APPLICATION OF Z-TRANSFORM IN SOLVING DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS.

WORKING Rule:

Step I :→ Take the Z-transform of both sides of the given difference equation.

II →:- Express $\bar{u}(z)$ {Z-transform of u_n} as a function of Z.

III →: Express this function in terms of the known functions.

IV →: Take the Inverse Z-transform on both sides which gives a sequence $\{u_n\}$. It will be the solution of the given linear difference equation.

NOTE →: here we used the Following Formula's more frequently.

(1) $Z[\{u_{n+1}\}] = z[\bar{u}(z) - u_0]$

(ii) $Z[\{u_{n+2}\}] = z^2[\bar{u}(z) - u_0 - u_1 z^{-1}]$

(3) $Z[\{u_{n+3}\}] = z^3[\bar{u}(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}]$

(4) $Z[\{a^n\}] = \frac{z}{z-a}$

EXAMPLES

(1) Solve $6u_{n+2} - u_{n+1} - u_n = 0$; given $u_0 = 0, u_1 = 1$

Sol:→ Let $Z[\{u_n\}] = \bar{u}(z)$

Taking Z-transform on both the sides of the given difference equation, we have

$$6Z[\{u_{n+2}\}] - Z[\{u_{n+1}\}] - Z[\{u_n\}] = 0$$

or $6z^2[\bar{u}(z) - u_0 - u_1 z^{-1}] - z[\bar{u}(z) - u_0] - \bar{u}(z) = 0$

$\Rightarrow \bar{u}(z)[6z^2 - z - 1] - (6z^2 - z)u_0 - 6z u_1 = 0$

put $u_0 = 0, u_1 = 1$

$\Rightarrow \bar{u}(z)[6z^2 - z - 1] = 6z$

$$\Rightarrow \bar{u}(z) = \frac{6z}{(6z^2 - z - 1)} = \frac{6z}{(3z+1)(2z-1)}$$

$$\Rightarrow \bar{u}(z) = \frac{6z}{3z(1 + \frac{1}{3z}) \times 2z(1 - \frac{1}{2z})}$$

$$\Rightarrow \bar{u}(z) = \frac{1/z}{(1 + \frac{1}{3z})(1 - \frac{1}{2z})}$$

Let $\dfrac{1/z}{(1 + \frac{1}{3z})(1 - \frac{1}{2z})} = \dfrac{A}{(1 - \frac{1}{2z})} + \dfrac{B}{(1 + \frac{1}{3z})}$

$\therefore A = \dfrac{2}{1 + 2/3} = \dfrac{6}{5}$ (putting $z = \frac{1}{2}$)

$B = \dfrac{3}{1 + 3/2} = -\dfrac{6}{5}$ (Putting $z = -1/3$)

$$\therefore \bar{u}(z) = \frac{6}{5}\left[\frac{1}{(1 - \frac{1}{2z})} - \frac{1}{(1 + \frac{1}{3z})}\right]$$

$\therefore \bar{u}(z) = \frac{6}{5}\left[\frac{2z}{2z-1}\right] - \frac{6}{5}\left[\frac{3z}{3z+1}\right]$

$\Rightarrow \bar{u}(z) = \frac{6}{5}\left[\frac{z}{z-1/2}\right] - \frac{6}{5}\left[\frac{z}{z+\frac{1}{3}}\right]$

$\Rightarrow Z^{-1}[\bar{u}(z)] = \frac{6}{5} Z^{-1}\left[\frac{z}{z-1/2}\right] - \frac{6}{5} Z^{-1}\left[\frac{z}{z-(\frac{-1}{3})}\right]$

$\Rightarrow \quad \{u_n\} = \frac{6}{5}\left[\left(\frac{1}{2}\right)^n - \left(-\frac{1}{3}\right)^n\right]$ Ans

(2) Solve $u_{n+2} - 3u_{n+1} - 4u_n = 3^n$.

Sol:→ Let $Z[\{u_n\}] = \bar{u}(z)$

Taking z-transform on both the sides

$Z[\{u_{n+2}\}] - 3Z[\{u_{n+1}\}] - 4Z[\{u_n\}] = Z[\{3^n\}]$

$\Rightarrow z^2[\bar{u}(z) - u_0 - u_1 z^{-1}] - 3z[\bar{u}(z) - u_0] - 4\bar{u}(z) = \frac{z}{z-3}$

$\Rightarrow (z^2-3z-4)\bar{u}(z) - (z^2-3z)u_0 - zu_1 = \frac{z}{z-3}$

$\Rightarrow (z-4)(z+1)\bar{u}(z) = \frac{z}{z-3} + (z^2-3z)u_0 + zu_1$

$\Rightarrow \frac{\bar{u}(z)}{z} = \left[\frac{1}{(z-3)(z-4)(z+1)}\right] + \left[\frac{(z-3)u_0}{(z-4)(z+1)}\right] + \frac{u_1}{(z-4)(z+1)}$

Resolving into partial fractions.

$\bar{u}(z) = \left\{-\frac{1}{4}\left(\frac{z}{z-3}\right) + \frac{1}{5}\left(\frac{z}{z-4}\right) + \frac{1}{20}\left(\frac{z}{z+1}\right)\right\}$

$+ u_0\left[\frac{1}{5}\left(\frac{z}{z-4}\right) + \frac{4}{5}\left(\frac{z}{z+1}\right)\right] + u_1\left[\frac{1}{5}\left(\frac{z}{z-4}\right) - \frac{1}{5}\left(\frac{z}{z+1}\right)\right]$

Now taking Inverse Z-transform

$\therefore\ Z^{-1}[\bar{u}(z)] = u_n = -\frac{1}{4}(3)^n + \frac{1}{5}(4)^n + \frac{1}{20}(-1)^n$
$+ u_0\left[\frac{1}{5}(4)^n + \frac{4}{5}(-1)^n\right] + u_1\left[\frac{1}{5}(4)^n - \frac{1}{5}(-1)^n\right]$

or $\{u_n\} = c_1(4)^n + c_2(-1)^n - \frac{1}{4}(3)^n$ Ans

where $c_1 = \left(\frac{1}{5} + \frac{u_0}{5} + \frac{u_1}{5}\right)$; $c_2 = \left(\frac{1}{20} + \frac{4u_0}{5} - \frac{u_1}{5}\right)$

(3) Solve $u_{n+1} + \frac{1}{4}u_n = \left(\frac{1}{4}\right)^n$, $n \geq 0$, $u(0) = 0$

Sol:- Taking Z-transform on both the sides

$$Z[\{u_{n+1}\}] + \frac{1}{4}Z[\{u_n\}] = Z\left[\left\{\frac{1}{4}\right\}^n\right]$$

$$\Rightarrow z[\bar{u}(z) - u_0] + \frac{1}{4}\bar{u}(z) = \frac{z}{z-(1/4)}$$

$$\Rightarrow \bar{u}(z)\left[z + \frac{1}{4}\right] - zu_0 = \frac{z}{z-1/4}$$

$$\Rightarrow \bar{u}(z) = \frac{z}{(z-\frac{1}{4})(z+\frac{1}{4})} \qquad [\because u(0)=0]$$

$$\Rightarrow \frac{\bar{u}(z)}{z} = \frac{1}{(z-1/4)(z+1/4)} \quad \text{——(1)}$$

Breaking into partial fractions

Let $\dfrac{1}{(z-\frac{1}{4})(z+\frac{1}{4})} = \dfrac{A}{z-1/4} + \dfrac{B}{z+1/4}$

$\therefore\ A = \dfrac{1}{\frac{1}{4}+\frac{1}{4}} = 2 \quad (\text{Put } z = \tfrac{1}{2})$

$B = \dfrac{1}{-\frac{1}{4}-\frac{1}{4}} = -2 \quad (\text{Put } z = -\tfrac{1}{2})$

$\therefore$ From (1) $\frac{\bar{u}(z)}{z} = 2 \cdot \frac{1}{z-1/4} - 2 \frac{1}{z+1/4}$

$\Rightarrow \bar{u}(z) = 2 \cdot \frac{z}{z-1/4} - 2 \cdot \frac{z}{z+\frac{1}{4}}$

Taking Inverse Z-transform.

$\Rightarrow z^{-1}\{\bar{u}(z)\} = 2 \cdot z^{-1}\left[\frac{z}{z-1/4}\right] - 2 z^{-1}\left[\frac{z}{z+1/4}\right]$

$\Rightarrow \{u_n\} = 2\left(\frac{1}{4}\right)^n - 2\left(-\frac{1}{4}\right)^n$ Ans

(4) Solve $6u_{n+2} + 5u_{n+1} - 4u_n = 6H(n)$;

$u_0 = u_1 = 0$; $n \geq 0$

where $H(n) = \begin{cases} 1; & n \geq 0 \\ 0; & n < 0 \end{cases}$ is a unit step-function.

Sol:- Taking Z-transform on both the sides.

$6Z[u_{n+2}] + 5Z[u_{n+1}] - 4Z[u_n] = 6Z[H(n)]$

or $6z^2[\bar{u}(z) - u_0 - u_1 z^{-1}] + 5z[\bar{u}(z) - u_0]$

$- \bar{u}(z) = 6\frac{z}{z-1}$ $\quad \left[\because Z\{H(n)\} = \frac{z}{z-1}\right]$

Use $u_0 = u_1 = 0$

or $(6z^2 + 5z - 1)\bar{u}(z) = \frac{6z}{z-1}$

$\therefore \bar{u}(z) = \frac{6z}{(6z-1)(z+1)(z-1)}$

or $\frac{\bar{u}(z)}{z} = \frac{6}{(6z-1)(z+1)(z-1)}$

$\therefore \frac{\bar{u}(z)}{z} = \left[\frac{-216}{35}\left(\frac{1}{6z-1}\right) + \frac{3}{7}\left(\frac{1}{z+1}\right) + \frac{3}{5}\left(\frac{1}{z-1}\right)\right]$

$\therefore \frac{\bar{u}(z)}{z} = \left[\frac{-216}{35} \times \frac{1}{6}\left(\frac{1}{z-1/6}\right) + \frac{3}{7}\left(\frac{1}{z+1}\right) + \frac{3}{5}\left(\frac{1}{z-1}\right)\right]$

Taking Inverse Z-transform

$\Rightarrow z^{-1}[\bar{u}(z)] = \frac{-36}{35} z^{-1}\left(\frac{z}{z-1/6}\right) + \frac{3}{7} z^{-1}\left(\frac{z}{z+1}\right) + \frac{3}{5} z^{-1}\left(\frac{z}{z-1}\right)$

$\Rightarrow \quad u_n = -\frac{36}{35}\left(\frac{1}{6}\right)^n + \frac{3}{7}(-1)^n + \frac{3}{5}(1)^n.$

(5) solve. $u_{n+2} - 6u_{n+1} + 8u_n = 2^n + 6n.$

soln:- Taking Z-transform of both sides

$Z[u_{n+2}] - 6Z[u_{n+1}] + 8Z[u_n] = Z(2^n) + 6Z(n)$

$\Rightarrow z^2[\bar{u}(z) - u_0 - u_1 z^{-1}] - 6z[\bar{u}(z) - u_0] + 8\bar{u}(z) = \frac{z}{z-2} + 6 \cdot \frac{z}{(z-1)^2} \quad \left[\because Z(n) = \frac{z}{(z-1)^2}\right.$

$\Rightarrow (z^2 - 6z + 8)\bar{u}(z) = \frac{z}{z-2} + \frac{6z}{(z-1)^2} + u_0(z^2 - 6z) + u_1 z$

or $(z-2)(z-4)\bar{u}(z) = \frac{z}{z-2} + \frac{6z}{(z-1)^2} + u_0(z^2 - 6z) + u_1 z.$

Let $u_0 = k_1$ and $u_1 = k_2$

$\therefore \bar{u}(z) = \frac{z}{(z-2)^2(z-4)} + \frac{6z}{(z-1)^2(z-2)(z-4)} + \frac{k_1(z^2 - 6z)}{(z-2)(z-4)} + \frac{k_2 z}{(z-2)(z-4)}$

Breaking in to Partial Fractions

or $\bar{u}(z) = \left[\frac{1}{4}\cdot\frac{z}{z-4} - \frac{1}{4}\frac{z}{z-2} - \frac{1}{2}\frac{z}{(z-2)^2}\right]$

$+ \left[\frac{8}{3}\frac{z}{z-1} + 2\frac{z}{(z-1)^2} - 3\cdot\frac{z}{z-2} + \frac{1}{3}\cdot\frac{z}{z-4}\right]$

$+ k_1\left[2\cdot\frac{z}{z-2} - \frac{z}{z-4}\right] + k_2\left[\frac{1}{2}\cdot\frac{z}{z-4} - \frac{1}{2}\cdot\frac{z}{z-2}\right]$

or $\bar{u}(z) = \frac{z}{z-4}\left(\frac{1}{4} + \frac{1}{3} - k_1 + \frac{k_2}{2}\right) + \frac{z}{z-2}\left(-\frac{1}{4} - 3\right.$

$\left. + 2k_1 - \frac{k_2}{2}\right) + \frac{z}{(z-2)^2}\left[-\frac{1}{2}\right] + \frac{8}{3}\cdot\frac{z}{z-1} + 2\cdot\frac{z}{(z-1)^2}$

or $\bar{u}(z) = \left(\frac{7}{12} - k_1 + \frac{k_2}{2}\right)\frac{z}{z-4} + \left(2k_1 - \frac{k_2}{2} - \frac{13}{4}\right)\frac{z}{z-2}$

$- \frac{1}{2}\frac{z}{(z-2)^2} + \frac{8}{3}\frac{z}{z-1} + 2\cdot\frac{z}{(z-1)^2}$

Let $\frac{7}{12} - k_1 + \frac{k_2}{2} = c_1$, $2k_1 - \frac{k_2}{2} - \frac{13}{4} = c_2$

$\therefore \bar{u}(z) = c_1\frac{z}{z-4} + c_2\frac{z}{z-2} - \frac{1}{2}\frac{z}{(z-2)^2} + \frac{8}{3}\frac{z}{z-1} + 2\frac{z}{(z-1)^2}$

Taking Inverse Z-transform

$u_n = c_1(4)^n + c_2(2)^n - \frac{1}{2}\{2^n n\} + \frac{8}{3}(1)^n + 2n$

$\left[\because Z(a^n n) = \frac{z}{(z-a)^2}\right]$