NUMERICAL ANALYSIS (NA)

INTRODUCTION:- The study of NA is aimed at providing convinient methods for obtaining useful solutions of problems of advanced learning in Science and technology.

when Exact/Analytical methods fails to give solution of problems then the Numerical approach is useful in those situations.

Numerical Methods are of repetative nature and in each step, better approximation of the exact solution is obtained.

FINITE DIFFERENCES & INTERPOLATION

FINITE DIFFERENCE :- The calculus of Finite differences deals with the changes that take place in the value of dependent variable due to finite changes in the dependent variable.

Finite difference

$$x: \quad x_0 \quad x_1 \quad x_2 \quad \cdots\cdots \quad x_{n-1} \quad x_n$$

$$y=f(x): \quad f(x_0) \quad f(x_1) \quad f(x_2) \quad \cdots\cdots \quad f(x_{n-1}) \quad f(x_n)$$

OPERATORS :-

1. Identity operator 'I'

$$I[f(x)] = f(x)$$

2. Forward shifting operator 'E'

$$E[f(x)] = f(x+h) \;;\; h \in R$$

where 'h' is the increment in x.

$$E^2[f(x)] = E[E f(x)] = E[f(x+h)] = f(x+2h)$$

Or In General-

$$\boxed{E^n f(x) = f(x+nh)} \quad \& \quad \boxed{E^0 = 1}$$

3. Backward shifting operator 'E^{-1}'

$E^{-1} f(x) = f(x-h)$; $h \in R$

where 'h' is decrement in x.

$E^{-2} f(x) = E^{-1}[E^{-1} f(x)] = E^{-1}[f(x-h)] = f(x-2h)$

Or In General -

$\boxed{E^{-n} = f(x-nh)}$; $n = 1, 2, 3, \dots$

* $EE^{-1} = E^{0} = 1$

4. Forward Difference Operator (Δ - Del)

$\boxed{\Delta f(x) = f(x+h) - f(x)}$ ——— (1)

$\Delta^2 f(x) = \Delta[\Delta f(x)] = \Delta[f(x+h) - f(x)]$

$\boxed{\Delta^2 f(x) = \Delta f(x+h) - \Delta f(x)} = f(x+2h) - f(x+h) - f(x+h) + f(x)$

$= f(x+2h) - 2f(x+h) + f(x)$

Or in General -

$\boxed{\Delta^n f(x) = \Delta^{n-1} f(x+h) - \Delta^{n-1} f(x)}$; $\Delta^0 = 1$

$n = 1, 2, 3, \dots$ ——— (2)

* $\because \Delta f(x) = f(x+h) - f(x) = E f(x) - f(x)$

$\Rightarrow \Delta f(x) = (E-1) f(x)$

$\Rightarrow \boxed{\Delta \equiv E-1}$ or $\boxed{E \equiv 1+\Delta}$

5. Backward Difference Operator (∇ - Nebula)

$\boxed{\nabla f(x) = f(x) - f(x-h)}$ ——— (1)

$\nabla^2 f(x) = \nabla[\nabla f(x)] = \nabla[f(x) - f(x-h)]$

$\boxed{\nabla^2 f(x) = \nabla f(x) - \nabla f(x-h)}$ ——— (2)

$= f(x) - f(x-h) - f(x-h) - f(x-2h)$

$= f(x) - 2f(x-h) - f(x-2h)$

Or in General -

$\boxed{\nabla^n f(x) = \nabla^{n-1} f(x) - \nabla^{n-1} f(x-h)}$; $\nabla^0 = 1$

$n = 1, 2, 3, \dots$

$*\quad \because \nabla f(x) = f(x) - f(x-h) = f(x) - E^{-1} f(x)$
$$= (1 - E^{-1}) f(x)$$
$$\Rightarrow \boxed{\nabla \equiv 1 - E^{-1}}$$

6. Operator e^{hD} $\quad \left(D \equiv \frac{d}{dx} \right)$

$$\because e^{hD} f(x) = \left[1 + hD + \frac{h^2}{2!} D^2 + \frac{h^3}{3!} D^3 + \cdots \right] f(x)$$
$$= f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$$
$$= f(x+h) \quad \text{[By Taylor Series Expansion]}$$
$$= E f(x)$$
$$\Rightarrow \boxed{e^{hD} \equiv E}$$

Some Relations between operators

(a) $\nabla E \equiv \Delta \equiv E \nabla$

Proof:- Let $f(x)$ is a f^n of x, then-
$$\nabla E f(x) = (1 - E^{-1}) E f(x)$$
$$= (E - E^{-1} E) f(x)$$
$$= (E - 1) f(x) = \Delta f(x)$$
$$\Rightarrow \nabla E \equiv \Delta$$

Similarly. $\quad E \nabla f(x) = E (1 - E^{-1}) f(x)$
$$= (E - 1) f(x) = \Delta f(x)$$
$$\Rightarrow E \nabla \equiv \Delta$$
$$\therefore \boxed{\nabla E = \Delta = E \nabla}$$

(b) $E \Delta = \Delta E$

Proof:- $\because E \Delta f(x) = E (E-1) f(x) = (E^2 - E) f(x)$
$$= (E - 1) E f(x)$$
$$= \Delta E f(x)$$
$$\Rightarrow \boxed{E \Delta \equiv \Delta E}$$

(c) $\nabla = 1 - E^{-1} = \Delta E^{-1}$

Proof:- $\because \nabla f(x) = (1 - E^{-1}) f(x) = (E - 1) E^{-1} f(x)$
$$= \Delta E^{-1} f(x)$$
$$\Rightarrow \boxed{\nabla \equiv 1 - E^{-1} \equiv \Delta E^{-1}}$$

(d) $\Delta - \nabla = \Delta\nabla$

Proof:- $\because (\Delta\nabla)f(x) = [(E-1)(1-E^{-1})]f(x)$

$= [E-1-1+E^{-1}]f(x)$

$= [(E-1)-(1-E^{-1})]f(x) = (\Delta-\nabla)f(x)$

$\Rightarrow \boxed{\Delta\nabla = \Delta - \nabla}$

(e) $(1+\Delta)(1-\nabla) = 1$

Proof:- $\because [(1+\Delta)(1-\nabla)]f(x) = [1-\nabla+\Delta-\Delta\nabla]f(x)$

$= [1+(\Delta-\nabla)-(\Delta-\nabla)]f(x)$ $\quad [\because \Delta\nabla = \Delta-\nabla]$

$= f(x)$

$\Rightarrow \boxed{(1+\Delta)(1-\nabla) = 1}$

(f) $\Delta+\nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

Proof:- $\because \left(\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}\right)f(x) = \left(\frac{\Delta^2-\nabla^2}{\Delta\nabla}\right)f(x)$

$= \left(\frac{(\Delta+\nabla)(\Delta-\nabla)}{(\Delta-\nabla)}\right)f(x) = (\Delta+\nabla)f(x)$

$\Rightarrow \boxed{\Delta+\nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}}$

Some Examples On Operators

Ex. 1 Show that-

$\Delta[f(x)g(x)] = f(x+h)\Delta g(x) + g(x)\Delta f(x)$

Proof:- $\because \Delta[f(x)g(x)] = f(x+h)g(x+h) - f(x)g(x)$

$= f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)$

$= f(x+h)[g(x+h)-g(x)] + g(x)[f(x+h)-f(x)]$

$= f(x+h)\Delta g(x) + g(x)\Delta f(x)$

Proved.

Ex.2 Show that-

$$\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\,\Delta f(x) - f(x)\,\Delta g(x)}{g(x+h)\,g(x)}$$

Proof:- $\because \Delta\left[\frac{f(x)}{g(x)}\right] = \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}$

$$= \frac{g(x)f(x+h) - f(x)g(x+h)}{g(x+h)g(x)}$$

$$= \frac{g(x)f(x+h) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x+h)\,g(x)}$$

$$= \frac{g(x)[f(x+h) - f(x)] - f(x)[g(x+h) - g(x)]}{g(x+h)\,g(x)}$$

$$= \frac{g(x)\,\Delta f(x) - f(x)\,\Delta g(x)}{g(x+h)\,g(x)}$$

Proved

Ex.3 Show that-

$$\Delta^r y_k = \nabla^r y_{k+r} \quad ; \; h=1$$

Proof:- $\because \nabla^r y_{k+r} = (1-E^{-1})^r y_{k+r}$

$$= [1 - {}^rc_1 E^{-1} + {}^rc_2 E^{-2} - {}^rc_3 E^{-3} + \cdots + (-1)^r E^{-r}]\, y_{k+r}$$

$$= y_{k+r} - {}^rc_1 E^{-1} y_{k+r} + {}^rc_2 E^{-2} y_{k+r} + {}^rc_3 E^{-3} y_{k+r} + \cdots$$

$$\cdots + (-1)^r E^{-r} y_{k+r}$$

$$= y_{k+r} - {}^rc_1\, y_{k+r-h} + {}^rc_2\, y_{k+r-2h} + \cdots + (-1)^r y_{k+r-rh}$$

put $\underline{h=1}$

$$\nabla^r y_{k+r} = y_{k+r} - {}^rc_1\, y_{k+r-1} + {}^rc_2\, y_{k+r-2} + \cdots + (-1)^r y_{k+r-r}$$

$$= E^r y_k - {}^rc_1 E^{r-1} y_k + {}^rc_2 E^{r-2} y_k - \cdots + (-1)^r y_k$$

$$= (E-1)^r y_k = \Delta^r y_k$$

$$\therefore \nabla^r y_{k+r} = \Delta^r y_k$$

Proved.

Ex.4 Show that-

$$\Delta\sqrt{u_x} = \frac{\Delta u_x}{\sqrt{u_x} + \sqrt{u_{x+h}}}$$

Proof:- $\because$ $$\Delta\sqrt{u_x} = \left(\sqrt{u_{x+h}} - \sqrt{u_x}\right) \times \frac{\sqrt{u_{x+h}} + \sqrt{u_x}}{\sqrt{u_{x+h}} + \sqrt{u_x}}$$

$$= \frac{u_{x+h} - u_x}{\sqrt{u_{x+h}} + \sqrt{u_x}} = \frac{\Delta u_x}{\sqrt{u_{x+h}} + \sqrt{u_x}}$$

Proved

Ex.5 Show that-

$$\left(\frac{\Delta^2}{E}\right)e^x \cdot \frac{E(e^x)}{\Delta^2 e^x} = e^x$$

Proof:- $\because$ $$\left(\frac{\Delta^2}{E}\right)e^x \cdot \frac{E(e^x)}{\Delta^2 e^x} = \left[\frac{(E-1)^2}{E}\right]e^x \cdot \frac{E(e^x)}{(E-1)^2 e^x}$$

$$= \left(\frac{E^2-2E+1}{E}\right)e^x \cdot \frac{e^{x+h}}{(E^2-2E+1)e^x}$$

$$= (E-2+E^{-1})\,e^x \cdot \frac{e^{x+h}}{e^{x+2h} - 2e^{x+h} + e^x}$$

$$= \frac{(e^{x+h} - 2e^x + e^{x-h})\,e^{x+h}}{e^{x+2h} - 2e^{x+h} + e^x}$$

$$\cancel{= e^{x}}$$

$$= \frac{(e^{x+2h} - 2e^{x+h} + e^x)\,e^x}{(e^{x+2h} - 2e^{x+h} + e^x)}$$

$$= e^x$$ Proved.

Difference Table:-

Let $y = f(x)$ is a function of 'x' and for the following values of x -

$x = x_0,\ x_0+h,\ x_0+2h,\ x_0+3h,\ x_0+4h,\ x_0+5h$

we have the data -

x:	x_0	x_0+h	x_0+2h	x_0+3h	x_0+4h	x_0+5h
$y=f(x)$:	$f(x_0)$	$f(x_0+h)$	$f(x_0+2h)$	$f(x_0+3h)$	$f(x_0+4h)$	$f(x_0+5h)$

Then we can construct a difference table with the help of operators Δ & ∇.

FORWARD DIFFERENCE TABLE

x	$f(x)$	$\Delta f(x_0)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
x_0	$f(x_0)$					
		$f(x_0+h) - f(x_0) = \boxed{\Delta f(x_0)}$				
			$\Delta f(x_0+h) - \Delta f(x_0) = \boxed{\Delta^2 f(x_0)}$			
x_0+h	$f(x_0+h)$					
				$\Delta^2 f(x_0+h) - \Delta^2 f(x_0) = \boxed{\Delta^3 f(x_0)}$		
		$f(x_0+2h) - f(x_0+h) = \Delta f(x_0+h)$				
			$\Delta f(x_0+2h) - \Delta f(x_0+h) = \Delta^2 f(x_0+h)$		$\Delta^3 f(x_0+h) - \Delta^3 f(x_0) = \boxed{\Delta^4 f(x_0)}$	
x_0+2h	$f(x_0+2h)$			$\Delta^2 f(x_0+2h) - \Delta^2 f(x_0+h) = \Delta^3 f(x_0+h)$		$\Delta^4 f(x_0+h) - \Delta^4 f(x_0) = \boxed{\Delta^5 f(x_0)}$
		$f(x_0+3h) - f(x_0+2h) = \Delta f(x_0+2h)$				
			$\Delta f(x_0+3h) - \Delta f(x_0+2h) = \Delta^2 f(x_0+2h)$		$\Delta^3 f(x_0+2h) - \Delta^3 f(x_0+h) = \Delta^4 f(x_0+h)$	
x_0+3h	$f(x_0+3h)$			$\Delta^2 f(x_0+3h) - \Delta^2 f(x_0+2h) = \Delta^3 f(x_0+2h)$		
		$f(x_0+4h) - f(x_0+3h) = \Delta f(x_0+3h)$				
			$\Delta f(x_0+4h) - \Delta f(x_0+3h) = \Delta^2 f(x_0+3h)$			
x_0+4h	$f(x_0+4h)$					
		$f(x_0+5h) - f(x_0+4h) = \Delta f(x_0+4h)$				
x_0+5h	$f(x_0+5h)$					

$$\because \Delta f(x) = f(x+h) - f(x)$$

BACKWARD DIFFERENCE TABLE

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$	$\nabla^5 f(x)$
x_0	$f(x_0)$					
		$f(x_0+h)-f(x_0) = \nabla f(x_0+h)$				
x_0+h	$f(x_0+h)$		$\nabla f(x_0+2h)-\nabla f(x_0+h) = \nabla^2 f(x_0+2h)$			
		$f(x_0+2h)-f(x_0+h) = \nabla f(x_0+2h)$		$\nabla^2 f(x_0+3h)-\nabla^2 f(x_0+2h) = \nabla^3 f(x_0+3h)$		
x_0+2h	$f(x_0+2h)$		$\nabla f(x_0+3h)-\nabla f(x_0+2h) = \nabla^2 f(x_0+3h)$		$\nabla^3 f(x_0+4h)-\nabla^3 f(x_0+3h) = \nabla^4 f(x_0+4h)$	
		$f(x_0+3h)-f(x_0+2h) = \nabla f(x_0+3h)$		$\nabla^2 f(x_0+4h)-\nabla^2 f(x_0+3h) = \nabla^3 f(x_0+4h)$		$\nabla^4 f(x_0+5h)-\nabla^4 f(x_0+4h) = \underline{\nabla^5 f(x_0+5h)}$
x_0+3h	$f(x_0+3h)$		$\nabla f(x_0+4h)-\nabla f(x_0+3h) = \nabla^2 f(x_0+4h)$		$\nabla^3 f(x_0+5h)-\nabla^3 f(x_0+4h) = \underline{\nabla^4 f(x_0+5h)}$	
		$f(x_0+4h)-f(x_0+3h) = \nabla f(x_0+4h)$		$\nabla^2 f(x_0+5h)-\nabla^2 f(x_0+4h) = \underline{\nabla^3 f(x_0+5h)}$		
x_0+4h	$f(x_0+4h)$		$\nabla f(x_0+5h)-\nabla f(x_0+4h) = \underline{\nabla f(x_0+5h)}$			
		$f(x_0+5h)-f(x_0+4h) = \underline{\nabla f(x_0+5h)}$				
x_0+5h	$\underline{f(x_0+5h)}$					

$$\because \quad \nabla f(x) = f(x) - f(x-h)$$

Ex: Construct Forward and Backward difference table for the following data -

x:	10	20	30	40	50	60	70
$f(x)$:	1	8	27	64	128	218	393

Sol:-

FORWARD DIFF: TABLE

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$	$\Delta^6 f(x)$
10	1						
		7					
20	8		12				
		19		6			
30	27		18		3		
		37		9		−13	
40	64		27		−10		83
		64		−1		70	
50	128		26		60		
		90		59			
60	218		85				
		175					
70	393						

BACKWARD DIFFERENCE TABLE

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$	$\nabla^5 f(x)$	$\nabla^6 f(x)$
10	1						
		7					
20	8		12				
		19		6			
30	27		18		3		
		37		9		−13	
40	64		27		−10		83
		64		−1		70	
50	128		26		60		
		90		59			
60	218		85				
		175					
70	393						

Fundamental Theorem of Finite Differences:-

Let $y = f(x)$ be a Polynomial of degree 'n' and the values of the independent variable 'x' are equally spaced, then n^{th} forward difference of $f(x)$ is constant [i.e. $\Delta^n f(x) = $ Constt.] and $(n+1)^{th}$ & higher forward differences of $f(x)$ are zero. [i.e. $\Delta^{n+1} f(x) = 0$]

The converse of this theorem is also true. i.e. The n^{th} forward difference of a tabulated f^n $f(x)$ is constant and higher order Forward differences (greater than n) are zero, when values of independent variable 'x' are equally spaced, then $y = f(x)$ is a Polynomial of degree 'n'.

Ex.1 Find the missing Value in the given data.

x: 2   4   6   8   10

$y(x)$: 5.6   8.6   13.9   –   35.6

(i) By using Difference Table

(ii) without using Difference Table.

Sol:- (i) Let the missing value is y_8, then the difference table is -

x	$y(x)$	$\Delta y(x)$	$\Delta^2 y(x)$	$\Delta^3 y(x)$	$\Delta^4 y(x)$
2	5.6				
		3			
			2.3		
4	8.6			$y_8 - 21.5$	
		5.3			$90.2 - 4y_8$
			$y_8 - 19.2$		
6	13.9			$68.7 - 3y_8$	
		$y_8 - 13.9$			
			$49.5 - 2y_8$		
8	y_8				
		$35.6 - y_8$			
10	35.6				

Let $y(x) = f(x)$ be a polynomial of some degree, for which the data is given. As we know that a polynomial of least degree with four terms is a third degree polynomial.
[In General we can say if $(n+1)$ terms of the polynomial $y = f(x)$ are given then $y = f(x)$ may be considered as a polynomial of degree 'n']

Here Four values of $y(x)$ are given, so $y(x)$ may be taken as a polynomial of degree 3. So By Fundamental Th. of Finite differences $\Delta^3 y(x)$ must be a constant and $\Delta^4 y(x) = 0$.

So from the difference table-

$$\Delta^4 y(x) = 90.2 - 4y_8 = 0 \quad \Rightarrow y_8 = 22.55 \quad \underline{\text{Ans.}}$$

(ii) $\because \Delta^4 y(x) = 0$

$$\Rightarrow (E-1)^4 y(x) = 0$$

$$\Rightarrow (E^4 - 4E^3 + 6E^2 - 4E + 1) y(x) = 0$$

$$\Rightarrow y(x+4h) - 4y(x+3h) + 6y(x+2h) - 4y(x+h) + y(x) = 0$$

As here $\underline{h=2}$, so

$$y(x+8) - 4y(x+6) + 6y(x+4) - 4y(x+2) + y(x) = 0$$

put $\underline{x=2}$, then

$$y(10) - 4y(8) + 6y(6) - 4y(4) + y(2) = 0$$

$$\Rightarrow 90.2 - 4y(8) = 0 \quad \Rightarrow y(8) = 22.55$$

Ans.

Ex.2 Find the missing terms of the given data-

x:	1	2	3	4	5	6	7	8
y:	1	8	–	64	–	216	343	512

Sol:- Let the missing terms are $y(3)$ and $y(5)$.
By using the above mentioned Concept, here 6 values are given so-

$$\Delta^6 y(x) = 0 \quad \text{———} (1)$$

$\Rightarrow (E-1)^6 y(x) = 0$

$\Rightarrow (E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1)\, y(x) = 0$

$\Rightarrow y(x+6h) - 6y(x+5h) + 15y(x+4h) - 20y(x+3h) - 6y(x+h) + y(x) + 15y(x+2h) = 0$

$\because$ here $h = 1$, so

$y(x+6) - 6y(x+5) + 15y(x+4) - 20y(x+3) - 6y(x+1) + y(x) + 15y(x+2) = 0$ ——— (2)

put $x = 1$, then

$y(7) - 6y(6) + 15y(5) - 20y(4) - 6y(2) + y(1) + 15y(3) = 0$

$\Rightarrow \quad y(3) + y(5) = 152$ ——— (3)

put $\underline{x=2}$ in (2), then.

$y(8) - 6y(7) + 15y(6) - 20y(5) - 6y(3) + y(2) + 15y(4) = 0$

$\Rightarrow \quad 10y(5) + 3y(3) = 1331$ ——— (4)

From (3) & (4)

$y(3) = 27$ and $y(5) = 125$

Factorial Notation :-

The Product of factors, in which the first factor is 'x' (say) and the successive factors decrease by a constant difference, is called as Factorial Notation of x. It is denoted by $x^{(n)}$, where 'n' is the degree of this factorial notation and $n \in I^+$.

[If $h=1$]

$x^{(1)} = x$

$x^{(2)} = x(x-1)$

$x^{(3)} = x(x-1)(x-2)$

$x^{(4)} = x(x-1)(x-2)(x-3)$

$\vdots$

$x^{(n)} = x(x-1)(x-2)(x-3)\cdots\cdots(x-\overline{n-1})$

\# $\Delta x^{(n)} = nh\, x^{(n-1)}$

$\Delta^2 x^{(n)} = n(n-1)h^2 \cdot x^{(n-2)}$

$\Delta^3 x^{(n)} = n(n-1)(n-2)\, h^3 \cdot x^{(n-3)}$

$\vdots$

$\Delta^n x^{(n)} = n(n-1)(n-2)\cdots\cdots 3.2.1\, h^n = h^n \cdot n!$

If $h=1$, then $\Delta^n x^{(n)} = n!$

Methods For Converting a Polynomial in Factorial Notation (when $h=1$)

(a) Direct Method:-

Ex: Express $f(x) = 2x^3 - 3x^2 + 3x - 10$ in Factorial Notation Form.

Sol:- The degree of the Factorial Notation is same as the degree of original Polynomial.

So Let $f(x) = 2x^3 - 3x^2 + 3x - 10 = Ax^{(3)} + Bx^{(2)} + Cx^{(1)} + D$

$$= Ax(x-1)(x-2) + Bx(x-1) + Cx + D \quad \text{--- (1)}$$

where A, B, C & D are Constants.

putting $x=0 \Rightarrow D=-10$

putting $x=1 \Rightarrow -8 = C+D \Rightarrow C = 2$

putting $x=2 \Rightarrow 0 = 2B+2C+D \Rightarrow B=3$

Comparing co-efficient of x^3 -

$\Rightarrow A = 2$

$\therefore f(x) = 2x^3 - 3x^2 + 3x - 10 = 2x^{(3)} + 3x^{(2)} + 2x^{(1)} - 10$

(b) Synthetic Division:-

Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$ be a Polynomial of degree 'n', Let the corresponding Factorial Notation of $f(x)$, $h=1$, is

$A_0x^{(n)} + A_1x^{(n-1)} + A_2x^{(n-2)} + \cdots + A_{n-1}x^{(1)} + A_n$.

The Value of $A_0, A_1, A_2, \cdots A_n$ Can be found by Synthetic division as-

1	a_0	a_1	a_2	$\cdots$	a_{n-2}	a_{n-1}	$\boxed{a_n} \to A_n$
	0	$1\cdot b_0$	$1\cdot b_1$		$1\cdot b_{n-3}$	$1\cdot b_{n-2}$	
2	$a_0=b_0$	a_1+b_0 $=b_1$	a_2+b_1 $=b_2$	$\cdots$	$a_{n-2}+b_{n-3}$ $=b_{n-2}$	$a_{n-1}+b_{n-2}$ $=\boxed{b_{n-1}} \to A_{n-1}$	
	0	$2\cdot b_0$	$2\cdot c_1$		$2c_{n-3}$		
3	b_0 $=c_0$	b_1+2c_0 $=c_1$	b_2+2c_1 $=c_2$	$\cdots$	$b_{n-2}+2c_{n-2}$ $=\boxed{c_{n-2}} \to A_{n-2}$		
$\vdots$							
$(n-2)$	n_0	n_1	n_2	$\boxed{n_3} \to A_3$			
	0	$(n-2)p_0$	$(n-2)p_1$				
$(n-1)$	n_0 $=p_0$	$n_1+(n-2)p_0$ $=p_1$	$n_2+(n-2)p_1$ $=\boxed{p_2} \to A_2$				
	0	$(n-1)q_0$					
n	p_0 $=q_0$	$p_1+(n-1)q_0$ $=\boxed{q_1} \to A_1$					
	0						
	q_0 $=\boxed{q_0} \to A_0$						

Explanation:-

a) write all the co-efficients of all powers of x, in the descending order of powers of x, issuing the co-efficient zero to the missing powers of x.

b) The last term (constant Only) is the Value of A_n.

c) Divide this row synthetically by 1 and find Third row by adding two rows.

d) The last term is the Value of A_{n-1}.

e) Divide this third row synthetically by 2 and find 5^{th} Row.

f) The last term is the Value of A_{n-2}.

g) continue the process untill we find all A_i; $i = 0, 1, 2, 3, \ldots n$

Ex. 1 Express $f(x) = 2x^3 - 3x^2 + 3x - 10$ in Factorial Notation.

Sol:- Let $f(x) = 2x^3 - 3x^2 + 3x - 10 = A_0 x^{(3)} + A_1 x^{(2)} + A_2 x^{(1)} + A_3$ —— (1)

where A_0, A_1, A_2 & A_3 can be obtained as-

$$
\begin{array}{c|cccc}
1 & 2 & -3 & 3 & \boxed{-10} \to A_3 \\
 & 0 & 2 & -1 & \\
\hline
2 & 2 & -1 & \boxed{2} \to A_2 & \\
 & 0 & 4 & & \\
\hline
3 & 2 & \boxed{3} \to A_1 & & \\
 & 0 & & & \\
\hline
 & \boxed{2} \to A_0 & & &
\end{array}
$$

$\therefore f(x) = 2x^3 - 3x^2 + 3x - 10 = 2x^{(3)} + 3x^{(2)} + 2x^{(1)} - 10$

Ex. 2 Express $f(x) = 3x^5 - x^3 + 2x^2 + 5$ in Factorial Notation, and find $\Delta^2 f(x)$.

Sol:- Let $f(x) = 3x^5 + 0 \cdot x^4 - x^3 + 2x^2 + 0 \cdot x + 5$

$= A_0 x^{(5)} + A_1 x^{(4)} + A_2 x^{(3)} + A_3 x^{(2)} + A_4 x^{(1)} + A_5$

where A_0, A_1, A_2, A_3, A_4 & A_5 can be obtained as-

$$
\begin{array}{c|cccccc}
1 & 3 & 0 & -1 & 2 & 0 & \boxed{5} \to A_5 \\
 & 0 & 3 & 3 & 2 & 4 & \\
\hline
2 & 3 & 3 & 2 & 4 & \boxed{4} \to A_4 & \\
 & 0 & 6 & 18 & 40 & & \\
\hline
3 & 3 & 9 & 20 & \boxed{44} \to A_3 & & \\
 & 0 & 9 & 54 & & & \\
\hline
4 & 3 & 18 & \boxed{74} \to A_2 & & & \\
 & 0 & 12 & & & & \\
\hline
5 & 3 & \boxed{30} \to A_1 & & & & \\
 & 0 & & & & & \\
\hline
 & \boxed{3} \to A_0 & & & & &
\end{array}
$$

$\therefore f(x) = 3x^5 - x^3 + 2x^2 + 5 = 3x^{(5)} + 30x^{(4)} + 74x^{(3)} + 44x^{(2)} + 4x^{(1)} + 5$

$\therefore \Delta f(x) = 15x^{(4)} + 120x^{(3)} + 222x^{(2)} + 88x^{(1)} + 4$

$\Delta^2 f(x) = 60x^{(3)} + 360x^{(2)} + 444x^{(1)} + 88$ Ans.

INTER POLATION

The word interpolation means "reading between lines". i.e. something searching between given values.

Let us consider a data for the f^n $y = f(x)$ as-

x: x_0 x_1 x_2 ...... x_{n-1} x_n

$y=f(x)$: $f(x_0)$ $f(x_1)$ $f(x_2)$ ...... $f(x_{n-1})$ $f(x_n)$

then the process of finding the value of $y = f(x)$ corresponding to any value of x, between x_0 and x_n is known as Interpolation.

There are three types of Interpolations-

(a) Interpolation with equal Interval.

(b) Interpolation with unequal Interval

(c) Central Interpolation.

INTERPOLATION $\bar{c}$ EQUAL INTERVAL

In this topic, we will discuss the methods of interpolation, when the values of independent variable are equally spaced.

1. Newton-Gregory Forward Difference Formula

Let $y = f(x)$ represents a function, which assumes the values $f(x_0)$, $f(x_0+h)$, $f(x_0+2h)$, ... $f(x_0+nh)$ at $(n+1)$ equidistant values $x = x_0, x_0+h, x_0+2h, \ldots, x_0+nh$.

Let $P_n(x)$ be a Polynomial of degree 'n' and may written in the form-

$$P_n(x) = A_0 + A_1(x-x_0) + A_2(x-x_0)(x-\overline{x_0+h}) + A_3(x-x_0)(x-\overline{x_0+h})(x-\overline{x_0+2h}) + \cdots + A_n(x-x_0)(x-\overline{x_0+h})\cdots(x-\overline{x_0+(n-1)h}) \quad \text{---}(1)$$

where $A_0, A_1, A_2, \ldots\ldots A_n$ are Constants and chosen such that-

$$P_n(x_0) = f(x_0)$$

$$P_n(x_0+h) = f(x_0+h)$$

$$P_n(x_0+2h) = f(x_0+2h)$$

$$\vdots$$

$$P_n(x_0+nh) = f(x_0+nh)$$

Putting $x = x_0$ in (1), then

$$\Rightarrow A_0 = f(x_0) \quad \text{——— (2)}$$

Putting $x = x_0+h$ in (1), then.

$$f(x_0+h) = A_0 + A_1 h = f(x_0) + A_1 h$$

$$\Rightarrow \boxed{A_1 = \frac{f(x_0+h) - f(x_0)}{h} = \frac{\Delta f(x_0)}{h}} \quad \text{——— (3)}$$

Putting $x = x_0+2h$ in (1) then.

$$f(x_0+2h) = A_0 + 2h A_1 + 2h^2 A_2$$

$$= f(x_0) + 2h \frac{\Delta f(x_0)}{h} + 2h^2 A_2$$

$$\Rightarrow A_2 = \frac{f(x_0+2h) - 2\Delta f(x_0) - f(x_0)}{h^2 . 2!}$$

$$= \frac{[f(x_0+2h) - f(x_0+h)] - [f(x_0+h) - f(x_0)]}{h^2 . 2!}$$

$$\Rightarrow \boxed{A_2 = \frac{\Delta f(x_0+h) - \Delta f(x_0)}{h^2 . 2!} = \frac{\Delta^2 f(x_0)}{h^2 . 2!}} \quad \text{——— (4)}$$

Similarly $A_3 = \dfrac{\Delta^3 f(x_0)}{h^3 . 3!}$ ——— (5) [putting $x = x_0+3h$]

$A_4 = \dfrac{\Delta^4 f(x_0)}{h^4\, 4!}$ ——— (6) [Putting $x = x_0+4h$]

$\vdots$

$A_n = \dfrac{\Delta^n f(x_0)}{h^n . n!}$ ——— (7) [putting $x = x_0+nh$]

Ex.2 For the following data, find $y(0.62)$

x:	0	0.2	0.4	0.6	0.8
$y=f(x)$:	0.3989	0.3910	0.3683	0.3332	0.2897

Sol:- Let us construct the difference table as.

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
0	0.3989				
		−0.0079			
			−0.0148		
0.2	0.3910				
		−0.0227		0.0024	
			−0.0124		0.0016
0.4	0.3683				
		−0.0351		0.0040	
			−0.0084		
0.6	0.3332				
		−0.0435			
0.8	0.2897				

Newton Gregory Backward Difference Formula is-

$$f(x) = f(x_0+nh) + u\nabla f(x_0+nh) + \frac{u(u+1)}{2!}\nabla^2 f(x_0+nh) + \cdots \quad \text{---(1)}$$

$$\text{where } u = \frac{x-(x_0+nh)}{h} = \frac{0.62-0.8}{0.2} = -0.9$$

$$\text{So } f(0.62) = 0.2897 + (-0.9)(-0.0435) + \frac{(-0.9)(-0.9+1)}{2!}(-0.0084)$$

$$+ \frac{(-0.9)(-0.9+1)(-0.9+2)}{3!}(0.0040)$$

$$+ \frac{(-0.9)(-0.9+1)(-0.9+2)(-0.9+3)}{4!}(0.0016)$$

$$\Rightarrow f(0.62) = 0.32914814 \approx 0.3291$$

Ex.3 The Following table gives the Population of a town during Last six censuses. Estimate the increase in the population during 1946 to 1948.

Year	x :	1911	1921	1931	1941	1951	1961
Population $f(x)$: (in 1000)		12	15	20	27	39	52

Sol:- Let us construct the difference table as-

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$	$\nabla^5 f(x)$
1911	12					
		3				
1921	15		2			
		5		0		
1931	20		2		3	
		7		3		$\underline{-10}$
1941	27		5		$\underline{-7}$	
		12		$\underline{-4}$		
1951	39		$\underline{-1}$			
		$\underline{13}$				
1961	$\underline{52}$					

The Newton Gregory Backward formula -

$$f(x) = f(x_0+nh) + u\nabla f(x_0+nh) + \frac{u(u+1)}{2!}\nabla^2 f(x_0+nh) + \cdots \quad \text{—— (1)}$$

$$\text{where } u = \frac{x-(x_0+nh)}{h}. \quad \text{—— (2)}$$

For $\underline{x = 1946}$

$$u = \frac{1946-1961}{10} = -1.5 \quad \text{—— (3)}$$

For $\underline{x = 1948}$

$$u = \frac{1948-1961}{10} = -1.3 \quad \text{—— (4)}$$

$$\therefore f(1946) = 52 + (-1.5)(13) + \frac{(-1.5)(-1.5+1)}{2!}(1) + \frac{(-1.5)(-1.5+1)(-1.5+2)}{3!}(-4)$$

$$+ \frac{(-1.5)(-1.5+1)(-1.5+2)(-1.5+3)}{4!}(-7)$$

$$+ \frac{(-1.5)(-1.5+1)(-1.5+2)(-1.5+3)(-1.5+4)}{5!}(-10)$$

$$\Rightarrow f(1946) = 32.34375 \approx 32 \text{ (in Thousand)}$$

$\therefore\ f(45) = 31 + 0.5(42) + \frac{0.5(0.5-1)}{2!}(9)$

$+ \frac{0.5(0.5-1)(0.5-2)}{3!}(-25)$

$+ \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!}(37)$

$\Rightarrow f(45) = 47.87 \approx 48$

No. of students getting marks between 40 & 45 are.

$f(45) - f(40) = 48 - 31 = 17.$

Ex.4 Find the Polynomial which take the following values.-

x:	0	1	2	3
$f(x)$:	1	0	1	10

Sol:- Let us construct the difference table as-

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			
		-1		
			2	
1	0			6
		1		
			8	
2	1			
		9		
3	10			

The Newton Gregory Forward difference formula-

$$f(x) = f(x_0) + u\,\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \cdots \quad \text{——— (1)}$$

$$\text{where } u = \frac{x - x_0}{h} = \frac{x-0}{1} = x$$

So $f(x) = 1 + x(-1) + \frac{x(x-1)}{2!}(2) + \frac{x(x-1)(x-2)}{3!}(6)$

$\Rightarrow f(x) = x^3 - 2x^2 + 1$ Ans.

INTERPOLATION $\bar{c}$ UNEQUAL INTERVAL

In this case the argument (x) are unequally spaced, so the various differences will be effected with the change in values of argument.

Divided Difference:-

Let $f(x_0), f(x_1), f(x_2), \dots f(x_n)$ are values of the function $y = f(x)$, corresponding to the values of $x = x_0, x_1, x_2, \dots x_n$, where values of x are not equally spaced.

The First Divided difference of $f(x)$ for x_0 & x_1 is defined as-

$$\underset{x_1}{\Delta} f(x_0) = f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Similarly-

$$\underset{x_2}{\Delta} f(x_1) = f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\underset{x_3}{\Delta} f(x_2) = f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

$$\vdots$$

$$\underset{x_n}{\Delta} f(x_{n-1}) = f(x_{n-1}, x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

Second Divided Differences-

$$\underset{x_1, x_2}{\Delta^2} f(x_0) = f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} = \frac{\underset{x_0}{\Delta} f(x_1) - \underset{x_1}{\Delta} f(x_1)}{x_2 - x_0}$$

$$\underset{x_2, x_3}{\Delta^2} f(x_1) = f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1} = \frac{\Delta f(x_2) - \Delta f(x_1)}{x_3 - x_1}$$

$$\vdots$$

$$\underset{n-1, n}{\Delta^2} f(x_{n-2}) = f(x_{n-2}, x_{n-1}, x_n) = \frac{f(x_{n-1}, x_n) - f(x_{n-2}, x_{n-1})}{x_n - x_{n-2}}$$

$$= \frac{\Delta f(x_{n-1}) - \Delta f(x_{n-2})}{x_n - x_{n-2}}$$

Similarly the n^{th} divided difference is-

$$\underset{x_1, x_2, \dots x_n}{\Delta^n} f(x_0) = \frac{f(x_1, x_2, \dots x_n) - f(x_0, x_1, \dots x_{n-1})}{x_n - x_0}$$

DIVIDED DIFFERENCE TABLE

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
x_0	$f(x_0)$				
		$\frac{f(x_1)-f(x_0)}{x_1-x_0} = \Delta f(x_0)$			
x_1	$f(x_1)$		$\frac{\Delta f(x_1)-\Delta f(x_0)}{x_2-x_0} = \Delta^2 f(x_0)$		
		$\frac{f(x_2)-f(x_1)}{x_2-x_1} = \Delta f(x_1)$		$\frac{\Delta^2 f(x_1)-\Delta^2 f(x_0)}{x_3-x_0} = \Delta^3 f(x_0)$	
x_2	$f(x_2)$		$\frac{\Delta f(x_2)-\Delta f(x_1)}{x_3-x_1} = \Delta^2 f(x_1)$		$\frac{\Delta^3 f(x_1)-\Delta^3 f(x_0)}{x_4-x_0} = \Delta^4 f(x_0)$
		$\frac{f(x_3)-f(x_2)}{x_3-x_2} = \Delta f(x_2)$		$\frac{\Delta^2 f(x_2)-\Delta^2 f(x_1)}{x_4-x_1} = \Delta^3 f(x_1)$	
x_3	$f(x_3)$		$\frac{\Delta f(x_3)-\Delta f(x_2)}{x_4-x_2} = \Delta^2 f(x_2)$		$\frac{\Delta^3 f(x_2)-\Delta^3 f(x_1)}{x_5-x_1} = \Delta^4 f(x_0)$
		$\frac{f(x_4)-f(x_3)}{x_4-x_3} = \Delta f(x_3)$		$\frac{\Delta^2 f(x_3)-\Delta^2 f(x_2)}{x_5-x_2} = \Delta^3 f(x_2)$	
x_4	$f(x_4)$		$\frac{\Delta f(x_4)-\Delta f(x_3)}{x_5-x_3} = \Delta^2 f(x_3)$		
		$\frac{f(x_5)-f(x_4)}{x_5-x_4} = \Delta f(x_4)$			
x_5	$f(x_5)$				

where $f(x_0), f(x_1), f(x_2) \dots f(x_5)$ are values of the f^n $y = f(x)$ at $x = x_0, x_1, x_2, \dots x_5$.

Newton's Divided Difference Formula:-

Let $f(x_0), f(x_1), f(x_2), \ldots f(x_n)$ are values of the f^n $y = f(x)$ at $x = x_0, x_1, \ldots x_n$, where $x_1, x_2, \ldots x_n$ are not equally spaced, then Newton's Divided Difference Formula is-

$$\boxed{\begin{aligned} f(x) = {} & f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) \\ & + (x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0) + (x-x_0)(x-x_1)(x-x_2)(x-x_3)\Delta^4 f(x_0) \\ & + \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})\Delta^n f(x_0) \end{aligned}}$$

Ex.1 Find $f(8)$ from the following data.

x: 4 5 7 10 11 13

$f(x)$: 48 100 294 900 1210 2028

Sol:- Let us construct Divided difference table-

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48				
		52			
			15		
5	100			1	
		97			0
			21		
7	294			1	
		202			0
			27		
10	900			1	
		310			
			33		
11	1210				
		409			
13	2028				

The Newton's Divided Formula is -

$$f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) + \cdots \quad (1)$$

$$\therefore f(8) = 48 + (8-4)52 + (8-4)(8-5)15 + (8-4)(8-5)(8-7)1 + (8-4)(8-5)(8-7)(8-10).0$$

$$\Rightarrow f(8) = 448$$

Ex.2 Find the Polynomial for the following data-

x: -1 1 2 3

$f(x)$: -21 15 12 3

Sol:- Let us construct the divided difference table as-

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
-1	-21			
		18		
1	15		-7	
		-3		1
2	12		-3	
		-9		
3	3			

The Newton's Divided difference Formula is-

$$f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) + \cdots$$

So $f(x) = -21 + (x+1)(18) + (x+1)(x-1)(-7) + (x+1)(x-1)(x-2)(1)$

$\Rightarrow f(x) = x^3 - 9x^2 + 17x + 6$

LAGRANGE'S INTERPOLATION FORMULA

Let the given data is as-

x : x_0 x_1 x_2 ...... x_n

$y = f(x)$: $f(x_0)$ $f(x_1)$ $f(x_2)$ ...... $f(x_n)$

then Value of $f(x)$ for any x is-

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots\cdots(x_0-x_n)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)\cdots\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots\cdots(x_1-x_n)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)} f(x_2)$$

$$+ \cdots\cdots + \frac{(x-x_0)(x-x_1)(x-x_2)\cdots\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})} f(x_n)$$

* This Formula can be applied for all values of x, which are equally spaced or not.

Ex. 1 Find f(10), from the following data -

x:	5	6	9	11
$f(x)$:	12	13	14	16

Sol:- Here $x_0 = 5,\ x_1 = 6,\ x_2 = 9,\ x_3 = 11$

& $f(x_0) = 12,\ f(x_1) = 13,\ f(x_2) = 14,\ f(x_3) = 16$

$\therefore$ the Lagrange's Formula is -

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} f(x_1)$$

$$+ \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} f(x_n)$$

$$\therefore f(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} \times 12 + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \times 13$$

$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \times 16$$

$\Rightarrow f(10) = 14.6667$

Ex. 2 Find the Polynomial for the data

x:	0	2	3	6
$f(x)$:	648	704	729	792

Sol:- Here $x_0 = 0,\ x_1 = 2,\ x_2 = 3,\ x_3 = 6$

& $f(x_0) = 648,\ f(x_1) = 704,\ f(x_2) = 729,\ f(x_3) = 792$

So by Lagrange's Formula -

$$f(x) = \frac{(x-2)(x-3)(x-6)}{(0-2)(0-3)(0-6)} \times 648 + \frac{(x-0)(x-3)(x-6)}{(2-0)(2-3)(2-6)} \times 704$$

$$+ \frac{(x-0)(x-2)(x-6)}{(3-0)(3-2)(3-6)} \times 729 + \frac{(x-0)(x-2)(x-3)}{(6-0)(6-2)(6-3)} \times 792$$

$\Rightarrow f(x) = -107x^3 + 1020x^2 - 1584x + 648$

INVERSE INTERPOLATION

LAGRANGE'S FORMULA

$$x = \frac{(f-f_1)(f-f_2)\cdots(f-f_n)}{(f_0-f_1)(f_0-f_2)\cdots(f_0-f_n)}x_0 + \frac{(f-f_0)(f-f_2)\cdots(f-f_n)}{(f_1-f_0)(f_1-f_2)\cdots(f_1-f_n)}x_1$$

$$+ \frac{(f-f_0)(f-f_1)(f-f_3)\cdots(f-f_n)}{(f_2-f_0)(f_2-f_1)(f_2-f_3)\cdots(f_2-f_n)}x_2 + \cdots\cdots$$

$$+ \frac{(f-f_0)(f-f_1)\cdots(f-f_{n-1})}{(f_n-f_0)(f_n-f_1)\cdots(f_n-f_{n-1})}x_n$$

where $f \to f(x)$, $f_0 \to f(x_0)$, $f_1 \leftrightarrow f(x_1), \cdots f_n \to f(x_n)$

<u>Ex.1</u> Find value of x for $f(x) = 7$, for the following data -

x:	1	3	4
$f(x)$:	4	12	19

<u>Sol</u>:- here $x_0 = 1$, $x_1 = 3$, $x_2 = 4$
& $f_0 = 4$, $f_1 = 12$, $f_2 = 19$ & $f = 7$

So By Lagranges formula for inverse interpolation -

$$x = \frac{(7-12)(7-19)}{(4-12)(4-19)} \times 1 + \frac{(7-4)(7-19)}{(12-4)(12-19)} \times 3$$

$$+ \frac{(7-4)(7-12)}{(19-4)(19-12)} \times 4$$

$$\Rightarrow x = 1.85714$$

CENTRAL INTERPOLATION

Newton's method which have been discussed earlier for equal interval are fundamental and are applicable to almost all cases of equal interval interpolation problems, but the rate of convergence of those is very slow, compare to the formulae used in central difference. Central difference formulae are based upon differences obtained from the values of f(x) on either side of origin.

u	x	$y = f(x) = y_x$
⋮	⋮	⋮
-4	x_{-4}	y_{-4}
-3	x_{-3}	y_{-3}
-2	x_{-2}	y_{-2}
-1	x_{-1}	y_{-1}
0	x_0	y_0
1	x_1	y_1
2	x_2	y_2
3	x_3	y_3
4	x_4	y_4
⋮	⋮	⋮

Operators

There are two operators, used in central difference -

(a) $\delta = E^{1/2} - E^{-1/2}$

(b) $\mu = \frac{1}{2}[E^{1/2} + E^{-1/2}]$

Some Results on operators

1. $\mu^2 = 1 + \frac{\delta^2}{4}$

Proof:- Let y_x is a f^n of x, then

$$\mu^2 y_x = \frac{1}{4}(E^{1/2} + E^{-1/2})^2 y_x$$
$$= \frac{1}{4}(E + E^{-1} + 2) y_x$$
$$= \frac{1}{4}[(E + E^{-1} - 2) + 4] y_x$$
$$= \frac{1}{4}[4 + (E^{1/2} - E^{-1/2})^2] y_x$$
$$= (1 + \frac{\delta^2}{4}) y_x$$

$$\Rightarrow \boxed{\mu^2 = 1 + \frac{\delta^2}{4}}$$

2. $\mu + \frac{\delta}{2} = E^{1/2}$

Proof:- $\because$ $(\mu + \frac{\delta}{2}) y_x = \frac{1}{2}[(E^{1/2} + E^{-1/2}) + (E^{1/2} - E^{-1/2})] y_x$

$$= E^{1/2} y_x$$

$$\Rightarrow \boxed{\mu + \frac{\delta}{2} = E^{1/2}}$$

3. $\mu - \frac{\delta}{2} = E^{-1/2}$

Proof:- $\because$ $(\mu - \frac{\delta}{2}) y_x = \frac{1}{2}[(E^{1/2} + E^{-1/2}) - (E^{1/2} - E^{-1/2})] y_x$

$$= E^{-1/2} y_x$$

$$\Rightarrow \boxed{\mu - \frac{\delta}{2} = E^{-1/2}}$$

4. $\mu\delta = \frac{1}{2}(\Delta + \nabla)$

Proof:- $\because (\mu\delta) y_x = \frac{1}{2}[(E^{1/2}+E^{-1/2})(E^{1/2}-E^{-1/2})] y_x$

$$= \frac{1}{2}[E - E^{-1}] y_x$$

$$= \frac{1}{2}[(E-1) + (1-E^{-1})] y_x$$

$$= \frac{1}{2}(\Delta + \nabla) y_x$$

$$\Rightarrow \boxed{\mu\delta = \frac{1}{2}(\Delta+\nabla)}$$

5. $\Delta = \frac{\delta^2}{2} + \delta\sqrt{1+\frac{\delta^2}{4}}$

Proof:- $\because \left(\frac{\delta^2}{2} + \delta\sqrt{1+\frac{\delta^2}{4}}\right) y_x = \left(\frac{(E+E^{-1}-2)}{2} + (E^{1/2}-E^{-1/2})\sqrt{1+\frac{E+E^{-1}-2}{4}}\right) y_x$

$$= \frac{1}{2}\left[E+E^{-1}-2 + (E^{1/2}-E^{-1/2})\sqrt{E+E^{-1}+2}\right] y_x$$

$$= \frac{1}{2}\left[E+E^{-1}-2 + (E^{1/2}-E^{-1/2})\sqrt{(E^{1/2}+E^{-1/2})^2}\right] y_x$$

$$= \frac{1}{2}[E+E^{-1}-2 + (E-E^{-1})] y_x$$

$$= \frac{1}{2}[2E-2] y_x = (E-1) y_x = \Delta y_x$$

$$\Rightarrow \boxed{\frac{\delta^2}{2} + \delta\sqrt{1+\frac{\delta^2}{4}} = \Delta}$$

6. $\boxed{\delta[f(x)g(x)] = \mu[f(x)]\,\delta[g(x)] + \mu[g(x)]\,\delta[f(x)]}$

Proof:- R.H.S.

$$\mu[f(x)]\,\delta[g(x)] + \mu[g(x)]\,\delta[f(x)] = \frac{1}{2}\left[f(x+\tfrac{h}{2}) + f(x-\tfrac{h}{2})\right] \times \left[g(x+\tfrac{h}{2}) - g(x-\tfrac{h}{2})\right]$$

$$+ \frac{1}{2}\left[g(x+\tfrac{h}{2}) + g(x-\tfrac{h}{2})\right]\left[f(x+\tfrac{h}{2}) - f(x-\tfrac{h}{2})\right]$$

$$= \frac{1}{2}\left[f(x+\tfrac{h}{2})g(x+\tfrac{h}{2}) - f(x+\tfrac{h}{2})g(x-\tfrac{h}{2}) + f(x-\tfrac{h}{2})g(x+\tfrac{h}{2}) - f(x-\tfrac{h}{2})g(x-\tfrac{h}{2})\right]$$

$$+ \frac{1}{2}\left[f(x+\tfrac{h}{2})g(x+\tfrac{h}{2}) - f(x-\tfrac{h}{2})g(x+\tfrac{h}{2}) + f(x+\tfrac{h}{2})g(x-\tfrac{h}{2}) - f(x-\tfrac{h}{2})g(x-\tfrac{h}{2})\right]$$

$$= f\left(x+\tfrac{h}{2}\right)g\left(x+\tfrac{h}{2}\right) - f\left(x-\tfrac{h}{2}\right)g\left(x-\tfrac{h}{2}\right)$$

$$= \left(E^{1/2} - E^{-1/2}\right)[f(x)\,g(x)]$$

$$= \delta\,[f(x)\,g(x)]$$

7. $$\delta\left[\frac{f(x)}{g(x)}\right] = \frac{\mu[g(x)]\,\delta[f(x)] - \mu[f(x)]\,\delta[g(x)]}{g\left(x-\frac{h}{2}\right)g\left(x+\frac{h}{2}\right)}$$

Proof:- R.H.S.

$$= \frac{\frac{1}{2}\left[g\left(x+\frac{h}{2}\right)+g\left(x-\frac{h}{2}\right)\right]\left[f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right)\right] - \frac{1}{2}\left[f\left(x+\frac{h}{2}\right)+f\left(x-\frac{h}{2}\right)\right]\left[g\left(x+\frac{h}{2}\right)-g\left(x-\frac{h}{2}\right)\right]}{g\left(x-\frac{h}{2}\right)g\left(x+\frac{h}{2}\right)}$$

$$= \frac{f\left(x+\frac{h}{2}\right)g\left(x-\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)g\left(x+\frac{h}{2}\right)}{g\left(x-\frac{h}{2}\right)g\left(x+\frac{h}{2}\right)}$$

$$= \frac{f\left(x+\frac{h}{2}\right)}{g\left(x+\frac{h}{2}\right)} - \frac{f\left(x-\frac{h}{2}\right)}{g\left(x-\frac{h}{2}\right)} = \left(E^{1/2}-E^{-1/2}\right)\left[\frac{f(x)}{g(x)}\right]$$

$$= \delta\left[\frac{f(x)}{g(x)}\right]$$

8. $$\mu\left[\frac{f(x)}{g(x)}\right] = \frac{\mu[f(x)]\,\mu[g(x)] - \frac{1}{4}\,\delta[f(x)]\,\delta[g(x)]}{g\left(x-\frac{h}{2}\right)g\left(x+\frac{h}{2}\right)}$$

Proof:- L.H.S. $$\mu\left[\frac{f(x)}{g(x)}\right] = \frac{1}{2}\left[\frac{f\left(x+\frac{h}{2}\right)}{g\left(x+\frac{h}{2}\right)} + \frac{f\left(x-\frac{h}{2}\right)}{g\left(x-\frac{h}{2}\right)}\right]$$

$$= \frac{1}{2}\left[\frac{f\left(x+\frac{h}{2}\right)g\left(x-\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)g\left(x+\frac{h}{2}\right)}{g\left(x+\frac{h}{2}\right)g\left(x-\frac{h}{2}\right)}\right] \qquad (1)$$

R.H.S.

$$= \frac{\frac{1}{4}\left[f\left(x+\frac{h}{2}\right)+f\left(x-\frac{h}{2}\right)\right]\left[g\left(x+\frac{h}{2}\right)+g\left(x-\frac{h}{2}\right)\right] - \frac{1}{4}\left[f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right)\right]\left[g\left(x+\frac{h}{2}\right)-g\left(x-\frac{h}{2}\right)\right]}{g\left(x-\frac{h}{2}\right)g\left(x+\frac{h}{2}\right)}$$

$$= \frac{\frac{1}{2}\left[f\left(x+\frac{h}{2}\right)g\left(x-\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)g\left(x+\frac{h}{2}\right)\right]}{g\left(x-\frac{h}{2}\right)g\left(x+\frac{h}{2}\right)} \qquad (2)$$

$(1) = (2) \Rightarrow$ L.H.S = R.H.S

9. $\mu[f(x)\cdot g(x)] = \mu[f(x)]\,\mu[g(x)] + \frac{1}{4}\,\delta[f(x)]\,\delta[g(x)]$

Proof:- R.H.S.

$$= \frac{1}{4}\left[f(x+\tfrac{h}{2}) + f(x-\tfrac{h}{2})\right]\left[g(x+\tfrac{h}{2}) + g(x-\tfrac{h}{2})\right]$$
$$+ \frac{1}{4}\left[f(x+\tfrac{h}{2}) - f(x-\tfrac{h}{2})\right]\left[g(x+\tfrac{h}{2}) - g(x-\tfrac{h}{2})\right]$$
$$= \frac{1}{2}\left[f(x+\tfrac{h}{2})\,g(x+\tfrac{h}{2}) + f(x-\tfrac{h}{2})\,g(x-\tfrac{h}{2})\right]$$
$$= \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)\left[f(x)\,g(x)\right]$$
$$= \mu[f(x)\,g(x)]$$

CENTRAL DIFFERENCE TABLE

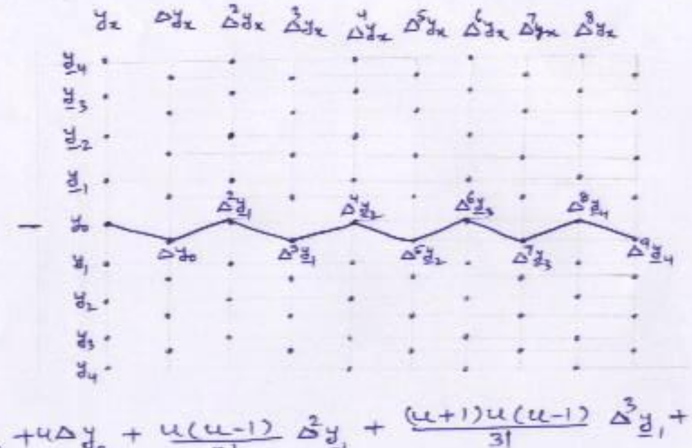
u	x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$	$\Delta^5 y_x$	$\Delta^6 y_x$	$\Delta^7 y_x$	$\Delta^8 y_x$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-4	x_{-4}	y_{-4}								
			Δy_{-4}							
-3	x_{-3}	y_{-3}		$\Delta^2 y_{-4}$						
			Δy_{-3}		$\Delta^3 y_{-4}$					
-2	x_{-2}	y_{-2}		$\Delta^2 y_{-3}$		$\Delta^4 y_{-4}$				
			Δy_{-2}		$\Delta^3 y_{-3}$		$\Delta^5 y_{-4}$			
-1	x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		$\Delta^6 y_{-4}$		
			Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$		$\Delta^7 y_{-4}$	
0	x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$		$\Delta^8 y_{-4}$
			Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$		$\Delta^7 y_{-3}$	
1	x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		$\Delta^6 y_{-2}$		
			Δy_1		$\Delta^3 y_0$		$\Delta^5 y_{-1}$			
2	x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$				
			Δy_2		$\Delta^3 y_1$					
3	x_3	y_3		$\Delta^2 y_2$						
			Δy_3							
4	x_4	y_4								
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

where $u = \dfrac{x - x_0}{h}$

CENTRAL DIFFERENCE FORMULAE

1. GAUSS FORWARD FORMULA

$$\boxed{u = \frac{x - x_0}{h}}$$

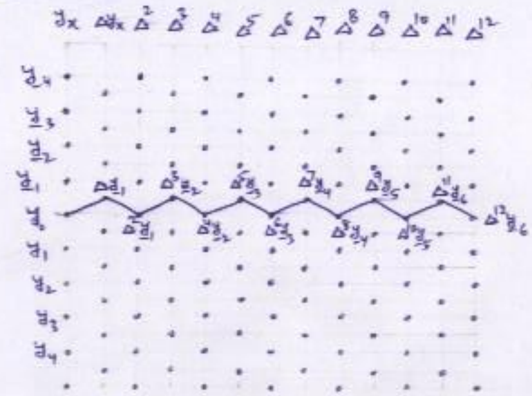


$$f(x) = y_x = y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2}$$
$$+ \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)(u-3)}{6!}\Delta^6 y_{-3}$$
$$+ \frac{(u+3)(u+2)(u+1)u(u-1)(u-2)(u-3)}{7!}\Delta^7 y_{-3} + \frac{(u+3)(u+2)(u+1)u(u-1)(u-2)(u-3)(u-4)}{8!}\Delta^8 y_{-4}$$
$$+ \frac{(u+4)(u+3)(u+2)(u+1)u(u-1)(u-2)(u-3)(u-4)}{9!}\Delta^9 y_{-4} + \cdots$$

$\boxed{\text{OR}}$

$$y_x = y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\Delta^3 y_{-1} + \frac{u(u^2-1^2)(u-2)}{4!}\Delta^4 y_{-2} + \frac{u(u^2-1^2)(u^2-2^2)}{5!}\Delta^5 y_{-2}$$
$$+ \frac{u(u^2-1^2)(u^2-2^2)(u-3)}{6!}\Delta^6 y_{-3} + \frac{u(u^2-1^2)(u^2-2^2)(u^2-3^2)}{7!}\Delta^7 y_{-3} + \frac{u(u^2-1^2)(u^2-2^2)(u^2-3^2)(u-4)}{8!}\Delta^8 y_{-4}$$
$$+ \frac{u(u^2-1^2)(u^2-2^2)(u^2-3^2)(u^2-4^2)}{9!}\Delta^9 y_{-4} + \cdots$$

2. GAUSS BACKWARD FORMULA



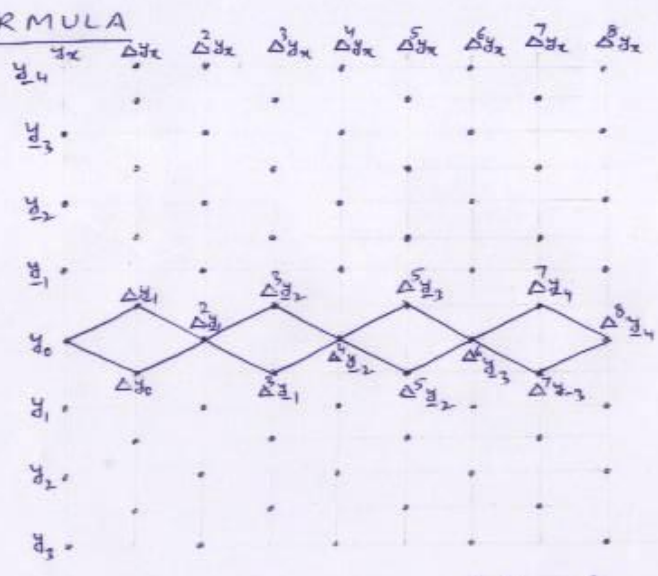
$$y_x = y_u = y_0 + u\,\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\Delta^3 y_{-2} + \frac{u(u^2-1^2)(u+2)}{4!}\Delta^4 y_{-2}$$

$$+ \frac{u(u^2-1^2)(u^2-2^2)}{5!}\Delta^5 y_{-3} + \frac{u(u^2-1^2)(u^2-2^2)(u+3)}{6!}\Delta^6 y_{-3} + \frac{u(u^2-1^2)(u^2-2^2)(u^2-3^2)}{7!}\Delta^7 y_{-4}$$

$$+ \frac{u(u^2-1^2)(u^2-2^2)(u^2-3^2)(u+4)}{8!}\Delta^8 y_{-4} + \cdots\cdots$$

$$\boxed{u = \frac{x - x_0}{h}}$$

3. STERLING FORMULA



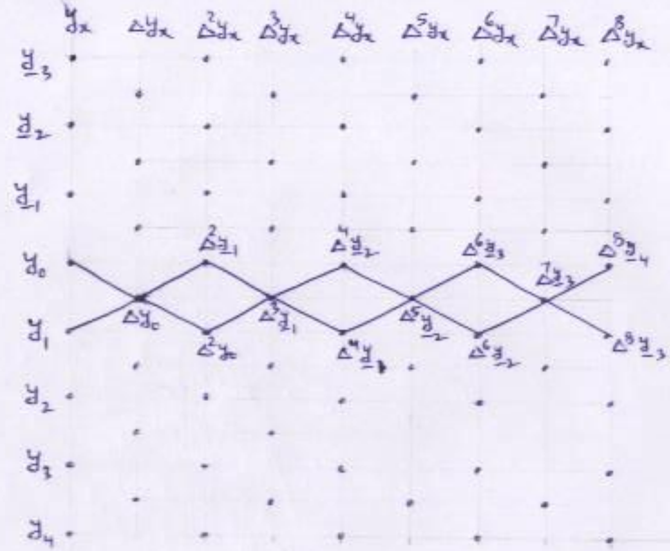
$$y_u = y_x = y_0 + u\left[\frac{\Delta y_{-1} + \Delta y_0}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\left[\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right] + \frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2}$$

$$+ \frac{u(u^2-1^2)(u^2-2^2)}{5!}\left[\frac{\Delta^5 y_{-3} + \Delta^5 y_{-2}}{2}\right] + \frac{u^2(u^2-1^2)(u^2-2^2)}{6!}\Delta^6 y_{-3}$$

$$+ \frac{u(u^2-1^2)(u^2-2^2)(u^2-3^2)}{7!}\left[\frac{\Delta^7 y_{-4} + \Delta^7 y_{-3}}{2}\right] + \frac{u^2(u^2-1^2)(u^2-2^2)(u^2-3^2)}{8!}\Delta^8 y_{-4} + \cdots$$

where $\boxed{u = \dfrac{x - x_0}{h}}$

4. Bessel's Formula



$$y_x = y_u = \left[\frac{y_0 + y_1}{2}\right] + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right] + \frac{\left(u-\frac{1}{2}\right)u(u-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{u(u^2-1^2)(u-2)}{4!}\left[\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right] + \frac{\left(u-\frac{1}{2}\right)u(u^2-1^2)(u-2)}{5!}\Delta^5 y_{-2}$$

$$+ \frac{u(u^2-1^2)(u^2-2^2)(u-3)}{6!}\left[\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2}\right] + \frac{\left(u-\frac{1}{2}\right)u(u^2-1^2)(u^2-2^2)(u-3)}{7!}\Delta^7 y_{-3} + \cdots\cdots$$

where $\boxed{u = \frac{x - x_0}{h}}$

Ex.1 Use Gauss Forward Central difference Formula to find y at $x = 30$, from the data-

x: 21 25 29 33 37

y_x: 18.47 17.81 17.11 16.34 15.52

Sol:- Let us construct the difference table as-

u	x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
-2	21	18.47				
			-0.66			
-1	25	17.81		-0.04		
			-0.70		-0.03	
0	29	17.11		-0.07		0.05
			-0.77		0.02	
1	33	16.34		-0.05		
			-0.82			
2	37	15.52				

The Gauss Forward Formula is-

$$y_x = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\Delta^3 y_{-1} + \cdots \quad \text{——(1)}$$

$$\text{where } u = \frac{x - x_0}{h} = \frac{30-29}{4} = 0.25$$

so from (1) at $x = 30$

$$y_{30} = 17.11 + 0.25(-0.77) + \frac{0.25(0.25-1)}{2!}(-0.07)$$
$$+ \frac{0.25\left[(0.25)^2 - 1^2\right]}{3!}(0.02) + \frac{0.25\left[(0.25)^2-1^2\right](0.25-2)}{4!}(0.05)$$

$\therefore\ y_{30} = 16.92413574 \approx 16.92$

Ex.2 Use sterling Formula to compute $y_{12.2}$ from the data -

x : 10 11 12 13 14

$10^5 y_x$: 23967 28060 31788 35200 38368

Sol:- Let us construct the difference table as-

u	x	$10^5 y_x = P_x$	ΔP_x	$\Delta^2 P_x$	$\Delta^3 P_x$	$\Delta^4 P_x$
-2	10	23967				
			4093			
				-365		
-1	11	28060				
			3728		49	
0	12	31788		-316		23
			3412		72	
1	13	35200		-244		
			3168			
2	14	38368				

By Sterling Formula -

$$P_x = P_u = P_0 + u\left[\frac{\Delta P_0 + \Delta P_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 P_{-1} + \cdots \qquad (1)$$

$$\text{where} \quad u = \frac{x - x_0}{h} = \frac{12.2 - 12}{1} = 0.2 \qquad (2)$$

$$\text{so } P_{12.2} = 10^5 y_{12.2} = 31788 + (0.2)\left[\frac{3728 + 3412}{2}\right] + \frac{(0.2)^2}{2!}(-316) + \frac{(0.2)[(0.2)^2 - 1^2]}{3!}\left[\frac{49+72}{2}\right] + \frac{(0.2)^2[(0.2)^2 - 1^2]}{4!}(23)$$

$$\Rightarrow 10^5 y_{12.2} = 32495.7072$$

$$\Rightarrow y_{12.2} = 0.324957072$$

Ex.3 Find value of $\log_{10} 337.5$ by Bessel's Formula for the data -

x:	310	320	330	340	350	360
$\log_{10} x$:	2.4914	2.5052	2.5185	2.5315	2.5441	2.5563

Sol:- Let $y_x = \log_{10} x$ and construct the difference table as-

u	x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$	$\Delta^5 y_x$
-2	310	2.4914					
			0.0138				
-1	320	2.5052		-0.0005			
			0.0133		0.0002		
0	330	2.5185		-0.0003		-0.0003	
			0.0130		-0.0001		0.0004
1	340	2.5315		-0.0004		0.0001	
			0.0126		0		
2	350	2.5441		-0.0004			
			0.0122				
3	360	2.5563					

The Bessel's Formula is-

$$y_x = \left[\frac{y_0+y_1}{2}\right] + \left(u-\tfrac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left[\frac{\Delta^2 y_1 + \Delta^2 y_0}{2}\right] + \cdots \quad \text{—(1)}$$

where $u = \dfrac{x-x_0}{h} = \dfrac{337.5-330}{10} = 0.75$

$$\therefore y_{337.5} = \log_{10} 337.5 = \left[\frac{2.5185+2.5315}{2}\right] + (0.75-0.5)(0.0130)$$

$$+ \frac{0.75(0.75-1)}{2!}\left[\frac{-0.0003-0.0004}{2}\right] + \frac{(0.75-0.5)(0.75)(0.75-1)}{3!}(-0.0001)$$

$$+ \frac{0.75[(0.75)^2-1^2](0.75-2)}{4!}\left[\frac{-0.0003+0.0001}{2}\right]$$

$$+ \frac{(0.75-0.5)(0.75)[(0.75)^2-1^2](0.75-2)}{5!}(0.0004)$$

$$\Rightarrow \log_{10} 337.5 = 2.528285642$$

NUMERICAL DIFFERENTIATION

Numerical differentiation is the process by which we can find the derivative of a f^n at some values of independent variable (x) when the set of values of f^n is given.

The Problem of differentiation is solved by first approximating the f^n by an interpolation formula and then differentiating this formula as many times as desired.

Methods of interpolation that we discussed earlier are in two forms either in terms of variable x or in terms of u, and u is defined in two ways -

(i) $u = \dfrac{x - x_0}{h}$ ——(1) or (ii) $u = \dfrac{x - (x_0 + nh)}{h}$ ——(2)

If we would like to find the first derivative of $y = f(x)$ with respect to x then - $\boxed{\dfrac{dy}{dx} = f'(x)}$ Simply, if f is of f^n x only.

But if y is a f^n of x and x is a f^n of u then -

$$\frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dx} \quad \text{——— (3)}$$

then from (1) & (2) it is obvious that

$$\frac{du}{dx} = \frac{1}{h} \quad \text{——— (4)}$$

$$\therefore \boxed{\frac{dy}{dx} = \frac{1}{h} \cdot \frac{df}{du}} \quad \text{——— (5)}$$

For Second and higher order derivatives obviously -

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\frac{d^2f}{du^2} \qquad \Big| \qquad \frac{d^2y}{dx^2} = f''(x)$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\cdot\frac{d^3f}{du^3} \qquad \Big| \qquad \frac{d^3y}{dx^3} = f'''(x)$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{d^ny}{dx^n} = \frac{1}{h^n}\frac{d^nf}{du^n} \qquad \Big| \qquad \frac{d^ny}{dx^n} = f^{(n)}(x)$$

Ex.1 Find $f'(8)$, $f''(8)$ and $f'''(8)$ by Newton's Divided difference formula from the data -

x :	6	7	9	12
$f(x)$:	1.556	1.690	1.908	2.158

Sol:- Let us construct the divided difference table -

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
6	1.556			
		0.134		
7	1.690		−0.00833	
		0.109		0.000532
9	1.908		−0.00514	
		0.0833		
12	2.158			

Sol:- The Newton's Divided difference formula is-

$$f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) + (x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0) + \cdots \quad \text{--- (1)}$$

$$\therefore f'(x) = \Delta f(x_0) + [2x - (x_0+x_1)]\Delta^2 f(x_0) + [3x^2 - 2x(x_0+x_1+x_2) + (x_1x_2 + x_2x_0 + x_0x_1)]\Delta^3 f(x_0) + \cdots \quad \text{--- (2)}$$

$$\therefore f'(8) = 0.134 + [2(8) - (6+7)](-0.00833) + [3(8)^2 - 2(8)(6+7+9) + \{(7)(9) + (9)(6) + (6)(7)\}](0.000532)$$

$$\Rightarrow f'(8) = -0.116432$$

$$f''(x) = 2\Delta^2 f(x_0) + [6x - 2(x_0+x_1+x_2)]\Delta^3 f(x_0) + \cdots$$

$$\therefore f''(8) = 2(-0.00833) + [6(8) - 2(6+7+9)](0.000532)$$

$$\Rightarrow f''(8) = 0.014532$$

$$f'''(x) = 6\Delta^3 f(x_0) + \cdots$$

$$\therefore f'''(8) = 6(0.000532) = 0.003192$$

Ex.2 Find $y'(0)$ and $y''(0)$ from the data-

x:	0	1	2	3	4	5
$y=f(x)$:	4	8	15	7	6	2

Sol:- Let us construct the difference table.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0	4					
		4				
			3			
1	8			-18		
		7			40	
			-15			-72
2	15			22		
		-8			-32	
			7			
3	7			-10		
		-1				
			-3			
4	6					
		-4				
5	2					

The Newton Gregory forward difference formula is-

$$y = f(x) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(x_0) + \frac{u(u-1)(u-2)(u-3)(u-4)}{5!}\Delta^5 f(x_0) + \cdots \quad (1)$$

where $u = \frac{x-x_0}{h} = \frac{0-0}{1} = 0$

$$y'(x) = f'(x) = \frac{1}{h}\Big[\Delta f(x_0) + \frac{2u-1}{2}\Delta^2 f(x_0) + \frac{3u^2-6u+2}{6}\Delta^3 f(x_0) + \frac{4u^3-18u^2+22u-6}{24}\Delta^4 f(x_0) + \frac{5u^4-40u^3+105u^2-100u+24}{120}\Delta^5 f(x_0) + \cdots$$

$$\therefore y'(0) = \frac{1}{1}\Big[4 - \frac{1}{2}(3) + \frac{2}{6}(-18) + \left(-\frac{6}{24}\right)40 + \frac{24}{120}(-172)\Big]$$

$$\Rightarrow y'(0) = -27.9$$

$$y''(x) = f''(x) = \frac{1}{h^2}\Big[\Delta^2 f(x_0) + (u-1)\Delta^3 f(x_0)$$
$$+ \frac{12u^3 - 36u + 22}{24}\Delta^4 f(x_0) + \frac{20u^3 - 120u^2 + 210u - 100}{120}\Delta^5 f(x_0)$$
$$+ \cdots \Big]$$

$$y''(0) = \frac{1}{1^2}\Big[3 - (-18) + \frac{22}{24}(40) - \frac{100}{120}(-72) \Big]$$

$$\Rightarrow y''(0) = 117.6667$$

——— x ——— x ———

NUMERICAL INTEGRATION

The process of evaluating a definite integral from a set of tabulated values of Integrand $y = f(x)$, is called Numerical Integration. This Process when applied to the function of single variable, is known as Qudrature.

The Problem of Numerical Integration, is solved by representing $y = f(x)$ by an interpolation formula and then integrating it between the given Limits.

Let $I = \int_a^b y\,dx$ ——— (1) be the integration which we want to evaluate.

where $y = f(x)$ takes values $y_0, y_1, y_2, \cdots y_n$ at $x = x_0, x_1, x_2, \cdots x_n$ respectively.

Let us divide the interval (a, b) in to 'n' sub intervals of length 'h' $(= \frac{b-a}{n})$, such that $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, ..., $x_n = x_0 + nh = b$

then

$$I = \int_{x_0}^{x_0+nh} y\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx \quad \text{Let } x = x_0 + hu \Rightarrow dx = h\,du$$

$$= h\int_0^n f(x_0 + hu)\,du \quad \text{——— (2)}$$

By Newton Gregory Forward difference formula -

$$f(x) = f(x_0 + hu) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \cdots$$

$$\approx y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \cdots$$

$$\therefore \quad I = h\int_0^n \Big[y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \frac{u(u-1)(u-2)(u-3)(u-4)}{5!}\Delta^5 y_0 + \frac{u(u-1)(u-2)(u-3)(u-4)(u-5)}{6!}\Delta^6 y_0 + \cdots \Big] du$$

$$\Rightarrow \int_{x_0}^{x_0+nh} y\,dx = nh\Big[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n\right)\frac{\Delta^4 y_0}{4!} + \left(\frac{n^5}{6} - 2n^4 + \frac{35n^3}{4} - \frac{50n^2}{3} + 12n\right)\frac{\Delta^5 y_0}{5!} + \left(\frac{n^6}{7} - \frac{15n^5}{6} + 17n^4 - \frac{225n^3}{4} + \frac{274n^2}{4} - 60n\right)\frac{\Delta^6 y_0}{6!} + \cdots \Big] \quad \text{---(3)}$$

which is known as Newton - Cotes Quadrature Formula. By Putting $n = 1, 2, 3, \ldots$ we can deduce some important rules for Quadrature

1. Trapezoidal Rule

Let us put $n = 1$ in (3) and taking the curve through (x_0, y_0) and (x_1, y_1) as a straight line i.e. a Polynomial of first degree, such that the differences of more than first order becomes zero, then -

$$\int_{x_0}^{x_0+h} y\,dx = h\left[y_0 + \tfrac{1}{2}\Delta y_0\right] = h\left[y_0 + \tfrac{1}{2}(y_1 - y_0)\right] = \frac{h}{2}(y_0 + y_1) \quad \text{---(i)}$$

similarly -

$$\int_{x_0+h}^{x_0+2h} y\,dx = 2h\left[y_1 + \tfrac{1}{2}\Delta y_1\right] = \frac{h}{2}(y_1 + y_2) \quad \text{----(ii)}$$

$\vdots$

$$\int_{x_0+(n-1)h}^{x_0+nh} y\,dx = \frac{h}{2}[y_{n-1} + y_n] \quad \text{-----(iii)}$$

Adding these 'n' integrals, we have.

$$\boxed{\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\cdots+y_{n-1})\right]}$$

which is Known as Trapezoidal Rule

As the Polynomial is taken of first degree (i.e. a straight line), so for this method, the number of intervals must be multiple of 1. i.e. $n = 1, 2, 3, \ldots$

2. Simpson's $\frac{1}{3}$ Rule:-

Putting $n=2$ in (3) and taking the curve through (x_0, y_0), (x_1, y_1) & (x_2, y_2) as a Parabola. (i.e. Polynomial of degree 2), such that the differences of more than order 2 Vanishes.

$$\text{so} \int_{x_0}^{x_0+2h} y\,dx = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

$$= 2h\left[y_0 + (y_1 - y_0) + \frac{1}{6}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right] \quad \text{------ (i)}$$

Similarly -

$$\int_{x_0+2h}^{x_0+4h} y\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right] \quad \text{------ (ii)}$$

$\vdots$

[n is even)

$$\int_{x_0+(n-2)h}^{x_0+nh} y\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right] \quad \text{------ (iii)}$$

Adding all these integrals-

$$\boxed{\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+y_5+\cdots+y_{n-1}) + 2(y_2+y_4+y_6+\cdots+y_{n-2})\right]}$$

which is Known as Simpson's $\frac{1}{3}$ Rule.

When we apply this method, the given interval must be divided in to even number of sub intervals, i.e. in multiple of 2.

i.e. $n = 2, 4, 6, 8, \ldots$

3. Simpson's $\frac{3}{8}$ Rule

Putting $n=3$ in (3) and taking the curve through (x_0,y_0), (x_1,y_1), (x_2,y_2) & (x_3,y_3) as a Polynomial of third degree (i.e. cubic Polynomial) such that the differences more than order three vanishes. then-

$$\int_{x_0}^{x_0+3h} y\,dx = 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right]$$

$$= 3h\left[y_0 + \frac{3}{2}(y_1-y_0) + \frac{3}{4}(y_2-2y_1+y_0) + \frac{1}{8}(y_3-3y_2+3y_1-y_0)\right]$$

$$= \frac{3}{8}h\left[y_0 + 3y_1 + 3y_2 + y_3\right] \quad \text{---- (i)}$$

Similarly.

$$\int_{x_0+3h}^{x_0+6h} y\,dx = \frac{3}{8}h\left[y_3 + 3y_4 + 3y_5 + y_6\right] \quad \text{---- (ii)}$$

$$\vdots$$

$$\int_{x_0+(n-3)h}^{x_0+nh} y\,dx = \frac{3}{8}h\left[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right] \quad \text{---- (iii)}$$

Adding all these integrals

$$\boxed{\int_{x_0}^{x_0+nh} y\,dx = \frac{3}{8}h\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+\cdots+y_{n-1}) + 2(y_3+y_6+y_9+\cdots+y_{n-3})\right]}$$

which is known as ~~Tr~~ Simpson's $\frac{3}{8}$ Rule.

when we apply this method, the given interval must be divided in to multiple number of 3. i.e. $n = 3, 6, 9, 12, \cdots$

Ex.1 Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ by using-

(i) Trapezoidal Rule

(ii) Simpson's $\frac{1}{3}$ Rule

(iii) Simpson's $\frac{3}{8}$ Rule

Hence obtain value of π in each case.

Sol:- As we know that for applying Trapezoidal Rule, the interval must be divided in to 'n' number of intervals, where n is a multiple of 1.

For applying Simpson's $\frac{1}{3}$ Rule, the interval must be divided into 'n' number of intervals, where n is a multiple of 2.

For Simpson's $\frac{3}{8}$ Rule, the n must be multiple of 3.

So when applying all the rules, then 'n' must be multiple of 1, 2 and 3. So Let $n=6$ intervals.

Hence $h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
x:	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
y:	$\frac{1}{1+0}$	$\frac{1}{1+\frac{1}{36}}$	$\frac{1}{1+\frac{1}{9}}$	$\frac{1}{1+\frac{1}{4}}$	$\frac{1}{1+\frac{4}{9}}$	$\frac{1}{1+\frac{25}{36}}$	$\frac{1}{1+1}$
	$=1$	$=0.97297$	$=0.9$	$=0.8$	$=0.692307$	$=0.59016$	$=0.5$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

1. TRAPEZOIDAL RULE

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\cdots+y_{n-1})\right]$$

$$\Rightarrow \int_0^1 \frac{1}{1+x^2}\,dx = \frac{1}{6\cdot 2}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{12}\left[(1+0.5) + 2(0.97297+0.9+0.8+0.692307+0.59016)\right]$$

$$= 0.7842395$$

2. SIMPSON'S $\frac{1}{3}$ RULE

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+y_5+\cdots) + 2(y_2+y_4+\cdots)\right]$$

$$\Rightarrow \int_0^1 \frac{1}{1+x^2}\,dx = \frac{1}{6\cdot 3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$$

$$= \frac{1}{18}\left[(1+0.5) + 4(0.97297 + 0.8 + 0.59016) + 2(0.9 + 0.692307)\right]$$

$$= 0.785396333$$

3. SIMPSON'S $\frac{3}{8}$ RULE

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3}{8}h\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+y_7+y_8+\cdots) + 2(y_3+y_6+y_9+\cdots)\right]$$

$$\Rightarrow \int_0^1 \frac{1}{1+x^2}\,dx = \frac{3}{8}\cdot\frac{1}{6}\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 2(y_3)\right]$$

$$= \frac{1}{16}\left[(1+0.5) + 3(0.972297 + 0.9 + 0.692307 + 0.59016) + 2(0.8)\right]$$

$$= 0.78539437$$

Value of π

As the given integral is -

$$\int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}1 - \tan^{-1}0$$

$$= \frac{\pi}{4}$$

$$\therefore\ \pi = 4\cdot\int_0^1 \frac{1}{1+x^2}\,dx$$

By Trapezoidal Rule.

$$\pi = 4[0.7842395] = 3.136958$$

By Simpson's $\frac{1}{3}$ Rule

$$\pi = 4[0.785396333] = 3.141585332$$

By Simpson's $\frac{3}{8}$ Rule

$$\pi = 4[0.78539437] = 3.14157748$$

Ex.2 Evaluate $\int_4^{5.2} \log_e x \, dx$ by-

(i) Simpson's $\frac{1}{3}$ Rule

(ii) Simpson's $\frac{3}{8}$ Rule

and compare the errors in both cases with actual value.

Sol:- To apply both the desired rules, Let us divide the interval (4, 5.2) in 6 equal intervals. So

$$h = \frac{5.2 - 4}{6} = 0.2$$

so we have following data -

x	$y = \log_e x$
$x_0 = 4$	$y_0 = 1.386294361$
$x_1 = 4.2$	$y_1 = 1.435084525$
$x_2 = 4.4$	$y_2 = 1.481604541$
$x_3 = 4.6$	$y_3 = 1.526056303$
$x_4 = 4.8$	$y_4 = 1.568615918$
$x_5 = 5.0$	$y_5 = 1.609437912$
$x_6 = 5.2$	$y_6 = 1.648658626$

Simpson's $\frac{1}{3}$ Rule

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+y_5+\cdots) + 2(y_2+y_4+\cdots)\right]$$

$$\Rightarrow \int_4^{5.2} \log_e x \, dx = \frac{0.2}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$$

$$= \frac{0.2}{3}\left[(1.386294361 + 1.648658626) + 4(1.435084525 + 1.526056303 + 1.609437912) + 2(1.481604541 + 1.568615918)\right]$$

$$= 1.827847258$$

Simpson's $\frac{3}{8}$ Rule

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+y_7+y_8+\cdots) + 2(y_3+y_6+y_9+\cdots)\right]$$

$$\Rightarrow \int_4^{5.2} \log_e x\,dx = \frac{3}{8}(0.2)\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 2(y_3)\right]$$

$$= \frac{3}{8}(0.2)\left[(1.386294361 + 1.648658626) + 3(1.435084525 + 1.481604541 + 1.568615918 + 1.609437912) + 2(1.526056303)\right]$$

$$= 1.827847071$$

Actual Value

$$\because \int_4^{5.2} \log_e x\,dx = \left[x(\log_e x - 1)\right]_4^{5.2}$$

$$= 1.8278474099$$

Error

$\because$ Error = Actual Value − Approximate Value.

By Simpson's $\frac{1}{3}$ Rule

Error = 1.8278474099 − 1.827847258 = 0.000000151

By Simpson's $\frac{3}{8}$ Rule

Error = 1.8278474099 − 1.827847071 = 0.000000338

NUMERICAL SOLUTION OF ODE

Under this topic we will discuss the methods for solving the Ordinary differential eq^n (ODE) of first order. i.e.

$$\frac{dy}{dx} = f(x,y)$$

For these methods it is desired that the equation having numerical co-efficients and the initial condition is given.

PICARD METHOD

Let us consider the first order ODE -

$$\frac{dy}{dx} = f(x,y) \quad \text{——— (1)}$$

with the initial cond^n -

$$y(x_0) = y_0 \quad \text{——— (2)}$$

Integrating (1) between x_0 and x —

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\,dx$$

$$\Rightarrow \quad y - y_0 = \int_{x_0}^{x} f(x,y)\,dx$$

$$\Rightarrow \quad y = y_0 + \int_{x_0}^{x} f(x,y)\,dx \quad \text{——— (3)}$$

For the first approximation, we replace y by y_0 in $f(x,y)$, for second approximation, we replace y by y_1 in $f(x,y)$, ... and so on.

So $$y_1 = y_0 + \int_{x_0}^{x} f(x,y_0)\,dx \quad \text{——— (4)}$$

$$y_2 = y_0 + \int_{x_0}^{x} f(x,y_1)\,dx \quad \text{——— (5)}$$

$$\vdots$$

$$\boxed{y_n = y_0 + \int_{x_0}^{x} f(x,y_{n-1})\,dx} \quad \text{——— (6)}$$

$n = 1, 2, 3, \ldots$

The Process is stopped, when two consecutive values of y, i.e. y_{i-1} & y_i are sensibly the same as per desired degree of accuracy.

Note:- This method can be applied only to those equations whose successive integrations can be performed easily.

Ex.1 Find value of y at $x = 0.2$ for the eq^n

$$\frac{dy}{dx} = x - y \quad ; \quad y(0) = 1$$

Sol:- Here $x_0 = 0$, $y_0 = 1$, $f(x, y) = x - y$

So By Picard Method -

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx = y_0 + \int_0^x [x - y_{n-1}]\,dx \quad n = 1, 2, 3, \ldots$$

First approximation $(n=1)$

$$y_1 = y_0 + \int_0^x [x - y_0]\,dx$$

$$= 1 + \int_0^x [x - 1]\,dx = 1 - x + \frac{x^2}{2}$$

$$\therefore y_1(0.2) = 1 - 0.2 + \frac{(0.2)^2}{2} = 0.82$$

Second approximation $(n=2)$

$$y_2 = y_0 + \int_0^x [x - y_1]\,dx$$

$$= 1 + \int_0^x [x - (1 - x + \frac{x^2}{2})]\,dx$$

$$= 1 - x + x^2 - \frac{x^3}{6}$$

$$\therefore y_2(0.2) = 1 - 0.2 + (0.2)^2 - \frac{(0.2)^3}{6} = 0.83867$$

Third approximation $(n=3)$

$$y_3 = y_0 + \int_0^x [x - y_2]\,dx$$

$$= 1 + \int_0^x [x - (1 - x + x^2 - \frac{x^3}{6})]\,dx$$

$$= 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{24}$$

$$\therefore y_3(0.2) = 1 - 0.2 + (0.2)^2 - \frac{(0.2)^3}{3} + \frac{(0.2)^4}{24} = 0.8374$$

$$= \int_0^x \left[x^2 + \frac{x^6}{9} + \frac{x^{14}}{3969} + \frac{2x^{10}}{189}\right] dx$$

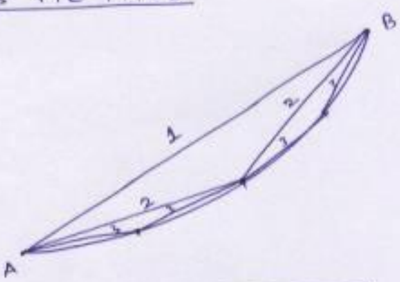
$$\Rightarrow y_3 = \frac{x^3}{3} + \frac{x^7}{63} + \frac{2x^{11}}{2079} + \frac{x^{15}}{59535}$$

$$\therefore y_3(0.4) = \frac{(0.4)^3}{3} + \frac{(0.4)^7}{63} + \frac{2(0.4)^{11}}{2079} + \frac{(0.4)^{15}}{59535} = 0.021359379$$

$$\because y_2(0.4) = y_3(0.4) = 0.02135$$

$$\Rightarrow y(0.4) = 0.02135.$$

2. <u>EULER'S METHOD</u>



$$\frac{dy}{dx} = f(x,y)$$

$$y(x_0) = y_0$$

$$\boxed{y_n = y_{n-1} + h\,f(x_{n-1}, y_{n-1})} \qquad n = 1, 2, 3, \ldots$$

$h \to$ Length of interval.

<u>Ex.1</u> Find approximate value of y at $x = 0.6$ for the equation-

$$\frac{dy}{dx} = 1 - 2xy \quad ; \quad y(0) = 0$$

Take $h = 0.2$

<u>Sol:-</u> Here $x_0 = 0$, $y_0 = 0$ and $f(x,y) = 1 - 2xy$

~~By Euler's Method~~

$\because h = 0.2$, so -

$$x_1 = 0.2, \quad x_2 = 0.4, \quad x_3 = 0.6$$

$$\therefore y(x_1) = y_1, \quad y(x_2) = y_2, \quad y(x_3) = y_3.$$

By Euler's Method-

$$y_n = y_{n-1} + h\,f(x_{n-1}, y_{n-1}) \qquad ; n = 1, 2, 3, \ldots$$

$$\Rightarrow y_n = y_{n-1} + h\left[1 - 2x_{n-1}y_{n-1}\right] \quad \text{——— (1)} \quad ; n = 1, 2, 3, \ldots$$

Fourth approximation $(n=4)$

$$y_4 = y_0 + \int_0^x [x - y_3]\, dx$$

$$= 1 + \int_0^x [x - (1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{24})]\, dx$$

$$= 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120}$$

$$\therefore\ y_4(0.2) = 1 - 0.2 + (0.2)^2 - \frac{(0.2)^3}{3} + \frac{(0.2)^4}{12} - \frac{(0.2)^5}{120} = 0.837464$$

As $y_3(0.2) = y_4(0.2) = 0.8374$ (up to Four decimals)

So $\underline{y(0.2) = 0.8374}$

Ex. 2 obtain the solution of $\frac{dy}{dx} = x^2 + y^2$; $y(0) = 0$ at $x = 0.4$, Correct up to Four Places of decimal.

Sol:- Here $x_0 = 0$, $y_0 = 0$ and $f(x,y) = x^2 + y^2$

By Picard method-

$$y_n = y_0 + \int_{x_0}^x f(x, y_{n-1})\, dx \quad ; n = 1, 2, \cdots$$

so $$y_n = y_0 + \int_0^x [x^2 + y_{n-1}^2]\, dx = \int_0^x [x^2 + y_{n-1}^2]\, dx$$

First Iteration $(n=1)$

$$y_1 = \int_0^x [x^2 + y_0^2]\, dx = \frac{x^3}{3}$$

$$\therefore\ y_1(0.4) = \frac{(0.4)^3}{3} = 0.0213333$$

Second Iteration $(n=2)$

$$y_2 = \int_0^x [x^2 + y_1^2]\, dx = \int_0^x [x^2 + (\frac{x^3}{3})^2]\, dx$$

$$= \frac{x^3}{3} + \frac{x^7}{63}$$

$$\therefore\ y_2(0.4) = \frac{(0.4)^3}{3} + \frac{(0.4)^7}{63} = 0.021359339$$

Third Iteration $(n=3)$

$$y_3 = \int_0^x [x^2 + y_2^2]\, dx = \int_0^x [x^2 + (\frac{x^3}{3} + \frac{x^7}{63})^2]\, dx.$$

$$= \int_0^x \left[x^2 + \frac{x^6}{9} + \frac{x^{14}}{3969} + \frac{2x^{10}}{189}\right] dx$$

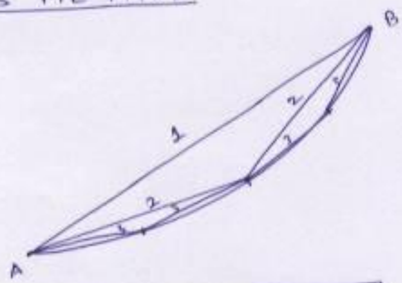
$$\Rightarrow y_3 = \frac{x^3}{3} + \frac{x^7}{63} + \frac{2x^{11}}{2079} + \frac{x^{15}}{59535}$$

$$\therefore y_3(0.4) = \frac{(0.4)^3}{3} + \frac{(0.4)^7}{63} + \frac{2(0.4)^{11}}{2079} + \frac{(0.4)^{15}}{59535} = 0.021359379$$

$$\because y_2(0.4) = y_3(0.4) = 0.02135$$

$$\Rightarrow y(0.4) = 0.02135.$$

2. <u>EULER'S METHOD</u>



$$\frac{dy}{dx} = f(x, y)$$
$$y(x_0) = y_0$$

$$\boxed{y_n = y_{n-1} + h f(x_{n-1}, y_{n-1})} \quad n = 1, 2, 3, \dots$$

$h \rightarrow$ Length of interval.

<u>Ex.1</u> Find approximate value of y at $x = 0.6$ for the equation-

$$\frac{dy}{dx} = 1 - 2xy \quad ; \; y(0) = 0$$

Take <u>$h = 0.2$</u>

<u>Sol:-</u> Here $x_0 = 0$, $y_0 = 0$ and $f(x, y) = 1 - 2xy$

~~By Euler's Method~~

$\because h = 0.2$, so -

$$x_1 = 0.2, \quad x_2 = 0.4, \quad x_3 = 0.6$$

$$\therefore y(x_1) = y_1, \quad y(x_2) = y_2, \quad y(x_3) = y_3.$$

By Euler's Method-

$$y_n = y_{n-1} + h f(x_{n-1}, y_{n-1}) \quad ; n = 1, 2, 3, \dots$$

$$\Rightarrow y_n = y_{n-1} + h\left[1 - 2x_{n-1} y_{n-1}\right] \quad \text{—— (1)} \quad ; n = 1, 2, 3, \dots$$

$\therefore\ y_1 = y_0 + h[1 - 2x_0 y_0] = 0 + 0.2[1-0] = 0.2$

$\Rightarrow y_1 = y(0.2) = 0.2$

$y_2 = y_1 + h[1 - 2x_1 y_1] = 0.2 + 0.2[1 - 2(0.2)(0.2)] = 0.384$

$\Rightarrow y_2 = y(0.4) = 0.384$

$y_3 = y_2 + h[1 - 2x_2 y_2] = 0.384 + 0.2[1 - 2(0.4)(0.384)]$

$\Rightarrow y_3 = y(0.6) = 0.52256$ Ans.

3. MODIFIED EULER's METHOD

$$\frac{dy}{dx} = f(x,y) \quad ; \quad y(x_0) = y_0$$

$$\boxed{\begin{aligned} y_n^{(1)} &= y_{n-1} + h f(x_{n-1}, y_{n-1}) \qquad n = 1,2,3,\cdots \\ y_n^{(m)} &= y_{n-1} + \frac{h}{2}[f(x_{n-1}, y_{n-1}) + f(x_n, y_n^{(m-1)})] \qquad n = 1,2,3,\cdots \\ & \qquad m = 1,2,3,\cdots \end{aligned}}$$

Ex.1 Find a solution of the equation -

$$\frac{dy}{dx} = x + |\sqrt{y}| \quad ; \quad y(0) = 1$$

at $x = 0.6$ by taking $h = 0.2$

Sol:- Here $x_0 = 0$, $y_0 = 1$, $f(x,y) = x + |\sqrt{y}|$

so $x_1 = 0.2$, $x_2 = 0.4$ and $x_3 = 0.6$

By Modified Euler's Method -

$$y_n^{(1)} = y_{n-1} + h f(x_{n-1}, y_{n-1})$$

$$y_n^{(m)} = y_{n-1} + \frac{h}{2}[f(x_{n-1}, y_{n-1}) + f(x_n, y_n^{(m-1)})]$$

$$m = 2,3,4,\cdots \quad ; \quad n = 1,2,3,\cdots$$

So $$y_n^{(1)} = y_{n-1} + h[x_{n-1} + |\sqrt{y_{n-1}}|] \qquad \text{—— (1)}$$

$$y_n^{(m)} = y_{n-1} + \frac{h}{2}[(x_{n-1} + |\sqrt{y_{n-1}}|) + (x_n + |\sqrt{y_n^{(m-1)}}|)] \qquad \text{—— (2)}$$

$$m = 2,3,4,\cdots$$
$$n = 1,2,3,\cdots$$

$\therefore\ y_1^{(1)} = y_0 + h[x_0 + |\sqrt{y_0}|] = 1 + 0.2[0+1] = 1.2$

$y_1^{(2)} = y_0 + \frac{h}{2}[(x_0 + |\sqrt{y_0}|) + (x_1 + |\sqrt{y_1^{(1)}}|)]$

$= 1 + \frac{0.2}{2}[(0+1) + (0.2 + |\sqrt{1.2}|)] = 1.22954$

$y_1^{(3)} = y_0 + \frac{h}{2}[(x_0 + |\sqrt{y_0}|) + (x_1 + |\sqrt{y_1^{(2)}}|)]$

$= 1 + \frac{0.2}{2}[(0+1) + (0.2 + |\sqrt{1.22954}|)] = 1.23088$

$y_1^{(4)} = y_0 + \frac{h}{2}[(x_0 + |\sqrt{y_0}|) + (x_1 + |\sqrt{y_1^{(3)}}|)]$

$= 1 + \frac{0.2}{2}[(0+1) + (0.2 + |\sqrt{1.23088}|)] = \underline{1.230945}$

$y_1^{(5)} = y_0 + \frac{h}{2}[(x_0 + |\sqrt{y_0}|) + (x_1 + |\sqrt{y_1^{(4)}}|)]$

$= 1 + \frac{0.2}{2}[(0+1) + (0.2 + |\sqrt{1.230945}|)] = \underline{1.230947}$

$\because\ y_1^{(4)} = y_1^{(5)} = 1.23094$

$\Rightarrow \boxed{y_1 = y(0.2) = 1.23094}$

$y_2^{(1)} = y_1 + h[x_1 + |\sqrt{y_1}|] = 1.23094 + 0.2[0.2 + |\sqrt{1.23094}|]$

$\Rightarrow y_2^{(1)} = 1.492835$

$y_2^{(2)} = y_1 + \frac{h}{2}[(x_1 + |\sqrt{y_1}|) + (x_2 + |\sqrt{y_2^{(1)}}|)]$

$= 1.23094 + \frac{0.2}{2}[(0.2 + |\sqrt{1.23094}|) + (0.4 + |\sqrt{1.492835}|)]$

$= 1.524069$

$y_2^{(3)} = y_1 + \frac{h}{2}[(x_1 + |\sqrt{y_1}|) + (x_2 + |\sqrt{y_2^{(2)}}|)] = 1.52534$

$y_2^{(4)} = y_1 + \frac{h}{2}[(x_1 + |\sqrt{y_1}|) + (x_2 + |\sqrt{y_2^{(3)}}|)] = \underline{1.525392}$

$y_2^{(5)} = y_1 + \frac{h}{2}[(x_1 + |\sqrt{y_1}|) + (x_2 + |\sqrt{y_2^{(4)}}|)] = \underline{1.525392}$

$\because\ y_2^{(4)} = y_2^{(5)} = 1.525392$

$\therefore\ \boxed{y_2 = y(0.4) = 1.525392}$

$y_3^{(1)} = y_2 + h[x_2 + |\sqrt{y_2}|] = 1.525392 + 0.2[0.4 + |\sqrt{1.525392}|]$

$= 1.852405$

$$y_3^{(2)} = y_2 + \frac{h}{2}\left[(x_2 + |\sqrt{y_2}|) + (x_3 + |\sqrt{y_3^{(1)}}|)\right]$$

$$= 1.525392 + \frac{0.2}{2}\left[(0.4 + |\sqrt{1.525392}|) + (0.6 + |\sqrt{1.852405}|)\right]$$

$$= 1.885$$

$$y_3^{(3)} = y_2 + \frac{h}{2}\left[(x_2 + |\sqrt{y_2}|) + (x_3 + |\sqrt{y_3^{(2)}}|)\right] = 1.886194$$

$$y_3^{(4)} = y_2 + \frac{h}{2}\left[(x_2 + |\sqrt{y_2}|) + (x_3 + |\sqrt{y_3^{(3)}}|)\right] = 1.886237$$

$$y_3^{(5)} = y_2 + \frac{h}{2}\left[(x_2 + |\sqrt{y_2}|) + (x_3 + |\sqrt{y_3^{(4)}}|)\right] = 1.886239$$

$\because$ $y_3^{(4)} = y_3^{(5)} = 1.88623$ (up to Five Decimals)

$\therefore$ $y_3 = y(0.6) = 1.88623$

4. RUNGE - KUTTA METHOD

The Differential equation-

$$\frac{dy}{dx} = f(x,y) \quad ; \quad y(x_0) = y_0$$

$$y_n = y_{n-1} + \frac{1}{6}[K_1 + 2(K_2+K_3) + K_4]$$
$$n = 1, 2, 3, \ldots$$

where -

$$K_1 = h f(x_{n-1}, y_{n-1})$$
$$K_2 = h f(x_{n-1} + \tfrac{h}{2}, y_{n-1} + \tfrac{K_1}{2})$$
$$K_3 = h f(x_{n-1} + \tfrac{h}{2}, y_{n-1} + \tfrac{K_2}{2})$$
$$K_4 = h f(x_{n-1} + h, y_{n-1} + K_3)$$
$$n = 1, 2, 3, \ldots$$

Ex.1 Find the value of y at $x = 0.4$ for the equation -

$$\frac{dy}{dx} = -2xy^2 \quad ; \quad y(0) = 1$$

by taking $h = 0.2$

Sol:- Here $x_0 = 0$, $y_0 = 1$, $f(x,y) = -2xy^2$

$h = 0.2$, so $x_1 = 0.2$ & $x_2 = 0.4$.

By Runge - Kutta Method -

$$y_n = y_{n-1} + \frac{1}{6}[K_1 + 2(K_2+K_3) + K_4]$$

where -

$$K_1 = h[-2x_{n-1} y_{n-1}^2]$$
$$K_2 = h[-2(x_{n-1} + \tfrac{h}{2})(y_{n-1} + \tfrac{K_1}{2})^2]$$
$$K_3 = h[-2(x_{n-1} + \tfrac{h}{2})(y_{n-1} + \tfrac{K_2}{2})^2]$$
$$K_4 = h[-2(x_{n-1} + h)(y_{n-1} + K_3)]$$
$$n = 1, 2, 3, \ldots$$

$\therefore \quad y_1 = y_0 + \frac{1}{6}[k_1 + 2(k_2+k_3) + k_4]$ ——— (1)

and $k_1 = h[-2x_0 y_0^2] = 0.2[-2(0)(1)] = 0$

$k_2 = h[-2(x_0+\frac{h}{2})(y_0+\frac{k_1}{2})^2] = -0.04$

$k_3 = h[-2(x_0+\frac{h}{2})(y_0+\frac{k_2}{2})^2] = -0.0384$

$k_4 = h[-2(x_0+h)(y_0+k_3)^2] = -0.0740$

So from (1)

$y_1 = 1 + \frac{1}{6}[0 + 2(-0.04 - 0.0384) - 0.0740]$

$\Rightarrow y_1 = y(0.2) = 0.9615$

$y_2 = y_1 + \frac{1}{6}[k_1 + 2(k_2+k_3) + k_4]$ ——— (2)

and $k_1 = h[-2x_1 y_1^2] = -0.0740$

$k_2 = h[-2(x_1+\frac{h}{2})(y_1+\frac{k_1}{2})^2] = -0.1026$

$k_3 = h[-2(x_1+\frac{h}{2})(y_1+\frac{k_2}{2})^2] = -0.0994$

$k_4 = h[-2(x_1+h)(y_1+k_3)^2] = -0.1189$

From (2)

$\Rightarrow y_2 = y(0.4) = 0.9615 + \frac{1}{6}[-0.0740 + 2(-0.1026 - 0.0994) - 0.1189]$

$\Rightarrow y_2 = y(0.4) = \underline{0.8620}$

5. MILNE'S METHOD

$$y' = \frac{dy}{dx} = f(x, y) \quad ; \quad y(x_0) = y_0$$

(a) PREDICTOR FORMULA

$$y_{n+1} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

$n = 3, 4, 5, \ldots$

(b) CORRECTOR FORMULA

$$y_{n+1}^{(m)} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y_{n+1}^{(m-1)'}\right]$$

$n = 3, 4, 5, \ldots$

$m = 1, 2, 3, \ldots$

* $y_{n+1}^{(0)}$ means the Predicted Value
* $y'_n = f(x_n, y_n)$

Ex.1 Find the solution of -

$$\frac{dy}{dx} = x - y^2 \quad ; \quad y(0) = 0$$

at $x = 0.8$. It is given that -

Find:-

x:	0.2	0.4	0.6
y:	0.02	0.0795	0.1762

Sol:- Here $x_0 = 0$, $y_0 = 0$, $h = 0.2$

and $x_1 = 0.2$, $y_1 = 0.02$

$x_2 = 0.4$, $y_2 = 0.0795$

$x_3 = 0.6$, $y_3 = 0.1762$

$x_4 = 0.8$, $y_4 = ?$

$\because \quad y'_n = x_n - y_n^2 \qquad \text{——} (1)$

$\therefore \quad y'_1 = x_1 - y_1^2 = 0.2 - (0.02)^2 = 0.1996$

$y'_2 = x_2 - y_2^2 = 0.4 - (0.0795)^2 = 0.39368$

$y'_3 = x_3 - y_3^2 = 0.6 - (0.1762)^2 = 0.56895$

By Predictor Formula

$$y_{n+1} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n] \quad \text{———} (2)$$

$$n = 3, 4, 5, \ldots$$

$$\therefore y_4 = y_0 + \frac{4h}{3}[2y'_1 - y'_2 + 2y'_3]$$

$$= 0 + \frac{4(0.2)}{3}[2(0.1996) - 0.39368 + 2(0.56895)]$$

$$\therefore y_4 = y(0.8) = 0.30491$$

Corrector Formula

$$y_{n+1}^{(m)} = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y_{n+1}^{(m-1)'}] \quad \text{———} (3)$$

$$n = 3, 4, 5, \ldots \; ; \; m = 1, 2, 3, \ldots$$

$$y_4^{(1)} = y_2 + \frac{h}{3}[y'_2 + 4y'_3 + y_4^{(0)'}]$$

$$y_4^{(1)} = y_2 + \frac{h}{3}[y'_2 + 4y'_3 + y'_4] \quad \text{———} (4)$$

From (1) $\quad y'_4 = x_4 - y_4^2 = 0.8 - (0.30491)^2$

$$= 0.70703$$

So from (4)

$$y_4^{(1)} = 0.0795 + \frac{0.2}{3}[0.39368 + 4(0.56895) + 0.70703]$$

$$= 0.30461$$

So $\quad y_4^{(1)'} = x_4 - (y_4^{(1)})^2 = 0.70721$

From (4)

$$y_4^{(2)} = y_2 + \frac{h}{3}[y'_2 + 4y'_3 + y_4^{(1)'}] = 0.30462$$

$$\because y_4^{(1)} = y_4^{(2)} = 0.3046$$

$$\therefore y_4 = y(0.8) = \underline{0.3046}$$

Ex.2 Solve the differential eq^n.

$$\frac{dy}{dx} = x^2(1+y) \quad ; \quad y(1) = 1$$

at $x = 1.4$.

Sol:- Let us subdivide the interval $(1, 1.4)$ into four parts, and $h = \frac{1.4-1}{4} = 0.1$

So here we have-

$x_0 = 1 \quad ; \quad y_0 = 1$

$x_1 = 1.1, \quad x_2 = 1.2, \quad x_3 = 1.3 \quad \& \quad x_4 = 1.4$

Here values of y_1, y_2 & y_3 are not known, so Let us approximate them with the help of Euler's Method as-

$\because$ Euler's Method is -

$$y_n = y_{n-1} + h f(x_{n-1}, y_{n-1})$$
$$n = 1, 2, 3, \cdots$$

$$\therefore \quad y_n = y_{n-1} + h\left[x_{n-1}^2 (1+y_{n-1})\right]$$

$$\therefore \quad y_1 = y_0 + h\left[x_0^2(1+y_0)\right]$$
$$= 1 + 0.1\left[1^2(1+1)\right] = 1.2$$

$$y_2 = y_1 + h\left[x_1^2(1+y_1)\right]$$
$$= 1.2 + 0.1\left[(1.1)^2(1+1.2)\right] = 1.4662$$

$$y_3 = y_2 + h\left[x_2^2(1+y_2)\right]$$
$$= 1.4662 + 0.1\left[(1.2)^2(1+1.4662)\right] = 1.8213$$

So we have values-

$x_0 = 1 \quad ; \quad y_0 = 1$

$x_1 = 1.1 \quad ; \quad y_1 = 1.2$

$x_2 = 1.2 \quad ; \quad y_2 = 1.4662$

$x_3 = 1.3 \quad ; \quad y_3 = 1.8213$

$x_4 = 1.4 \quad ; \quad y_4 = ?$

$\because$ The given eq^n is-

$$y' = \frac{dy}{dx} = x^2(1+y) \quad \text{or} \quad y_n' = x_n^2(1+y_n)$$

$$\therefore \quad y_1' = x_1^2(1+y_1) = (1.1)^2[1+1.2] = 2.662$$

$$y_2' = x_2^2(1+y_2) = (1.2)^2[1+1.4662] = 3.5513$$

$$y_3' = x_3^2(1+y_3) = (1.3)^2[1+1.8213] = 4.768$$

By Predictor Formula

$$y_4 = y_0 + \frac{4}{3}h[2y_1' - y_2' + 2y_3']$$

$$= 1 + \frac{4}{3}(0.1)[2(2.662) - 3.5513 + 2(4.768)]$$

$$y_4 = 2.5078 = y_4^{(0)} \quad \text{So } y_4^{(0)'} = x_4^2(1+y_4^{(0)}) = 6.8753$$

By Corrector Formula

$$y_4^{(m)} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4^{(m-1)'}]$$

$$\text{so } y_4^{(1)} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4^{(0)'}]$$

$$= 1.4662 + \frac{0.1}{3}[3.5513 + 4(4.768) + 6.8753]$$

$$= 2.4495$$

$$\therefore \quad y_4^{(1)'} = x_4^2(1+y_4^{(1)}) = 6.76102$$

$$y_4^{(2)} = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4^{(1)'}]$$

$$= 1.4662 + \frac{0.1}{3}[3.5513 + 4(4.768) + 6.76102]$$

$$= 2.4495$$

$$\because \quad y_4^{(1)} = y_4^{(2)} = 2.4495$$

$$\text{so } y_4 = y(1.4) = 2.4495.$$

NUMERICAL SOLUTION OF ALGEBRAIC & TRANS. EQN.

ALGEBRAIC & TRANSCENDENTAL EQ^N.

An expression of the form-

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n \; ; a_0 \neq 0, \; n \in I^+$$

is known a polynomial of degree 'n'. where $a_0, a_1, a_2, \ldots a_n$ are constants.

The eq^n $f(x) = 0$ is called a polynomial eq^n or Algebraic eq^n.

If $f(x)$ contains some other functions such as logarithmic, exponential, Trigonometric etc., then the eq^n $f(x) = 0$ is called as Transcendental eq^n.

Solution :- The value of x which satisfy eq^n $f(x) = 0$ —(1) is called solution or zeros or roots of the equation.

Geometrically a root of the eq^n $f(x) = 0$ is that value of x, where the graph $y = f(x)$ crosses x-axis.

Properties of Equations :-

1. An equation of degree 'n' has exactly 'n' roots. (Real or Complex)
2. complex root appear with its complex conjugate. i.e. $\alpha - i\beta$ must appear with $\alpha + i\beta$.
3. If $f(x)$ is continuous in $[a, b]$ and
   (i) $f(a) \cdot f(b) < 0$, then atleast One or an odd number of roots of the eq^n $f(x) = 0$ lying between a & b.
   (ii) $f(a) \cdot f(b) > 0$, then either no roots or even number of roots lying

between a & b.

Descart's Rule of Sign:-

(a) The eq^n $f(x)=0$ Cannot have the number of positive roots more than the number of changes of signs (+ive to −ive or −ive to +ive) in $f(x)$.

(b) The eq^n $f(x)=0$ Cannot have the number of negative roots more than the number of changes of signs in $f(-x)$.

Ex. Let us consider the eq^n-

$$f(x) = 3x^6 - 4x^5 + 5x^2 - 6x - 7 = 0$$

Sign of the terms are-

+ − + − −

1 2 3

As the total number of sign changes are 3 for $f(x)=0$, so the eq^n does not have positive roots more than 3.

$$\therefore f(-x) = 3x^6 + 4x^5 + 5x^2 + 6x - 7 = 0$$

\+ + + + −

1

As the total number of sign changes are 1, so the eq^n $f(x)=0$ does not have negative roots more than 1.

ROOT FINDING

1. GRAPHICAL METHOD

Let the eq^n $f(x)=0$ ——— (1)

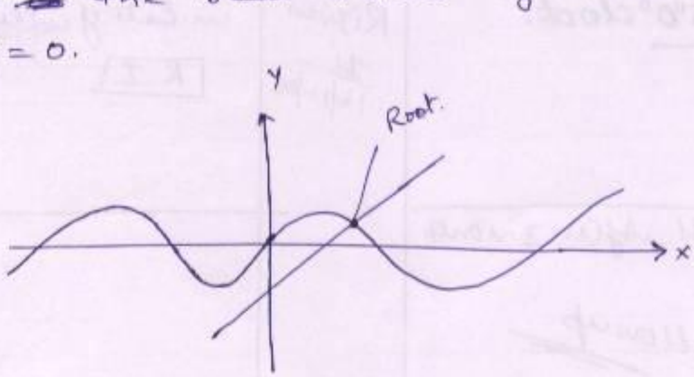
which can be written as -

$$f_1(x) = f_2(x) \quad \text{———} (2)$$

Let us plot graphs of two functions -

$$y = f_1(x) \quad \text{and} \quad y = f_2(x)$$

where the curves of functions intersect, that are the real roots of the eq^n $f(x)=0$.



2. BISECTION METHOD

Let the eq^n is $f(x)=0$ ——— (1)

Let a & b points are such that

$$f(a) \cdot f(b) < 0$$

$\Rightarrow$ The root of (1) lies in (a,b)

The first approximation of the root is given by -

$$x^{(1)} = \frac{a+b}{2} \quad \text{———} (2)$$

* If $f(x^{(1)})=0$, it means $x = x^{(1)}$ is the correct root of eq^n $f(x)=0$.
* If $f(x^{(1)}) \neq 0$ then the root lies between $(a, x^{(1)})$ or $(x^{(1)}, b)$ according to the

situation, whether $f(x^{(1)})$ is +ive or −ive.
Let the root lies in $(a, x^{(1)})$ then the Second approximation of the root is-

$$x^{(2)} = \frac{a + x^{(1)}}{2}$$

Again If $f(x^{(2)}) = 0$, then $x^{(2)}$ is the correct root of $f(x)=0$. Otherwise the root lies in either $(a, x^{(2)})$ or $(x^{(2)}, x^{(1)})$ according to situation whether $f(x^{(2)})$ is +ive or −ive.

Again we bisect the obtained interval as before and obtain the next approximation. Continue the process of bisection until the root is found to the desired degree of accuracy.

<u>Ex.1</u> Find the root of the equation-

$$x^3 - 9x + 1 = 0$$

correct up to three places of decimal.

<u>Sol:-</u> Here $f(x) = x^3 - 9x + 1 = 0$ ——— (1)

$\because f(0) = 1 > 0$

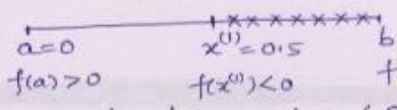
$f(1) = -7 < 0$

$\Rightarrow$ Root of (1) lies in $(0,1) \equiv (a,b)$

By Bisection Method-

$$x^{(1)} = \frac{a+b}{2} = \frac{0+1}{2} = 0.5$$

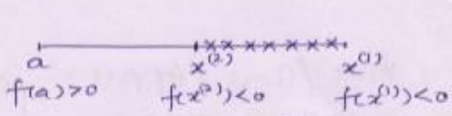
$$f(x^{(1)}) = -3.375 < 0$$



So the root lies in $(0, 0.5) \equiv (a, x^{(1)})$

$$x^{(2)} = \frac{a + x^{(1)}}{2} = \frac{0 + 0.5}{2} = 0.25$$

$$f(x^{(2)}) = -1.1234 < 0$$

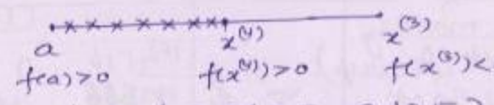


So the root lies in $(0, 0.25) \equiv (a, x^{(2)})$

$x^{(3)} = \frac{a + x^{(2)}}{2} = 0.125$; $f(x^{(3)}) = -0.123 < 0$

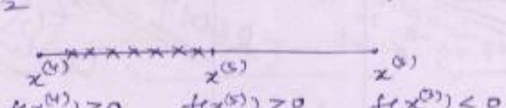
so root lies in $(0, 0.125) \equiv (a, x^{(3)})$

$x^{(4)} = \frac{a + x^{(3)}}{2} = 0.0625$; $f(x^{(4)}) = 0.4377 > 0$



$\Rightarrow$ root lies in $(0.0625, 0.125) \equiv (x^{(4)}, x^{(3)})$

$x^{(5)} = \frac{x^{(4)} + x^{(3)}}{2} = 0.0938$; $f(x^{(5)}) = 0.1566 > 0$

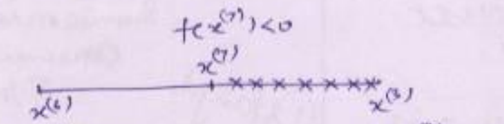


$\Rightarrow$ root lies in $(0.0938, 0.125) \equiv (x^{(5)}, x^{(3)})$

$x^{(6)} = \frac{x^{(5)} + x^{(3)}}{2} = 0.1094$; $f(x^{(6)}) = 0.0167$

$\Rightarrow$ root lies in $(0.1094, 0.125) \equiv (x^{(6)}, x^{(3)})$

$x^{(7)} = \frac{x^{(6)} + x^{(3)}}{2} = 0.1172$; $f(x^{(7)}) = -0.5319 < 0$



$\Rightarrow$ root lies in $(0.1094, 0.1172) \equiv (x^{(6)}, x^{(7)})$

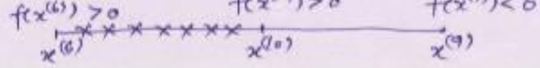
$x^{(8)} = \frac{x^{(6)} + x^{(7)}}{2} = 0.1133$; $f(x^{(8)}) = -0.0182 < 0$

$\Rightarrow$ root lies in $(0.1094, 0.1133) = (x^{(6)}, x^{(8)})$

$x^{(9)} = \frac{x^{(6)} + x^{(8)}}{2} = 0.1114$; $f(x^{(9)}) = -0.0012 < 0$

$\Rightarrow$ root lies in $(0.1094, 0.1114) \equiv (x^{(6)}, x^{(9)})$

$x^{(10)} = \frac{x^{(6)} + x^{(9)}}{2} = 0.1104$; $f(x^{(10)}) = 0.1109 > 0$



$\Rightarrow$ root lies in $(0.1104, 0.1114) \equiv (x^{(10)}, x^{(9)})$

$x^{(11)} = \dfrac{x^{(10)} + x^{(9)}}{2} = 0.1109$; $f(x^{(11)}) = 0.00326 > 0$

$\Rightarrow$ root lies in $(0.1109, 0.1114) \equiv (x^{(11)}, x^{(9)})$

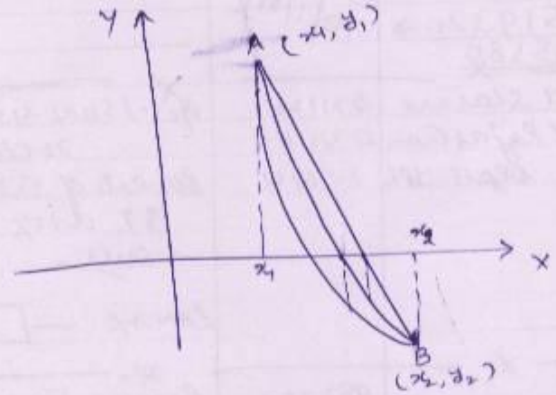
$x^{(12)} = \dfrac{x^{(11)} + x^{(9)}}{2} = 0.1112$; $f(x^{(12)}) = 0.000575 > 0$

$\Rightarrow$ root lies in $(0.1112, 0.1114) \equiv (x^{(12)}, x^{(9)})$

$\because x^{(9)} - x^{(12)} = 0.1114 - 0.1112 = 0.0002$

$\Rightarrow$ which means the root $x = 0.111$ is Correct up to Third decimal place.

3. REGULA FALSI METHOD



Let us find two points x_1 and x_2 such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$ are of opposite signs. i.e. The line joining $A(x_1, y_1)$ & $B(x_2, y_2)$ Crosses x-axis.

The equation of the line AB is-

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

The line cuts the x-axis at $x = x_3$ (say) then $y = 0$, so-

$$-y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x_3 - x_1)$$

or $\boxed{x^{(1)} = x_3 = \dfrac{x_1 y_2 - x_2 y_1}{y_2 - y_1}}$

or $\boxed{x^{(1)} = x_3 = \dfrac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}}$

which is first approximation of the root. Now the root lies in $(x_1, x^{(1)})$ or $(x^{(1)}, x_2)$ according to situation that the value of $f(x^{(1)})$ is +ive or -ive. Get new interval (x_1, x_2) and repeat the same process to get $x^{(2)}$ and other approximation till the root is found to the desired degree of accuracy.

Ex. Find a real root of the equation-

$$x \log_{10} x - 1.2 = 0$$

Sol:- Here $f(x) = x \log_{10} x - 1.2 = 0$ ——— (1)

$f(1) = -1.2 < 0$

$f(2) = -0.59794 < 0$

$f(3) = 0.23136 > 0$

$\Rightarrow$ root lies in $(2, 3) \equiv (x_1, x_2)$

By Regula Falsi Method-

$$x^{(1)} = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{2(0.23136) - 3(-0.59794)}{0.23136 + 0.59794}$$

$\Rightarrow x^{(1)} = 2.72102$

& $f(x^{(1)}) = -0.01709 < 0$

So the root lies in $(2.72102, 3) \equiv (x_1, x_2)$

$$x^{(2)} = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{2.72102\,(0.23136) - 3(-0.01709)}{0.23136 + 0.01709}$$

$\Rightarrow x^{(2)} = 2.74021$

& $f(x^{(2)}) = -0.00038 < 0$

$\Rightarrow$ root lies in $(2.74021, 3) \equiv (x_1, x_2)$

$$x^{(3)} = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{2.74021\,(0.23136) - 3(-0.00038)}{0.23136 + 0.00038}$$

$\Rightarrow x^{(3)} = 2.74064$

& $f(x^{(3)}) = -0.0000053 < 0$

$\Rightarrow$ root lies in $(2.74064, 3) \equiv (x_1, x_2)$

$$x^{(4)} = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{2.74064(0.23136) - 3(-0.0000053)}{0.23136 + 0.0000053}$$

$\Rightarrow x^{(4)} = 2.74065$

& $f(x^{(4)}) = -0.000000034 < 0$

Here we can observe that

as $x \to 2.7406......$ then $f(x) \longrightarrow 0$

$\therefore$ it is quite obvious that

$x^{(3)} = x^{(4)} = 2.7406$ (Correct to 3 Decimal places)

$\therefore$ $\underline{x = 2.7406}$.

4. NEWTON RAPHSON MEHOD

Let x_0 be the approximate value of the root of $f(x)=0$ and $x_1 = x_0 + h$ be the exact value of the root of $f(x) = 0$, where h is a very small quantity. (i.e. $h \to 0$)

So $f(x_1) = f(x_0 + h) = 0$ ——— (1)

By Taylor Series expansion -

$$f(x_0+h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots\cdots = 0$$

neglecting second & higher order terms of h -

$$f(x_0) + h f'(x_0) = 0$$

$$\Rightarrow \quad h = -\frac{f(x_0)}{f'(x_0)} \quad ; \quad f'(x_0) \neq 0$$

$$\therefore \quad x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)} \quad \text{———} (2)$$

which is a closer approximation of the root of $f(x) = 0$. Similarly starting with x_1, the next better approximation x_2 is given by -

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \quad \text{———} (3)$$

Similarly the $(n+1)^{th}$ better approximation of the root is given by -

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \; ; \; f'(x_n) \neq 0 \\ n = 0, 1, 2, 3, \cdots}$$

Ex. 1 Find the real root of the equation -

$$3x = \cos x + 1$$

Sol:- Let $f(x) = 3x - \cos x - 1 = 0$ ——— (1)

$$f'(x) = 3 + \sin x \quad \text{———} (2)$$

$f(0) = -2 < 0$

$f(1) = 1.459698 > 0$

$\Rightarrow$ Root of (1) lies in $(0, 1)$.

Let the initial value of the root of (1) is $x = x_0 = 0$

By Newton Raphson method -

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad ; \quad n = 0, 1, 2, \cdots$$

$$\therefore\ x_{n+1} = x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$$

$$\Rightarrow x_{n+1} = \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n}$$

$n = 0, 1, 2, \cdots$

$$\therefore\ x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} = \frac{2}{3} = 0.6667$$

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1} = 0.6075$$

$$x_3 = \frac{x_2 \sin x_2 + \cos x_2 + 1}{3 + \sin x_2} = 0.6071$$

$$x_4 = \frac{x_3 \sin x_3 + \cos x_3 + 1}{3 + \sin x_3} = 0.6071$$

$\because\ x_3 = x_4 = 0.6071$ [up to Four decimal places]

So $\underline{x = 0.6071}$

NUMERICAL SOLUTION of ALG. SIMULTANEOUS EQN

(A) Direct Methods:- By these methods, the exact solution of sys^m of eq^n can be obtained.

1. GAUSS ELIMINATION METHOD:-

Let us consider the sys^m of eq^n as-

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1 \quad \text{—— (i)}$$
$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2 \quad \text{—— (ii)}$$
$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3 \quad \text{—— (iii)}$$

In this method, the given sys^m of eq^n will be reduced to an upper triangular sys^m and then the unknowns can be found easily.

For the purpose, firstly we eliminate x_1 from (ii) & (iii)

$$(ii) \rightarrow (ii) + \left(-\frac{a_{21}}{a_{11}}\right) \times (i) \quad \&$$
$$(iii) \rightarrow (iii) + \left(-\frac{a_{31}}{a_{11}}\right) \times (i)$$

So
$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1 \quad \text{—— (iv)}$$
$$a'_{22} x_2 + a'_{23} x_3 = b'_2 \quad \text{—— (v)}$$
$$a'_{32} x_2 + a'_{33} x_3 = b'_3 \quad \text{—— (vi)}$$

Now Let us eliminate x_2 from (vi) with the help of (v).

$$(vi) \rightarrow (vi) + \left(-\frac{a'_{32}}{a'_{22}}\right) \times (v)$$, So

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1 \quad \text{—— (vii)}$$
$$a'_{22} x_2 + a'_{23} x_3 = b'_2 \quad \text{—— (viii)}$$
$$a''_{33} x_3 = b''_3 \quad \text{—— (ix)}$$

Now the sys^m of eq^n is in upper triangular form and can be solved by back Substitution.
Here a_{11}, a'_{22} and a''_{33} which are supposed to be non-zero, are known as pivot elements.

PIVOTING:- If any one of the pivot elements vanish or becomes very small, compared to other elements (co-efficients of variables) in that row, then we rearrange the remaining rows so as to obtain a non vanishing pivot or to avoid the multiplication with a large number. This process is called pivoting, which is of two types.

(a) Partial Pivoting:-
— In the first stage of elimination, the first column is searched for the largest element in magnitude and brought as the first pivot by interchanging the first eq^n with the eq^n that having the first pivot.

Ex. Let us consider the sys^m of eq^n as-

$$x_1 + x_2 + x_3 = 6$$
$$3x_1 + 3x_2 + 4x_3 = 20$$
$$2x_1 + 4x_2 + 3x_3 = 13$$

As for the first column or the co-efficient of x_1 are 1, 3 & 2 and 3 is the largest in magnitude, so let us write this eq^n firstly.

so the sys^m is -

$$3x_1 + 3x_2 + 4x_3 = 20$$
$$x_1 + x_2 + x_3 = 6$$
$$2x_1 + 4x_2 + 3x_3 = 13$$

- In the second elimination stage, the Second column is searched for largest element in magnitude among rest of equations, and this element is brought as Second Pivot ~~element~~ by interchange of second eq^n with the eq^n that having second Pivot.

So in above Sys^m, the magnitude of +4 is greater than magnitude of -1, so Let us write the eq^n as -

$$3x_1 + 3x_2 + 4x_3 = 20$$
$$2x_1 + 4x_2 + 3x_3 = 13$$
$$x_1 + x_2 + x_3 = 6$$

- This process is continued untill we arrived at the Last pivot.

(b) Complete Pivoting :- This pivoting requires not only an interchange of equations but also an interchange of the position of variables.

So after Complete pivoting, the above Sys^m Can be written as -

$$4x_3 + 3x_2 + 3x_1 = 20$$
$$3x_3 + 4x_2 + 2x_1 = 13$$
$$x_3 + x_2 + x_1 = 6$$

Ex.1 Solve the sys^m of eq^n -

$$x_1 + 2x_2 + 2x_3 = 6$$
$$3x_1 + 3x_2 + 4x_3 = 20$$
$$2x_1 + 3x_2 + 3x_3 = 13$$

Sol:- After Partial Pivoting, the system can be written as-

$$3x_1 + 3x_2 + 4x_3 = 20 \quad \text{---} (1)$$
$$2x_1 + 3x_2 + 3x_3 = 13 \quad \text{---} (2)$$
$$x_1 + 2x_2 + 2x_3 = 6 \quad \text{---} (3)$$

$(2) \rightarrow (2) + \left(-\frac{2}{3}\right) \times (1)$

$(3) \rightarrow (3) + \left(-\frac{1}{3}\right) \times (1)$

so
$$3x_1 + 3x_2 + 4x_3 = 20 \quad \text{---} (4)$$
$$+x_2 + \frac{1}{3}x_3 = -\frac{1}{3} \quad \text{---} (5)$$
$$x_2 + \frac{2}{3}x_3 = -\frac{2}{3} \quad \text{---} (6)$$

$(6) \rightarrow (6) + (-1) \times 5$

so
$$3x_1 + 3x_2 + 4x_3 = 20 \quad \text{---} (7)$$
$$x_2 + \frac{1}{3}x_3 = -\frac{1}{3} \quad \text{---} (8)$$
$$\frac{1}{3}x_3 = -\frac{1}{3} \quad \text{---} (9)$$

From (9) $x_3 = -1$

From (8) $x_2 = 0$

From (7) $x_1 = \frac{16}{3}$

2. GAUSS-JORDAN METHOD:-

Let us consider the system of eq^n as-

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \quad \text{---} (1)$$

which can be written as-

$$Ax = B \quad \text{---} (2)$$

Ex.1 Solve the sys^m of eq^n -

$$x_1 + 2x_2 + 2x_3 = 6$$
$$3x_1 + 3x_2 + 4x_3 = 20$$
$$2x_1 + 3x_2 + 3x_3 = 13$$

Sol:- After Partial Pivoting, the system can be written as -

$$3x_1 + 3x_2 + 4x_3 = 20 \quad \text{———} (1)$$
$$2x_1 + 3x_2 + 3x_3 = 13 \quad \text{———} (2)$$
$$x_1 + 2x_2 + 2x_3 = 6 \quad \text{———} (3)$$

$(2) \rightarrow (2) + \left(-\frac{2}{3}\right) \times (1)$

$(3) \rightarrow (3) + \left(-\frac{1}{3}\right) \times (1)$

so
$$3x_1 + 3x_2 + 4x_3 = 20 \quad \text{———} (4)$$
$$+ x_2 + \frac{1}{3}x_3 = -\frac{1}{3} \quad \text{———} (5)$$
$$x_2 + \frac{2}{3}x_3 = -\frac{2}{3} \quad \text{———} (6)$$

$(6) \rightarrow (6) + (-1) \times 5$

so
$$3x_1 + 3x_2 + 4x_3 = 20 \quad \text{———} (7)$$
$$x_2 + \frac{1}{3}x_3 = -\frac{1}{3} \quad \text{———} (8)$$
$$\frac{1}{3}x_3 = -\frac{1}{3} \quad \text{———} (9)$$

From (9) $x_3 = -1$

From (8) $x_2 = 0$

From (7) $x_1 = \frac{16}{3}$

2. GAUSS-JORDAN METHOD:-

Let us consider the system of eq^n as -

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \quad \text{———} (1)$$

which can be written as -

$$Ax = B \quad \text{———} (2)$$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The Corresponding Augmented matrix is-

$$[A|B] = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

By Matrix Row operations, we will Convert the Part of A in to a diagonal matrix-

$$[A|B] \approx \left[\begin{array}{ccc|c} d_{11} & 0 & 0 & b_1' \\ 0 & d_{22} & 0 & b_2' \\ 0 & 0 & d_{33} & b_3' \end{array}\right]$$

$\therefore$ the corresponding sys^m of eq^n is-

$$d_{11} x_1 = b_1'$$
$$d_{22} x_2 = b_2'$$
$$d_{33} x_3 = b_3'$$

and by which we can obtain the Values of variables.

Ex.1 Solve -

$$x_1 + 2x_2 + x_3 = 8$$
$$2x_1 + 3x_2 + 4x_3 = 20$$
$$4x_1 + 3x_2 + 2x_3 = 16$$

Sol:- The given sys^m of eq^n can be written as-

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 20 \\ 16 \end{bmatrix}$$

or $Ax = B$

where $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 4 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 8 \\ 20 \\ 16 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The augmented matrix -

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 2 & 3 & 4 & 20 \\ 4 & 3 & 2 & 16 \end{array}\right]$$

$$\left.\begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{array}\right\} = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -1 & 2 & 4 \\ 0 & -5 & -2 & -16 \end{array}\right] \text{ or } \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -1 & 2 & 4 \\ 0 & 5 & 2 & 16 \end{array}\right]$$

$$\left.\begin{array}{l} R_1 \to R_1 + 2R_2 \\ R_2 \to R_2 - R_3 \end{array}\right\} = \left[\begin{array}{ccc|c} 1 & 0 & 5 & 16 \\ 0 & -6 & 0 & -12 \\ 0 & 5 & 2 & 16 \end{array}\right] \text{ or } \left[\begin{array}{ccc|c} 1 & 0 & 5 & 16 \\ 0 & 1 & 0 & 2 \\ 0 & 5 & 2 & 16 \end{array}\right]$$

$$R_3 \to R_3 - 5R_2 \} = \left[\begin{array}{ccc|c} 1 & 0 & 5 & 16 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 6 \end{array}\right] \text{ or } \left[\begin{array}{ccc|c} 1 & 0 & 5 & 16 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$R_1 \to R_1 - 5R_3 \} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

so the sys^m of eq^n is -

$x_1 = 1$
$x_2 = 2$
$x_3 = 3$

which is the solution.

3. FACTORIZATION
OR
TRIANGULARIZATION
OR
DE COMPOSITION
} METHOD

For this method, it is essential that in the eq^n $AX = B$, the co-efficient matrix 'A' must be a square matrix.

This method is based upon the fact that 'A' can be factorized into the form of LU, Provided that all leading minors of 'A' are non zero. [L and U are Lower and Upper Triangular Matrices].

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

then its leading/Principal minors are-

$$A_1 = |a_{11}| = a_{11} \neq 0$$

$$A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$$

$$\& \quad A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{33} & a_{33} \end{vmatrix} \neq 0$$

Let the sys^m of eq^n is-

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \quad \text{———} (1)$$

which can be written as-

$$AX = B \quad \text{———} (2)$$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ & $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Let $A = LU$ ——— (3)

where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ & $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

So $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

or $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{11} l_{21} & u_{12} l_{21} + u_{22} & u_{13} l_{21} + u_{23} \\ u_{11} l_{31} & u_{12} l_{31} + u_{22} l_{32} & u_{13} l_{31} + u_{23} l_{32} + u_{33} \end{bmatrix}$

By comparing these matrices, we can find values of all u's and l's.

By (2) & (3)

$LUX = B$ ——— (4)

Again Let $UX = Y$ ——— (5)

where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

By (4) & (5)

$LY = B$ ——— (6)

$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

By which the values of y_1, y_2 & y_3 can be obtained. Then from (5)

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

By which the values of x_1, x_2 & x_3 can be obtained easily.

Ex.1 Solve-

$$2x_1 + 3x_2 + x_3 = 9$$
$$x_1 + 2x_2 + 3x_3 = 6$$
$$3x_1 + x_2 + 2x_3 = 8$$

Sol:- The given sysm of eqn can be written as-

$$AX = B \quad \text{——— (1)}$$

where $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Let $A = LU$ ——— (2), which gives-

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{11}l_{21} & u_{12}l_{21} + u_{22} & u_{13}l_{21} + u_{23} \\ u_{11}l_{31} & u_{12}l_{31} + u_{22}l_{32} & u_{13}l_{31} + u_{23}l_{32} + u_{33} \end{bmatrix}$$

$\therefore$ $u_{11} = 2$, $u_{12} = 3$, $u_{13} = 1$

$u_{11}l_{21} = 1 \Rightarrow l_{21} = 1/2$

$u_{11}l_{31} = 3 \Rightarrow l_{31} = 3/2$

$u_{12}l_{21} + u_{22} = 2 \Rightarrow u_{22} = 1/2$

$u_{12}l_{31} + u_{22}l_{32} = 1 \Rightarrow l_{32} = -7$

$u_{13}l_{21} + u_{23} = 3 \Rightarrow u_{23} = 5/2$

$u_{13}l_{31} + u_{23}l_{32} + u_{33} = 2 \Rightarrow u_{33} = 18.$

So $L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix}$ & $U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}$

From (1) & (2) -

$$LUX = B \quad \text{——— (3)}$$

Let $Ux = Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ ——— (4)

so $LY = B$ ——— (5)

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

which gives-

$y_1 = 9$ ——— (6)

$\frac{1}{2} y_1 + y_2 = 6$ ——— (7)

$\frac{3}{2} y_1 - 7y_2 + y_3 = 8$ ——— (8)

which gives-

$y_1 = 9, \quad y_2 = 3/2 \quad \& \quad y_3 = 5$

Now from (4)

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 3/2 \\ 5 \end{bmatrix}$$

or
$$2x_1 + 3x_2 + x_3 = 9$$
$$\frac{1}{2} x_2 + \frac{5}{2} x_3 = \frac{3}{2}$$
$$18 x_3 = 5$$

which gives-

$x_3 = \frac{5}{18}, \quad x_2 = \frac{29}{18} \quad \& \quad x_1 = \frac{35}{18}.$

B. ITERATIVE METHOD :-

An iterative method is that in which, we start from an approximation to the exact solution and obtain better and better approximations from a cycle of computations, which are repeated as often as may be necessary to achieve the desired degree of accuracy.

1. JACOBI METHOD

Let us consider a sys^m of eq^n –

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n = b_3 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{array}\right\} \text{——(1)}$$

where -

$$\left.\begin{array}{l} |a_{11}| > |a_{12}| + |a_{13}| + \cdots + |a_{1n}| \\ |a_{22}| > |a_{21}| + |a_{23}| + \cdots + |a_{2n}| \\ |a_{33}| > |a_{31}| + |a_{32}| + \cdots + |a_{3n}| \\ \vdots \\ |a_{nn}| > |a_{n1}| + |a_{n2}| + \cdots + |a_{nn}| \end{array}\right\} \text{——(2)}$$

i.e. the coefficients of leading diagonals are large as compared to others. Then the sys^m (1) is Known as "Diagonally Dominant System".

If the system is not diagonally dominent, then we will convert it in to a diagonally dominant sys^m otherwise it can't be solved by iterative methods.

Then, the n^{th} approximation of the solution is given by-

$$x_1^{(n)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(n-1)} - a_{13}x_3^{(n-1)} - \cdots - a_{1m}x_m^{(n-1)}\right]$$

$$x_2^{(n)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(n-1)} - a_{23}x_3^{(n-1)} - \cdots - a_{2m}x_m^{(n-1)}\right]$$

$$\vdots$$

$$x_m^{(n)} = \frac{1}{a_{mm}}\left[b_m - a_{m1}x_1^{(n-1)} - a_{m2}x_2^{(n-1)} - \cdots - a_{mm-1}x_{m-1}^{(n-1)}\right]$$

$$n = 1, 2, 3, \cdots$$

In the absence of any better estimate for $x_1^{(0)}, x_2^{(0)}, \ldots x_m^{(0)}$, these can be taken as zero.

Ex.1 Solve-

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

up to Four iterations.

Sol:- As the given sys^m of eq^n is diagonally dominant, so we can proceed for the iterative procedure.

By Jacobi's Method, the n^{th} approximation of solution of sys^m of eq^n can be given by-

$$x^{(n)} = \frac{1}{20}\left[17 - y^{(n-1)} + 2z^{(n-1)}\right]$$

$$y^{(n)} = \frac{1}{20}\left[-18 - 3x^{(n-1)} + z^{(n-1)}\right]$$

$$z^{(n)} = \frac{1}{20}\left[25 - 2x^{(n-1)} + 3y^{(n-1)}\right] \quad n = 1, 2, 3, \ldots$$

Let the initial approximations are-

$$x^{(0)} = y^{(0)} = z^{(0)} = 0$$

<u>First Iteration ($n=1$)</u>

$$x^{(1)} = \frac{1}{20}\left[17 - y^{(0)} + 2z^{(0)}\right] = \frac{1}{20}\left[17 - 0 + 0\right] = 0.85$$

$$y^{(1)} = \frac{1}{20}\left[-18 - 3x^{(0)} + z^{(0)}\right] = \frac{1}{20}\left[-18 - 0 + 0\right] = -0.9$$

$$z^{(1)} = \frac{1}{20}\left[25 - 2x^{(0)} + 3y^{(0)}\right] = \frac{1}{20}\left[25 - 0 + 0\right] = 1.25$$

<u>Second Iteration ($n=2$)</u>

$$x^{(2)} = \frac{1}{20}\left[17 - y^{(1)} + 2z^{(1)}\right] = \frac{1}{20}\left[17 - (-0.9) + 2(1.25)\right] = 1.02$$

$$y^{(2)} = \frac{1}{20}\left[-18 - 3x^{(1)} + z^{(1)}\right] = \frac{1}{20}\left[-18 - 3(0.85) + 1.25\right] = -0.965$$

$$z^{(2)} = \frac{1}{20}\left[25 - 2x^{(1)} + 3y^{(1)}\right] = \frac{1}{20}\left[25 - 2(0.85) + 3(-0.9)\right] = 1.1515$$

<u>Third Iteration ($n=3$)</u>

$$x^{(3)} = \frac{1}{20}\left[17 - y^{(2)} + 2z^{(2)}\right] = \frac{1}{20}\left[17 - (-0.965) + 2(1.1515)\right] = 1.0134$$

$$y^{(3)} = \frac{1}{20}\left[-18 - 3x^{(2)} + z^{(2)}\right] = \frac{1}{20}\left[-18 - 3(1.02) + 1.1515\right] = -0.9954$$

$$z^{(3)} = \frac{1}{20}\left[25 - 2x^{(2)} + 3y^{(2)}\right] = \frac{1}{20}\left[25 - 2(1.02) + 3(-0.965)\right] = 1.0032$$

<u>Fourth Iteration ($n=4$)</u>

$$x^{(4)} = \frac{1}{20}\left[17 - y^{(3)} + 2z^{(3)}\right] = \frac{1}{20}\left[17 - (-0.9954) + 2(1.0032)\right] = 1.0009$$

$$y^{(4)} = \frac{1}{20}\left[-18 - 3x^{(3)} + z^{(3)}\right] = \frac{1}{20}\left[-18 - 3(1.0134) + 1.0032\right] = -1.0018$$

$$z^{(4)} = \frac{1}{20}\left[25 - 2x^{(3)} + 3y^{(3)}\right] = \frac{1}{20}\left[25 - 2(1.0134) + 3(-0.9954)\right] = 0.9993$$

So after Four Iterations,

$x = 1.0009 \approx 1$

$y = -1.0018 \approx -1$

$z = 0.9993 \approx 1$

2. GAUSS- SEIDAL METHOD

As the sys^m of eq^n was diagonally dominant and then the Jacobi's method was given. Similarly, if the given sys^m is diagonally dominent then Gauss- Seidal method is given by -

$$x_1^{(n)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(n-1)} - a_{13}x_3^{(n-1)} - \cdots - a_{1m}x_m^{(n-1)}\right]$$

$$x_2^{(n)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(n)} - a_{23}x_3^{(n-1)} \cdots - a_{2m}x_m^{(n-1)}\right]$$

$$\vdots$$

$$x_m^{(n)} = \frac{1}{a_{mm}}\left[b_m - a_{m1}x_1^{(n)} - a_{m2}x_2^{(n)} - \cdots - a_{mm-1}x_{m-1}^{(n)}\right]$$

$$n = 1, 2, 3, 4, \ldots$$

In this method, the Latest Value of the variable is used, so this method is better than Jacobi's Method. The Convergence of this method is twice as of Jacobi's Method.

The Process of iteration is stopped when we achieve the desired degree of accuracy.

Ex.1 Solve -

$$10x + y + z = 12$$
$$2x + 10y + z = 13$$
$$2x + 2y + 10z = 14$$

Correct up to three decimal places.

Sol:- As the given Sys^m is diagonally dominent, so ~~let~~ the Gauss - Seidal method is given by -

$$x^{(n)} = \frac{1}{10}\left[12 - y^{(n-1)} - z^{(n-1)}\right]$$

$$y^{(n)} = \frac{1}{10}\left[13 - 2x^{(n)} - z^{(n-1)}\right]$$

$$z^{(n)} = \frac{1}{10}\left[14 - 2x^{(n)} - 2y^{(n)}\right]$$

$$n = 1, 2, 3, \ldots$$

Let the initial approximation is-

$$x^{(0)} = y^{(0)} = z^{(0)} = 0$$

First Iteration $(n=1)$

$$x^{(1)} = \frac{1}{10}\left[12 - y^{(0)} - z^{(0)}\right] = \frac{1}{10}\left[12 - 0 - 0\right] = 1.2$$

$$y^{(1)} = \frac{1}{10}\left[13 - x^{(1)} - z^{(0)}\right] = \frac{1}{10}\left[13 - 1.2 - 0\right] = 1.06$$

$$z^{(1)} = \frac{1}{10}\left[14 - 2x^{(1)} - 2y^{(1)}\right] = \frac{1}{10}\left[14 - 2(1.2) - 2(1.06)\right] = 0.948$$

Second Iteration $(n=2)$

$$x^{(2)} = \frac{1}{10}\left[12 - y^{(1)} - z^{(1)}\right] = \frac{1}{10}\left[12 - 1.06 - 0.948\right] = 0.9992$$

$$y^{(2)} = \frac{1}{10}\left[13 - x^{(2)} - z^{(1)}\right] = \frac{1}{10}\left[13 - 0.9992 - 0.948\right] = 1.00536$$

$$z^{(2)} = \frac{1}{10}\left[14 - 2x^{(2)} - 2y^{(2)}\right] = \frac{1}{10}\left[14 - 2(0.9992) - 2(1.00536)\right] = 0.99909$$

Third Iteration $(n=3)$

$$x^{(3)} = \frac{1}{10}\left[12 - y^{(2)} - z^{(2)}\right] = \frac{1}{10}\left[12 - 1.00536 - 0.99909\right] = 0.99956$$

$$y^{(3)} = \frac{1}{10}\left[13 - x^{(3)} - z^{(2)}\right] = \frac{1}{10}\left[13 - 0.99956 - 0.99909\right] = 1.00018$$

$$z^{(3)} = \frac{1}{10}\left[14 - 2x^{(3)} - 2y^{(3)}\right] = \frac{1}{10}\left[14 - 2(0.99956) - 2(1.00018)\right] = 1.00005$$

Fourth Iteration $(n=4)$

$$x^{(4)} = \frac{1}{10}\left[12 - y^{(3)} - z^{(3)}\right] = \frac{1}{10}\left[12 - 1.00018 - 1.00005\right] = 0.99998$$

$$y^{(4)} = \frac{1}{10}\left[13 - x^{(4)} - z^{(3)}\right] = \frac{1}{10}\left[13 - 0.99998 - 1.00005\right] = 0.999999$$

$$z^{(4)} = \frac{1}{10}\left[14 - 2x^{(4)} - 2y^{(4)}\right] = \frac{1}{10}\left[14 - 2(0.99998) - 2(0.999999)\right] = 1.000004$$

After Four Iterations, we can conclude.

$x = 0.99998 \approx 1$

$y = 0.999999 \approx 1$

$z = 1.000004 \approx 1$

DIFFERENCE EQUATIONS

Difference Eqⁿ :- An eqⁿ which contains a relation between dependent & Independent variable and the successive differences of the dependent variable, is called a difference eqⁿ, as-

$$F(x, y_x, \Delta y_x, \Delta^2 y_x, \dots \Delta^n y_x) = \phi(x)$$

$$\text{or } F(x, y_x, y_{x+h}, y_{x+2h}, \dots y_{x+nh}) = \phi(x)$$

$$\text{or } F(x, y_x, Ey_x, E^2 y_x, \dots E^n y_x) = \phi(x)$$

Ex. $\Delta y_{x+1} + \Delta y_x = 2 \;;\; h=1$ ——— (1)

or $(E-1) y_{x+1} + (E-1) y_x = 2$

$y_{x+2} - y_{x+1} + y_{x+1} - y_x = 2$

or $y_{x+2} - y_x = 2$ ——— (2)

or $(E^2-1) y_x = 2$ ——— (3)

Note :- we will take $h=1$ througout this chapter

Order of Difference Eqⁿ

$$\text{Order} = \frac{\text{Largest Arguement} - \text{Smallest Arguement}}{h}$$

So for above example -

$$\text{order} = \frac{x+2-x}{1} = 2$$

Note :- when we find the order of difference eqⁿ, then it should be expressed in the form, free from Δ's & E's.

Degree of Difference Eqⁿ :- The highest power of the dependent variable (with any argument) is the degree of difference eqⁿ.

$\underline{Ex. 1}$ $y_x^4 \cdot y_{x+1}^3 - 2y_x y_{x+2}^2 + 4y_{x+3}^3 = \phi(x)$

Here highest power of dependent variable is 4, so the degree of the eq^n is 4.

Solution of Difference Eq^n:- A relation bet^n independent and dependent variable which satisfy the difference eq^n, is known as Sol^n of that difference eq^n.

In this Lesson we will solve only

Linear Difference Eq^n with Constant Co-efficients.

A difference equation is said to be LDE if it contains $y_x, y_{x+1}, y_{x+2}, \ldots$ in first degree and are not multiplied together.

A LDE -

$$a_0 y_{x+n} + a_1 y_{x+n-1} + a_2 y_{x+n-2} + \cdots + a_n y_x = \phi(x) \quad \text{——(1)}$$

where $a_0, a_1, a_2, \ldots a_n$ are constants.

(1) Can be written as -

$$a_0 E^n y_x + a_1 E^{n-1} y_x + a_2 E^{n-2} y_x + \cdots + a_n y_x = \phi(x)$$

$$\text{or } (a_0 E^n + a_1 E^{n-1} + a_2 E^{n-2} + \cdots + a_n) y_x = \phi(x)$$

$$\text{or } f(E) y_x = \phi(x) \quad \text{——(2)}$$

The complete sol^n of (2) is given by-

$$y_x = C.F. + P.I. \quad \text{——(3)}$$

Obtaining C.F.:-

Find Auxillary eq^n-

$$f(E) = 0 \quad \text{——(1)}$$

Find the roots of (1), Let these are. $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$. Then we can write C.F. according to following Rules -

R-1 - If roots are real and unequal then.

$$C.F. = c_1\lambda_1^x + c_2\lambda_2^x + \dots + c_n\lambda_n^x$$

where $c_1, c_2, \dots c_n$ are constants.

R-2 - If roots are real and equal.

Case-1 Let $\lambda_1 = \lambda_2$ then

$$C.F. = (c_1 + c_2 x)\lambda_1^x + c_3\lambda_3^x + c_4\lambda_4^x + \dots + c_n\lambda_n^x$$

Case-2 Let $\lambda_1 = \lambda_2 = \lambda_3$ then

$$C.F. = (c_1 + c_2 x + c_3 x^2)\lambda_1^x + c_4\lambda_4^x + \dots + c_n\lambda_n^x$$

R-3 - If roots are complex and unequal

Let roots are $\alpha \pm i\beta$ then

$$C.F. = r^x[c_1\cos\theta x + c_2\sin\theta x]$$

where $r = \sqrt{\alpha^2+\beta^2}$

$$\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

R-4 If roots are complex and equal

(a) Let roots are $\alpha \pm i\beta$, $\alpha \pm i\beta$ then.

$$C.F. = r^x[(c_1+c_2x)\cos\theta x + (c_3+c_4x)\sin\theta x]$$

where r and θ has the same value.

(b) Let roots are $\alpha \pm i\beta$, $\alpha \pm i\beta$ & $\alpha \pm i\beta$ then-

$$C.F. = r^x[(c_1+c_2x+c_3x^2)\cos\theta x + (c_4+c_5x+c_6x^2)\sin\theta x]$$

where r and θ has the same values.

i.e.

$$r = \sqrt{\alpha^2+\beta^2}$$

$$\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

Note:- If $\phi(x) = 0$ then the solution of eqn contains C.F. Only.

Ex.1 Solve-

$$y_{x+2} - 8y_{x+1} + 15y_x = 0$$

Sol:- The given eq^n can be written as-

$$(E^2 - 8E + 15)y_x = 0$$

The A.E. is- $E^2 - 8E + 15 = 0$

$$\Rightarrow E = 3, 5$$

$$\therefore \; y_x = c_1 3^x + c_2 5^x$$

Ex.2 Solve-

$$y_{x+3} - 4y_{x+2} + 5y_{x+1} - 2y_x = 0$$

Sol:- The given eq^n can be written as-

$$(E^3 - 4E^2 + 5E - 2)y_x = 0$$

The A.E. is- $E^3 - 4E^2 + 5E - 2 = 0$

$$\Rightarrow (E-1)^2(E-2) = 0$$

$$\Rightarrow E = 1, 1, 2$$

$$\therefore \; y_x = (c_1 + c_2 x) \cdot 1^x + c_3 2^x$$

Ex.3 Solve-

$$y_{x+2} + 2y_{x+1} + 4y_x = 0$$

Sol:- Given eq^n can be written as-

$$(E^2 + 2E + 4)y_x = 0$$

The A.E. is- $E^2 + 2E + 4 = 0$

$$\Rightarrow E = -1 \pm i\sqrt{3}$$

$$\therefore \; y_x = r^x[c_1 \cos\theta x + c_2 \sin\theta x]$$

where $r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = -\frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

$$\therefore \; y_x = 2^x\left[c_1 \cos\frac{2\pi}{3}x + c_2 \sin\frac{2\pi}{3}x\right]$$

Ex.1 Solve-

$$y_{x+2} - 8y_{x+1} + 15y_x = 0$$

Sol:- The given eq^n can be written as-

$$(E^2 - 8E + 15)y_x = 0$$

The A.E. is- $E^2 - 8E + 15 = 0$

$\Rightarrow E = 3, 5$

$\therefore\ y_x = c_1 3^x + c_2 5^x$

Ex.2 Solve-

$$y_{x+3} - 4y_{x+2} + 5y_{x+1} - 2y_x = 0$$

Sol:- The given eq^n can be written as-

$$(E^3 - 4E^2 + 5E - 2)y_x = 0$$

The A.E. is- $E^3 - 4E^2 + 5E - 2 = 0$

$\Rightarrow (E-1)^2(E-2) = 0$

$\Rightarrow E = 1, 1, 2$

$\therefore\ y_x = (c_1 + c_2 x)\, 1^x + c_3 2^x$

Ex.3 Solve-

$$y_{x+2} + 2y_{x+1} + 4y_x = 0$$

Sol:- Given eq^n can be written as-

$$(E^2 + 2E + 4)y_x = 0$$

The A.E. is- $E^2 + 2E + 4 = 0$

$\Rightarrow E = -1 \pm i\sqrt{3}$

$\therefore\ y_x = r^x[c_1 \cos\theta x + c_2 \sin\theta x]$

where $r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = -\frac{\pi}{3}$ or $\frac{2\pi}{3}$

$\therefore\ y_x = 2^x\left[c_1 \cos\frac{2\pi}{3}x + c_2 \sin\frac{2\pi}{3}x\right]$

Ex. 4 Show that the difference eq^n -

$$I_{m+1} - \left(2 + \frac{r_0}{r}\right) I_m + I_{m-1} = 0$$

has the solution $I_m = I_0 \dfrac{\sinh(n-m)\alpha}{\sinh(n-1)\alpha}$

If $I_1 = I_0$ and $I_n = 0$

and $\alpha = 2\sinh^{-1}\left(\frac{1}{2}\sqrt{\frac{r_0}{r}}\right)$

Sol:- The given eq^n can be written as -

$$\left[E^2 - \left(2 + \frac{r_0}{r}\right)E + 1\right] I_{m-1} = 0 \quad \text{———} (1)$$

A.E. $E^2 - \left(2 + \frac{r_0}{r}\right)E + 1 = 0$ ——— (2)

$\because \alpha = 2\sinh^{-1}\left(\frac{1}{2}\sqrt{\frac{r_0}{r}}\right) \Rightarrow \sqrt{\frac{r_0}{r}} = 2\sinh\frac{\alpha}{2}$

$\Rightarrow \frac{r_0}{r} = 4\sinh^2\frac{\alpha}{2}$ ——— (3)

so from (2) -

$E^2 - \left(2 + 4\sinh^2\frac{\alpha}{2}\right)E + 1 = 0$ ——— (4)

$\because \cosh 2\alpha = 1 + 2\sinh^2\alpha$

$\therefore E^2 - 2\cosh\alpha \cdot E + 1 = 0$

$$\Rightarrow E = \frac{2\cosh\alpha \pm \sqrt{4\cosh^2\alpha - 4}}{2}$$

$= \cosh\alpha \pm \sinh\alpha = e^{\alpha}, e^{-\alpha}$

$\therefore$ Sol^n of (1) is -

$I_{m-1} = c_1(e^{\alpha})^{m-1} + c_2(e^{-\alpha})^{m-1}$ ——— (5)

put $m = 2$

$\Rightarrow I_1 = c_1 e^{\alpha} + c_2 e^{-\alpha}$ ——— (6)

$\Rightarrow I_0 = c_1 e^{\alpha} + c_2 e^{-\alpha}$ ——— (7)

Put $m = n+1$

$I_n = c_1(e^{\alpha})^n + c_2(e^{-\alpha})^n = 0$ ——— (8)

or $I_n = c_1 e^{n\alpha} + c_2 e^{-n\alpha} = 0$

From (7) & (8)

$$\frac{c_1}{I_0 e^{-n\alpha}} = \frac{c_2}{-I_0 e^{n\alpha}} = \frac{1}{e^{-\alpha(n-1)} - e^{\alpha(n-1)}}$$

$$\Rightarrow c_1 = -\frac{I_0 e^{-n\alpha}}{2\sinh(n-1)\alpha} \quad \& \quad c_2 = \frac{I_0 e^{n\alpha}}{2\sinh(n-1)\alpha}$$

$\therefore$ from (5)

$$I_{m-1} = -\frac{I_0 e^{-n\alpha}}{2\sinh(n-1)\alpha} \cdot e^{\alpha(m-1)} + \frac{I_0 e^{n\alpha}}{2\sinh(n-1)\alpha} \cdot e^{-\alpha(m-1)}$$

$$\text{or } I_m = \frac{-I_0 e^{-n\alpha} e^{m\alpha}}{2\sinh(n-1)\alpha} + \frac{I_0 e^{n\alpha} e^{-m\alpha}}{2\sinh(n-1)\alpha}$$

$$= \frac{I_0}{2\sinh(n-1)\alpha}\left[e^{(n-m)\alpha} - e^{-(n-m)\alpha}\right]$$

$$I_m = \frac{I_0 \sinh(n-m)\alpha}{\sinh(n-1)\alpha}$$

Proved.

Ex.5 Solve- $y_{n+3} + 16 y_{n-1} = 0$

Sol:- The given eq^n can be written as-

$$(E^4 + 16)\, y_{n-1} = 0 \quad \text{——— (1)}$$

A.E. $E^4 + 16 = 0$

$$\text{or } \left(\frac{E}{2}\right)^4 = -1 \quad \Rightarrow \quad \frac{E}{2} = (-1)^{1/4} \quad \text{——— (2)}$$

So by Demoivre's Th. -

$$\frac{E}{2} = \cos(2m+1)\frac{\pi}{4} + i\sin(2m+1)\frac{\pi}{4}$$

$m = 0, 1, 2, 3.$

$$\Rightarrow E = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right),\ 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right),$$
$$2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right),\ 2\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$$

$$\Rightarrow E = \sqrt{2}(1 \pm i),\ -\sqrt{2}(1 \pm i)$$

$\therefore$ Sol^n of (1) is-

$$y_{n-1} = r_1^{n-1}\left[c_1\cos(n-1)\theta_1 + c_2\sin(n-1)\theta_1\right] + r_2^{n-1}\left[c_3\cos(n-1)\theta_2 + c_4\sin(n-1)\theta_2\right] \quad \text{——— (3)}$$

where $r_1 = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = 2$

$r_2 = \sqrt{(-\sqrt{2})^2 + (-\sqrt{2})^2} = 2$

$\theta_1 = \tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{\pi}{4}$

$\theta_2 = \tan^{-1}\left(\frac{-\sqrt{2}}{-\sqrt{2}}\right) = \frac{5\pi}{4}$

$\therefore$ soln of (1) is -

$$y_{n-1} = 2^{n-1}\left[c_1 \cos(n-1)\frac{\pi}{4} + c_2 \sin(n-1)\frac{\pi}{4}\right] + 2^{n-1}\left[c_3 \cos(n-1)\frac{5\pi}{4} + c_4 \sin(n-1)\frac{5\pi}{4}\right]$$

or $y_n = 2^n\left[c_1 \cos\frac{n\pi}{4} + c_2 \sin\frac{n\pi}{4}\right] + 2^n\left[c_3 \cos\frac{5n\pi}{4} + c_4 \sin\frac{5n\pi}{4}\right]$

——— x ———

Finding P.I.

Let the Difference eqn be-

$$f(E)\, y_x = \phi(x) \quad \text{———} (1)$$

(A) $\phi(x) = b^x$

$$\boxed{P.I. = \frac{1}{f(E)} \cdot b^x}$$

Case-I $\quad f(b) \neq 0$

$$P.I. = \frac{1}{f(b)} \cdot b^x$$

Case-II $\quad f(b) = 0$

(i) If $f(E) = (E-b)^r$ then.

$$\frac{1}{f(E)} \cdot b^x = \frac{1}{(E-b)^r} = \frac{x(x-1)(x-2)\cdots(x-\overline{r-1})\, b^{x-r}}{r!}$$

(ii) If $f(E) = f_1(E)\cdot(E-b)^r \quad ; \quad f_1(b) \neq 0$

$$\frac{1}{f(E)} \cdot b^x = \frac{1}{f_1(E)\,(E-b)^r} \cdot b^x = \frac{1}{f(b)} \cdot \frac{x(x-1)\cdots(x-\overline{r-1})}{r!}\, b^{x-r}$$

Ex.1 solve-

$$y_{x+2} - 5y_{x+1} + 6y_x = 5^x$$

Sol:- Given eq^n can be written as-

$$(E^2 - 5E + 6)\,y_x = 5^x \quad \text{---} (1)$$

A.E. $E^2 - 5E + 6 = 0 \quad \Rightarrow (E-2)(E-3) = 0$

$\Rightarrow E = 2, 3$

$\therefore$ C.F. $= c_1 2^x + c_2 3^x \quad \text{---} (2)$

$$\text{P.I.} = \frac{1}{(E-2)(E-3)} \cdot 5^x$$

$$= \frac{1}{(5-2)(5-3)} \cdot 5^x = \frac{1}{6} \cdot 5^x$$

$\therefore$ Sol^n of (1) is -

$$y_x = c_1 \cdot 2^x + c_2 \cdot 3^x + \frac{1}{6} \cdot 5^x$$

Ex.2 solve-

$$y_{x+2} - 4y_x = 2^x$$

Sol:- Given eq^n can be written as-

$$(E^2 - 4)\,y_x = 2^x \quad \text{---} (1)$$

A.E. $E^2 - 4 = 0 \quad \Rightarrow E = \pm 2$

$\therefore$ C.F. $= c_1 2^x + c_2 (-2)^x$

$$\text{P.I.} = \frac{1}{(E-2)(E+2)} \cdot 2^x$$

$$= \frac{1}{(2+2)} \cdot x \cdot 2^{x-1} \qquad [\because E-2=0 \text{ at } E=b=2]$$

$$= x \cdot 2^{x-3}$$

$\therefore$ sol^n of (1) is -

$$y_x = c_1 2^x + c_2 (-2)^x + x \cdot 2^{x-3}$$

Ex.3 solve-

$$y_{x+2} - 3y_{x+1} + 2y_x = b^x$$

Sol:- Given eq^n can be written as-

$$(E^2 - 3E + 2)\,y_x = b^x$$

A.E. $E^2 - 3E + 2 = 0 \quad \Rightarrow (E-1)(E-2) = 0$

$\Rightarrow E = 1, 2$

$\therefore$ C.F. $= c_1 \cdot 1^x + c_2 \cdot 2^x = c_1 + c_2 \cdot 2^x$

$$P.I. = \frac{1}{(E-1)(E-2)} \cdot b^x$$

Case - I $\quad b \neq 1, b \neq 2$

$$P.I. = \frac{1}{(b-1)(b-2)} \cdot b^x$$

$\therefore$ Sol^n is -

$$y_x = c_1 + c_2 \cdot 2^x + \frac{b^x}{(b-1)(b-2)}$$

Case - II $\quad b = 1$

$$P.I. = \frac{1}{(E-1)(E-2)} \cdot 1^x$$

$$= \frac{1}{(1-2)} \cdot x \cdot 1^{x-1} = -x$$

$\therefore$ Sol^n is -

$$y_x = c_1 + c_2 \cdot 2^x - x$$

Case - III $\quad b = 2$

$$\therefore P.I. = \frac{1}{(E-1)(E-2)} \cdot 2^x$$

$$= \frac{1}{(2-1)} \cdot x \cdot 2^{x-1} = x \cdot 2^{x-1}$$

$\therefore$ Sol^n is -

$$y_x = c_1 + c_2 \cdot 2^x + x \cdot 2^{x-1}$$

(B) $\phi(x) = \sin bx$ or $\cos bx$

(a) $\phi(x) = \cos bx$

$$P.I. = \frac{1}{f(E)} \cos bx$$

$$= \frac{1}{f(E)} \cdot \left(\frac{e^{ibx} + e^{-ibx}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{f(E)} \cdot (e^{ib})^x + \frac{1}{f(E)} (e^{-ib})^x\right]$$

$$= \frac{1}{2}\left[\frac{1}{f(E)} \cdot a_1^x + \frac{1}{f(E)} a_2^x\right]$$

where $a_1 = e^{ib}$ & $a_2 = e^{-ib}$

Case-I If $f(a_1) \neq 0$ & $f(a_2) \neq 0$ then-

$$P.I. = \frac{1}{2}\left[\frac{a_1^x}{f(a_1)} + \frac{a_2^x}{f(a_2)}\right]$$

Case-II If $f(a_1) = 0$ or $f(a_2) = 0$ then proceed as in (A) — Case II

Aliter

$$P.I. = \frac{1}{f(E)} \cos bx$$

$$= \text{Real pt. of } \frac{1}{f(E)} (e^{ib})^x$$

$$= \text{Real pt. of } \frac{1}{f(E)} a_1^x \quad ; a_1 = e^{ib}$$

which can be solved as in (A)

(b) $\phi(x) = \sin bx$

$$P.I. = \frac{1}{f(E)} \sin bx$$

$$= \frac{1}{f(E)} \left\{\frac{e^{ibx} - e^{-ibx}}{2i}\right\}$$

$$= \frac{1}{2i}\left[\frac{1}{f(E)} (e^{ib})^x - \frac{1}{f(E)} (e^{-ib})^x\right]$$

$$= \frac{1}{2i}\left[\frac{1}{f(E)} a_1^x - \frac{1}{f(E)} a_2^x\right]$$

where $a_1 = e^{ib}$ & $a_2 = e^{-ib}$.

It can be solved as in (A)

Aliter

$$P.I. = \frac{1}{f(E)} \sin bx$$

$$= \text{Im. pt. of } \frac{1}{f(E)} e^{ibx}$$

$$= \text{Im. pt. of } \frac{1}{f(E)} \cdot a^x \qquad [a = e^{ib}]$$

which can be solved as in (A).

Ex.1 Solve-

$$y_{x+2} - 7y_{x+1} + 12y_x = \cos x$$

Sol:- Given eq^n can be written as-

$$(E^2 - 7E + 12)\, y_x = \cos x \quad \text{———} (1)$$

A.E. $E^2 - 7E + 12 = 0 \Rightarrow (E-3)(E-4) = 0$

$$\Rightarrow E = 3, 4$$

$$\therefore \; C.F. = c_1 3^x + c_2 4^x$$

$$P.I. = \frac{1}{E^2 - 7E + 12} \cos x$$

$$= \frac{1}{(E-3)(E-4)} \cdot \left[\frac{e^{ix} + e^{-ix}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{e^{ix}}{(e^i - 3)(e^i - 4)} + \frac{e^{-ix}}{(e^{-i} - 3)(e^{-i} - 4)}\right]$$

$$= \frac{1}{2}\left[\frac{e^{ix}(e^{-i} - 3)(e^{-i} - 4) + e^{-ix}(e^i - 3)(e^i - 4)}{(e^i - 3)(e^{-i} - 3)(e^i - 4)(e^{-i} - 4)}\right]$$

$$= \frac{1}{2}\left[\frac{e^{ix}(e^{-2i} - 7e^{-i} + 12) + e^{-ix}(e^{2i} - 7e^i + 12)}{[1 - 3(e^i + e^{-i}) + 9][1 - 4(e^i + e^{-i}) + 16]}\right]$$

$$= \frac{1}{2}\left[\frac{(e^{i(x-2)} + e^{-i(x-2)}) - 7(e^{i(x-1)} + e^{-i(x-1)}) + 12(e^{ix} + e^{-ix})}{(10 - 6\cos 1)(17 - 8\cos 1)}\right]$$

$$= \frac{1}{2}\left[\frac{2\cos(x-2) - 14\cos(x-1) + 24\cos x}{(10 - 6\cos 1)(17 - 8\cos 1)}\right]$$

$$= \frac{\cos(x-2) - 7\cos(x-1) + 12\cos x}{(10 - 6\cos 1)(17 - 8\cos 1)}$$

$$\therefore \; y_x = c_1 3^x + c_2 \cdot 4^x + \frac{\cos(x-2) - 7\cos(x-1) + 12\cos x}{(10 - 6\cos 1)(17 - 8\cos 1)}$$

Aliter

$$P.I. = \frac{1}{(E-3)(E-4)} \cos x$$

$$= \text{Real pt. of } \frac{1}{(E-3)(E-4)} \cdot e^{ix}$$

$$= \text{Real pt. of } \frac{e^{ix}}{(e^{i}-3)(e^{i}-4)} \times \frac{(e^{-i}-3)(e^{-i}-4)}{(e^{-i}-3)(e^{-i}-4)}$$

$$= \text{Real pt. of } \frac{e^{ix}(e^{-2i}-7e^{-i}+12)}{(e^{i}-3)(e^{-i}-3)(e^{i}-4)(e^{-i}-4)}$$

$$= \text{Real pt. of } \frac{e^{i(x-2)}-7e^{i(x-1)}+12e^{ix}}{(10-6\cos 1)(17-8\cos 1)}$$

$$= \frac{\cos(x-2)-7\cos(x-1)+12\cos x}{(10-6\cos 1)(17-8\cos 1)}$$

$$\therefore\ y_x = c_1 \cdot 3^x + c_2 \cdot 4^x + \frac{\cos(x-2)-7\cos(x-1)+12\cos x}{(10-6\cos 1)(17-8\cos 1)}$$

Ex.2 Solve-

$$y_{n+2} + y_n = \sin\frac{n\pi}{2}$$

Sol:- Given eq^n can be written as-

$$(E^2+1)\,y_n = \sin\frac{n\pi}{2} \quad \text{———} (1)$$

A.E. $E^2+1=0 \Rightarrow E = \pm i$

$$\therefore\ C.F. = r^n[c_1\cos n\theta + c_2\sin n\theta] \quad \text{———} (2)$$

where $r = \sqrt{0+1} = 1$

$$\theta = \tan^{-1}\left(\frac{1}{0}\right) = \frac{\pi}{2}$$

$$\therefore\ C.F. = c_1\cos\frac{n\pi}{2} + c_2\sin\frac{n\pi}{2}$$

$$P.I. = \frac{1}{E^2+1} \cdot \sin\frac{n\pi}{2}$$

$$= \text{Im. pt. of } \frac{1}{E^2+1} \cdot \left(e^{\frac{i\pi}{2}}\right)^n$$

$$= \text{Im. pt. of } \frac{1}{(E+i)(E-i)} \cdot \left(e^{\frac{i\pi}{2}}\right)^n$$

$= \text{Im. pt. of } \frac{1}{(E+e^{i\pi/2})(E-e^{i\pi/2})} (e^{i\pi/2})^n$

$= \text{Im. pt. of } \frac{1}{(e^{i\pi/2}+e^{i\pi/2})} \cdot n(e^{i\pi/2})^{(n-1)}$

$= \text{Im. pt. of } \frac{n e^{\frac{i(n-1)\pi}{2}}}{2e^{\frac{i\pi}{2}}}$

$= \text{Im. pt. of } \frac{n[\cos(n-1)\frac{\pi}{2} + i\sin(n-1)\frac{\pi}{2}]}{2i}$

$= -\frac{n}{2}\cos(n-1)\frac{\pi}{2}$

$\therefore \quad y_n = \text{C.F.} + \text{P.I.}$

(C) $\phi(x) = P_n(x)$

$\text{P.I.} = \frac{1}{f(E)} P_n(x) = \frac{1}{f(1+\Delta)} P_n(x)$

$= [f(1+\Delta)]^{-1} F^{(n)}(x)$

where $F^{(n)}(x) \rightarrow$ Factorial Function of $P_n(x)$.

Ex.1 Solve-

$y_{n+2} - 4y_n = n^2+n-2$

Sol:- Given eqn can be written as-

$(E^2-4)y_n = n^2+n-2$ ——— (1)

A.E. $E^2-4=0 \Rightarrow E = \pm 2$

$\therefore \text{C.F.} = c_1 2^x + c_2(-2)^x$

$\text{P.I.} = \frac{1}{E^2-4}(n^2+n-2)$

$= \frac{1}{(1+\Delta)^2-4}(n^2+n-2)$

$= \frac{1}{1+\Delta^2+2\Delta-4}(n^2+n-2)$

$= -\frac{1}{3}\frac{1}{\left(1-\frac{\Delta^2+2\Delta}{3}\right)}(n^2+n-2)$

$$= -\frac{1}{3}\left[1 - \frac{\Delta^2+2\Delta}{3}\right]^{-1} (n^{(2)} + 2n^{(1)} - 2)$$

$$= -\frac{1}{3}\left[1 + \frac{\Delta^2+2\Delta}{3} + \left(\frac{\Delta^2+2\Delta}{3}\right)^2 + \cdots\right] (n^{(2)} + 2n^{(1)} - 2)$$

$$= -\frac{1}{3}\left[1 + \frac{2}{3}\Delta + \frac{7}{9}\Delta^2 + \cdots\right] (n^{(2)} + 2n^{(1)} - 2)$$

$$= -\frac{1}{3}\left[n^{(2)} + 2n^{(1)} - 2 + \frac{2}{3}(2n^{(1)} + 2) + \frac{7}{9}(2)\right]$$

$$= -\frac{1}{3}\left[n^2 + n - 2 + \frac{4}{3}(n+1) + \frac{14}{9}\right]$$

$$= -\frac{1}{3}\left[n^2 + \frac{7}{9}n + \frac{8}{9}\right]$$

$$= -\frac{1}{27}(9n^2 + 7n + 8)$$

$$\therefore \quad y_n = c_1 2^n + c_2 \cdot (-2)^n - \frac{1}{27}(9n^2 + 7n + 8)$$

(D) $\phi(x) = b^x \cdot P_n(x)$

$$P.I. = \frac{1}{f(E)} b^x \cdot P_n(x)$$

$$= b^x \cdot \frac{1}{f(bE)} \cdot P_n(x)$$

Further it can be solved as in (C)

Ex.1 Solve-

$$y_{n+2} - 2y_{n+1} + y_n = n^2 \cdot 2^n$$

Sol:- Given eq^n can be written as-

$$(E^2 - 2E + 1) y_n = n^2 \cdot 2^n$$

$$\text{or} \quad (E-1)^2 y_n = n^2 \cdot 2^n \quad \text{———} (1)$$

$$A.E. \quad (E-1)^2 = 0 \quad \Rightarrow E = 1, 1$$

$$\therefore \quad C.F. = c_1 + c_2 n$$

$$P.I. = \frac{1}{(E-1)^2} \cdot n^2 \cdot 2^n$$

$$= 2^n \cdot \frac{1}{(2E-1)^2} \cdot n^2$$

$$= 2^n \cdot \frac{1}{[2(1+\Delta)-1]^2} \cdot n^2$$

$$= 2^n \cdot \frac{1}{(1+2\Delta)^2} \cdot n^2$$

$$= 2^n \cdot [1+2\Delta]^{-2} (n^{(2)} + n^{(1)})$$

$$= 2^n [1 - 4\Delta + 3(2\Delta)^2 - \cdots] (n^{(2)} + n^{(1)})$$

$$= 2^n [n^{(2)} + n^{(1)} - 4(2n^{(1)}+1) + 12(2)]$$

$$= 2^n [n^2 - 8n + 20]$$

$$\therefore\ y_n = c_1 + c_2 n + 2^n [n^2 - 8n + 20]$$

(E) $\phi(x) = K$ (constant)

$$P.I. = \frac{1}{f(E)} K = K \cdot \frac{1}{f(E)} \cdot 1^x$$

Further it can be solved as (A)

Ex. 1 Solve- $y_{x+2} - 5y_{x+1} + 6y_x = 36$

Sol:- Given eqn can be written as-

$$(E^2 - 5E + 6) y_x = 36 \quad \text{——— (1)}$$

A.E. $E^2 - 5E + 6 = 0 \Rightarrow$

$$\Rightarrow (E-2)(E-3) = 0 \Rightarrow E = 2, 3$$

$$\therefore\ C.F. = c_1 2^x + c_2 3^x$$

$$P.I. = \frac{1}{(E-2)(E-3)} \cdot 36 = 36 \cdot \frac{1}{(E-2)(E-3)} \cdot 1^x$$

$$= \frac{36}{(1-2)(1-3)} \cdot 1^x$$

$$= 18 \cdot 1^x = 18$$

$$\therefore\ y_x = c_1 2^x + c_2 3^x + 18$$