagram are to be divided by this constant $\frac{125}{48\,EI}$ to get the actual values of the ILD ordinates which are shown in Figure 11.32 (g).

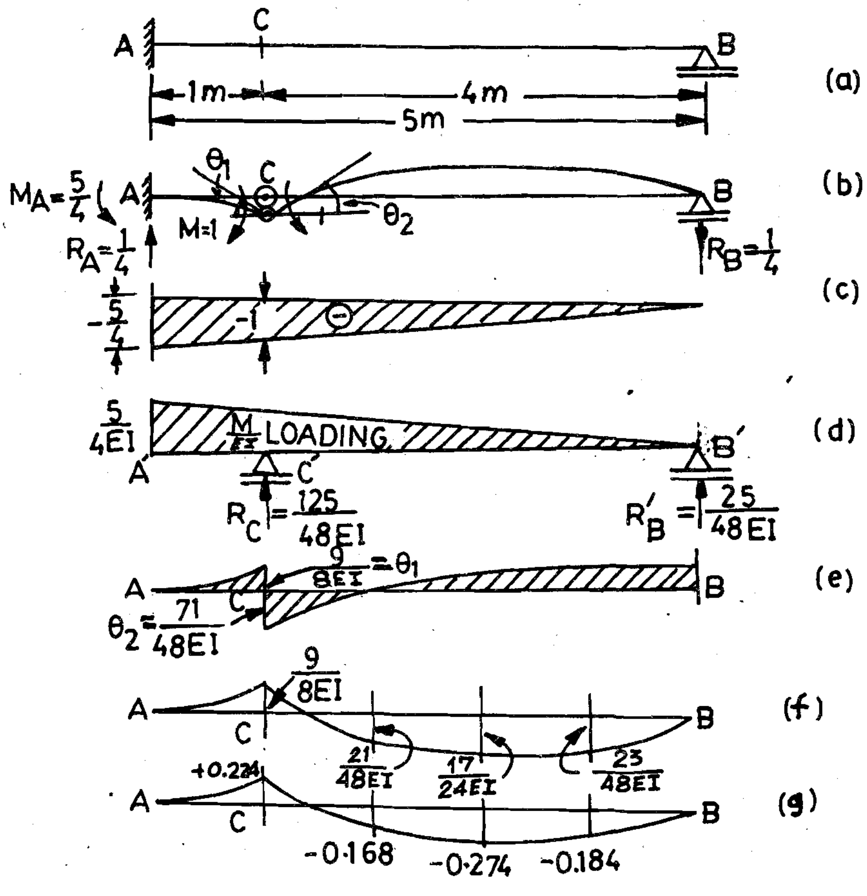


Figure 11.32

11.8 LIMITATIONS OF THE ENERGY METHODS

Energy methods are applicable to the structures of materials which follow Hooke's law and the entire system obeys the law of superposition. These methods are not applicable in the event when the stresses and displacements are not linear functions of the applied loads and the principle of superposition does not apply. Also the displacements must be very small as not to substantially cause a change in the geometry of the structure.

11.9 SUMMARY

Energy principles and various energy methods in structural analysis giving emphasis on indeterminate types are discussed in this unit. The basic concepts of conservation of energy is applicable to elastic bodies as well. Since deformation are imposed on the system, the internal work done by a structural system is a negative quantity and are restrained by its internal forces (or stress system). The external work done by the external forces is positive. Actually, the algebraic sum of external and internal work done must be zero.

11.10 KEY WORDS

Strain Energy

The strain energy is the work done by the internal forces due to distortion or displacement of the body. It may be represented by internal work done or potential energy. In elastic bodies, strain energy is released on removal of the loads or deformation. The general equation of strain energy is as follows :

$$U = \int \frac{S^2 ds}{2AE} + \int \frac{Q_x^2 ds}{2A_{rx}G} + \int \frac{Q_y^2 ds}{2A_{ry}G} + \int \frac{M_x^2 ds}{2EI_x} + \int \frac{M_y^2 ds}{2EI_y} + \int \frac{T^2 ds}{2GK}$$

where,

S = axial force

= Biaxial shear forces

M_x, M_y = Biaxial bending moments

T = Torsion

A = Area of cross section of the member

A_{rx}, A_{ry} = Reduced area of the cross section in the xz and yz planes

ds = Elemental length along the direction of force or stress

E = Modulus of elasticity

G = Shear modulus

K = A constant based on cross sectional shape of the twisting member known as form factor

I_x, I_y = Moment of inertia of the cross section about the neutral axis parallel to x and y axes

Principle of Virtual Work

If a structure is in equilibrium under the action of a set of external forces and is subjected to a set of displacements compatible with the constraints of the structure, the total work done by the external and internal forces during the displacement must be zero.

Castigliano's Theorem-I

The partial derivative of the strain energy U of a linearly elastic structure represented in terms of displacements with respect to any displacement Δ_j at coordinate j is equal to the force P_j at coordinate j. Mathematically,

$$\frac{\partial U}{\partial \Delta_j} = P_j$$

Castigliano's Theorem-II

The partial derivative of the strain energy of a linearly elastic structure represented in terms of forces with respect to any force P_j at coordinate j is equal to the displacement Δ_φ at coordinate j. Mathematically,

$$\frac{\partial U}{\partial P_j} = \Delta_j$$

Minimum Energy Theorem or Principle of Least Work

In any and every case of statically indeterminate structure where an indefinite number of different values of the redundant forces and displacements satisfy the conditions of statical equilibrium, their actual values are those that render the total strain energy stored to a minimum.

Therefore, $\quad \dfrac{\partial U}{\partial X} = 0$ and $\dfrac{\partial^2 U}{\partial X^2}$ is positive

where X = redundant force.

Maxwell's Reciprocal Theorem

In a linearly elastic structure in static equilibrium, the displacement at coordinate i due to a unit force acting at coordinate j is equal to the displacement at coordinate j due to unit force at coordinate i. Mathematically,

$$K_{ij} = K_{ji}$$

Generalised Reciprocal Theorem or Betti's Theorem

In a linearly elastic structure in static equilibrium, subjected to two system of forces, the virtual work done by the first system of forces during the displacements caused by the second system of forces is equal to the virtual work done by the second system of forces during the displacements caused by the first system of forces.

Mathematically,

$$U = P_1\Delta'_1 + P_2\Delta'_2 + P_3\Delta'_3 = P_1'\Delta_1 + P_2'\Delta_2 + P_3'\Delta_3$$

where,

P_1, P_2, P_3 = first system of forces at coordinate 1, 2 and 3.

P_1', P_2', P_3' = second system of forces at coordinate 1, 2 and 3.

$\Delta_1, \Delta_2, \Delta_3$ = Displacement due to second system forces at coordinate 1, 2 and 3.

$\Delta'_1, \Delta'_2, \Delta'_3$ = Displacement due to first system forces at coordinate 1, 2 and 3.

11.11 ANSWERS TO SAQs

SAQ 1

Total strain energy of the beam,

$$U = \int_0^2 (Rx - 10x^2)\frac{dx}{2EI} + \int_2^6 [Rx - 10x^2 - 200(x-2)]^2 \frac{dx}{2EI}$$

According to minimum energy principle, $\frac{\partial U}{\partial R} = 0$

Thus, we get, $R = 148.704$ t

SAQ 2

Let reaction at B = tension in BD = R

Tension in AD and CD each = P and length of BD = l

We have, $P = \frac{W - R}{2\cos\theta}$

Strain energy, $U = \sum \frac{S^2 l}{2AE} = \frac{R^2 l}{2AE} + \left(2P^2 \times \frac{l}{\cos\theta} \times \frac{1}{2AE}\right)$

Putting the value of P, we get, $U = \frac{R^2 l}{2AE} + \frac{(W-R)^2 l}{4AE\cos^3\theta}$

According to minimum energy principle, $\frac{\partial U}{\partial R} = 0$

Thus, we get, $R = \frac{W}{1 + 2\cos^3\theta}$ and $P = \frac{W\cos^2\theta}{1 + 2\cos^3\theta}$.

SAQ 3

Take the member BD as redundant. Considering compressive force as positive and tensile force as negative.

Thus, $\sum PKl = -Wl\sqrt{2}(1 + \sqrt{2})$

and $\sum K^2 l = 2l(1 + \sqrt{2})$

$$\therefore \text{ Correcting factor, } X = -\frac{\sum PKl}{\sum K^2 l} = -\frac{W\sqrt{2}}{2}$$

Applying the formula $F = P + XK$, we get the forces in the various members as given in the last column of Table 11.10.

Table 11.10

Member	P	K	l	PKl	K^2l	$F = P + XK$
AB	0	$-\frac{1}{\sqrt{2}}$	l	0	$\frac{l}{2}$	$-\frac{w}{2}$
BC	+ w	$-\frac{1}{\sqrt{2}}$	l	$-\frac{wl}{\sqrt{2}}$	$\frac{l}{2}$	$+\frac{w}{2}$
CD	+ w	$-\frac{1}{\sqrt{2}}$	l	$-\frac{wl}{\sqrt{2}}$	$\frac{l}{2}$	$+\frac{w}{2}$
DA	0	$-\frac{1}{\sqrt{2}}$	l	0	$\frac{l}{2}$	$-\frac{w}{2}$
AC	$-w\sqrt{2}$	+1	$l\sqrt{2}$	$-2wl$	$l\sqrt{2}$	$-\frac{w\sqrt{2}}{2}$
BD	0	+1	$l\sqrt{2}$	0	$l\sqrt{2}$	$+\frac{w\sqrt{2}}{2}$

SAQ 4

Take the member BC as redundant. Considering the compressive force as positive and tensile force as negative.

$$\text{Correcting factor } X = -\frac{\sum PKl}{\sum K^2 l} = -\frac{-160 - 60\sqrt{2}}{3 + 4\sqrt{2}} = +20\sqrt{2}$$

Applying the formula $F = P + XK$, we get the forces in the various members as given in the last column of Table 11.11.

Table 11.11

Member	P (N)	K (N)	l (metre)	PKl	K^2l	$F = P + KX$
AC	− 20	$-\frac{1}{\sqrt{2}}$	2	$+20\sqrt{2}$	1	− 40 N
CE	$-20\sqrt{2}$	0	$2\sqrt{2}$	0	0	$-20\sqrt{2}$
ED	+ 20	0	2	1	0	+ 20
DB	+ 60	$-\frac{1}{\sqrt{2}}$	2	$-60\sqrt{2}$	1	+ 40
DC	+ 20	$-\frac{1}{\sqrt{2}}$	2	$-20\sqrt{2}$	1	0
AD	$-40\sqrt{2}$	+ 1	$2\sqrt{2}$	− 160	$2\sqrt{2}$	$-20\sqrt{2}$
BC	0	+ 1	$2\sqrt{2}$	0	$2\sqrt{2}$	$+20\sqrt{2}$
			$\sum =$	$-160 - 60\sqrt{2}$	$3 + 4\sqrt{2}$	

SAQ 5

(a) Let the horizontal reaction at $A = H \leftarrow$ be taken as redundant.

Thus, we get, $H_D = (2 \times 4 - H) = (8 - H) \leftarrow$

and $\sum M_A = 0$, it gives $V_a = 4\text{ N}\uparrow$ and $V_a = 4\text{ N}\downarrow$

According to minimum energy principle, $\frac{\partial U}{\partial H} = 0$

Thus, we get,

$$\sum \int \frac{M \frac{\partial M}{\partial H} ds}{2EI} = 0$$

For the frame given in this problem, value of M in members AB, DC and BC is given in Table 11.12.

Table 11.12

Member	M	$\frac{\partial M}{\partial H}$	Limits of Integration
AB	$Hy - y^2$	y	0 to 4
DC	$-(8 - H)y = (H - 8)y$	y	0 to 4
BC	$4H - 4x - 16$	y	0 to 4

For this frame,

$$\int_0^4 (Hy - y^2)\, y\, dy + \int_0^4 (H - 8)\, y^2\, dy + 4\int_0^4 (4H - 4x - 16)\, dx = 0$$

It gives, $H = 5.8$ N

Thus, $H_A = 5.8 \text{ N} \leftarrow$ and $H_D = 2.2 \text{ N} \leftarrow$ and

Member AB

$$M = Hy - y^2$$

Thus, $M_A = 0$ and $M_B = 7.20$ N m

For maximisation of M, $\frac{dM}{dy} = 0$

It gives, $y = 2.9$ m and $(M)_{29} = +8.41$ N m

Member DC

$$M = Hy - 8y$$

Thus, $M_D = 0$ and $M_C = -8.8$ N m

Member BC

$$M = 4H - 4x - 16$$

$M_B = +7.2$ N m and $M_C = -8.8$ N m and

If the point of contraflexure is at x from B, then

$(M)_x = 4H - 4x - 16 = 0$ which gives $x = 1.8$ m

Now, after calculating the bending moment ordinates, we can draw the bending moment diagram (BMD) as given in Figure 11.33.

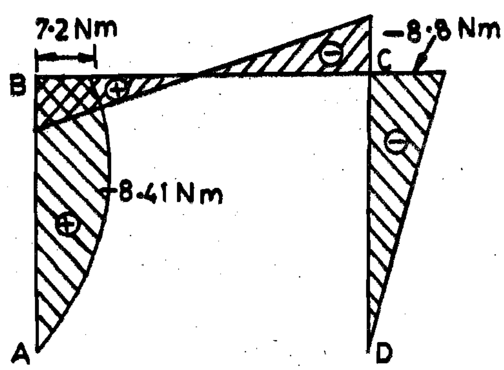


Figure 11.33 : Bending Moment Diagram

(b) Firstly, we consider the one-half portion of the ring ABC as equilibrium. Now we draw the free body diagram of ABC.

Moment at any section x making an angle θ with the horizontal is as follows :

$$M = \frac{WR}{2}(1 - \cos\theta) - M_0$$

Strain energy stored by the semicircular ring ABC

$$U = \int_0^{\frac{\pi}{2}} 2 \times \frac{M^2 ds}{2EI} = \int_0^{\frac{\pi}{2}} 2 \times \left(\frac{WR}{2}(1-\cos\theta) - M_0\right)^2 \frac{R\,d\theta}{2EI}$$

We assume M_0 as redundant and by the principle of minimum energy, $\frac{\partial U}{\partial M_0} = 0$

Thus, $$\int_0^{\pi/2} 2\left[\frac{WR}{2}(1-\cos\theta) - M_0\right](-1)\frac{R\,d\theta}{EI} = 0$$

or, $$\frac{WR}{2}\left(\frac{\pi}{2} - 1\right) - M_0\left(\frac{\pi}{2}\right) = 0$$

or, $$M_0 = \frac{WR}{2\pi}(\pi - 2)$$

$\therefore$ **BM at any section,** $M = \frac{WR}{2}(1 - \cos\theta) - \frac{WR}{2Pi}(\pi - 2)$

$\therefore$ BM at A, i.e. at $\theta = 0$,

$$M_A = (-)\frac{WR}{2\pi}(\pi - 2) \quad [\text{Max}^m - \text{ve}]$$

BM at B, i.e. at $\theta = \frac{\pi}{2}$,

$$M_B = (+)\frac{WR}{\pi} \quad [\text{Max}^m + \text{ve}]$$

Now at the point of contraflexure, $M = 0$

Thus, $$(1 - \cos\theta) - \left(1 - \frac{2}{\pi}\right) = 0$$

or, $$\cos\theta = \frac{2}{\pi}, \text{ i.e. } \theta = \cos^{-1}\left(\frac{2}{\pi}\right) \text{ i.e.,}$$

Now, after calculating the bending moment ordinates, we can draw the bending moment diagram (BMD) as given in Figure 11.34.

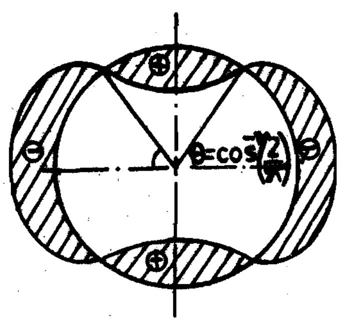


Figure 11.34 : Bending Moment Diagram

Here, contraction in the length of vertical diameter = 2 × vertical deflection of B.

Again, the strain energy stored by the semi-circular part ABC of the ring

$$U = \int_0^{\pi/2} 2 \times \frac{W^2R^2}{4\pi^2}[4 + \pi^2\cos^2\theta - 4\pi\cos\theta]\frac{R\,d\theta}{2EI} = \frac{W^2R^3}{16\pi EI}(\pi^2 - 8)$$

According to Castigliano's Second Theorem, $\dfrac{\partial U}{\partial W} = \delta$

$\therefore$ Vertical deflection of $B = \dfrac{\partial U}{\partial W} = \dfrac{2WR^3}{16\pi EI}(\pi^2 - 8) = \dfrac{WR^3}{8\pi EI}(\pi^2 - 8)$

$\therefore$ Contraction in the length of vertical diameter $= \dfrac{WR^3(\pi^2 - 8)}{4\pi EI}$

Now, we impose two equal and opposite horizontal forces H at A and C. So free body diagram of the semicircular ring is as given in Figure 11 35

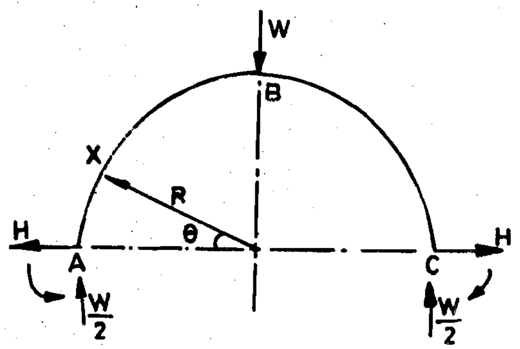


Figure 11.35 : Bending Moment Diagram

Bending moment at any section X making an angle θ with the horizontal

$$M = \frac{WR}{2}(1 - \cos\theta) - M_0 + HR\sin\theta$$

or

$$M = \frac{WR}{2}(1 - \cos\theta) - \frac{WR}{2\pi}(\pi - 2) + HR\sin\theta$$

$\therefore$ Strain energy stored by the semicircular ring ABC

$$U = 2\int_0^{\pi/2} \frac{M^2 R\, d\theta}{2EI} = \frac{R}{EI}\int_0^{\pi/2} M^2 d\theta$$

According to Castigliano's Second Theorem,

$$\text{Displacement of C relative to } A = \frac{\partial U}{\partial H} = \frac{R}{EI}\int_0^{\pi/2} 2M\left(\frac{\partial M}{\partial H}\right) d\theta$$

$$\frac{\partial U}{\partial H} = \frac{2R}{EI}\int_0^{\pi/2}\left[\frac{WR}{2}(1 - \cos\theta)\,\frac{WR}{2\pi}(\pi - 2) + HR\sin\theta\right] R\sin\theta\, d\theta$$

Since, $\dfrac{\partial M}{\partial H} = R\sin\theta]$

$$\frac{\partial U}{\partial H} = \frac{2R^2}{EI}\left[\frac{WR}{2}\times\frac{1}{2} - \frac{WR}{2\pi}(\pi - 2)1 + HR\frac{\pi}{4}\right]$$

Putting the actual value of $H = 0$, we get

$$\frac{\partial U}{\partial H} = \frac{2R^2}{EI}\left[\frac{WR}{4} - \frac{WR(\pi - 2)}{2\pi}\right]$$

$$\frac{\partial U}{\partial H} = \frac{WR^3}{2\pi EI}(4 - \pi)$$

Since, the expansion in the length of horizontal diameter is equal to the horizontal displacement of C relative to A.

$\therefore$ Required expansion $= \dfrac{WR^3}{2\pi EI}(4 - \pi)$

SAQ 6

Let the horizontal thrust at A and B each = H be taken as redundant.

Here, $\sum M_A = 0$

It gives $V_b = 2.5\text{ t}\uparrow$ and $V_a = 7.5\text{ t}\uparrow$

We know that for framed structures total strain energy is as follows :

$$U = \sum \int \frac{M^2\, ds}{2EI}$$

According to minimum energy principle, $\frac{\partial U}{\partial H} = 0$

Thus, we get,

$$\sum \int \frac{M \frac{\partial M}{\partial H} ds}{EI} = 0$$

For the frame given in this problem, value of M in members AB, DC and BC is given in Table 11.13.

Table 11.13

Member	M	$\frac{\partial M}{\partial H}$	Limits of Integration
AD	$7.5 \times \frac{4}{5} x - H \times \frac{3}{5} x = 6x - 0.6Hx$	$-0.6x$	0 to 5
DC	$7.5(x+5)\frac{4}{5} - H(x+5)\frac{3}{5} - 10 \times \frac{4}{5} x$ $= 30 - 2x - 0.6H(x+5)$	$-0.6(x+5)$	0 to 5
BC	$2.5 \times \frac{4}{5} x - H \times \frac{3}{5} x = 2x - 0.6Hx$	$-0.6x$	0 to 10

Here the members are of uniform flexural rigidity, i.e. EI = constant.

Thus, we get,

$$\sum \int M \frac{\partial M}{\partial H} ds = 0$$

Thus, we get on putting the values

$$\int_0^5 (-0.6x)(6x - 0.6Hx)\, dx + \int_0^5 (-0.6)(x+5)[30 - 2x - 0.6H(x+5)]\, dx +$$

$$\int_0^{10} (-0.6x)(2x - 0.6Hx)\, dx = 0$$

It gives, $H = 4.583$ t

Now, we can calculate the bending moment ordinates as follows :

BM at A $= M_A = 0$

BM at B $= M_B = 0$

BM at D $= M_D = 7.5 \times 4 - \frac{55}{12} \times 3 = +16.25$ t m

BM at C $= M_C = = 7.5 \times 8 - 10 \times 4 - \frac{55}{12} \times 6 = -7.5$ t m

Now, after calculating the bending moment ordinates, we can draw the bending moment diagram (BMD) as given in Figure 11.36.

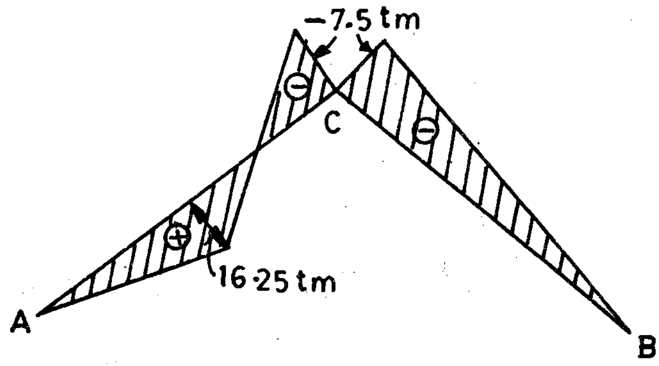


Figure 11.36 : Bending Moment Diagram

SAQ 7

From Table 11.9 of two sets of forces and corresponding displacements, we get

$\sum P\Delta' = 5\,(0.003) + 3\,(0.001) + 1\,(0.002) - 4\,(-\,0.001) + 2\Delta_9' = 0.024 + 2\Delta_9'$

$\sum P'\Delta = 10\,(0.001) + 4\,(0.002) + 5\,(0.002) + 2\,(0.001) + 1\,(0.002) = 0.032$

According to Betti's law,

$$\sum P\Delta' = \sum P'\Delta$$

Thus, $\quad 0.024 + 2\Delta'_9 = 0.032$

We get, $\quad \Delta'_9 = 0.004$ radian