Convolution Theorem.

Statement :- If $Z^{-1}[\bar{u}(z)] = \{u_n\}$ and $Z^{-1}[\bar{v}(z)] = \{v_n\}$ Then

$$Z^{-1}[\bar{u}(z)\cdot\bar{v}(z)] = \sum_{m=-\infty}^{\infty} u_m v_{n-m}.$$

$$= u_n * v_n$$

where The symbol $*$ denotes The convolution operation.

Proof :-

<u>Proof:-</u> since by definition

$$Z\left[\sum_{m=-\infty}^{\infty} u_m V_{n-m}\right] = \sum_{n=-\infty}^{\infty}\left[\sum_{m=-\infty}^{\infty} u_m V_{n-m} z^{-n}\right]$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} u_m V_{n-m} z^{-n}$$

Note

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} u_m V_{n-m} z^{-n+m-m}$$

$$= \sum_{m=-\infty}^{\infty} u_m z^{-m} \cdot \sum_{n=-\infty}^{\infty} V_{n-m} z^{-[n-m]}$$

putting $n-m=k$

$$= \sum_{m=-\infty}^{\infty} u_m z^{-m} \cdot \sum_{k=-\infty}^{\infty} V_k z^{-k}$$

$$= Z(u_m)\cdot Z(V_k)$$

$$= \bar{u}(z)\cdot \bar{V}(z)$$

$$\therefore \quad Z^{-1}[\bar{u}(z)\cdot \bar{V}(z)] = \sum_{m=-\infty}^{\infty} u_m V_{n-m}.$$

<u>NOTE</u> For a casual sequence $(n \geq 0)$ we use convolution Theorem is the following form.

$$Z^{-1}[\bar{u}(z)\cdot \bar{V}(z)] = \sum_{m=0}^{n} u_m V_{n-m} = u_n * V_n$$

EXAMPLE ① Find $Z^{-1}\left[\frac{z^2}{(z-\alpha)(z-\beta)}\right]$ by convolution Theorem

Sol→ since $Z^{-1}\left[\frac{z}{z-\alpha}\right]=\alpha^n$ and $Z^{-1}\left[\frac{z}{z-\beta}\right]=\beta^n$.

By convolution Theorem, $\quad \left(\because Z(a^n)=\frac{z}{z-a}\right)$

$$Z^{-1}\left[\frac{z^2}{(z-\alpha)(z-\beta)}\right]=\sum_{m=0}^{n}\alpha^m\beta^{n-m}$$

$$=\beta^n\sum_{m=0}^{n}\left(\frac{\alpha}{\beta}\right)^m$$

$$=\beta^n\left[1+\left(\frac{\alpha}{\beta}\right)+\left(\frac{\alpha}{\beta}\right)^2+\cdots+\left(\frac{\alpha}{\beta}\right)^n\right]$$

Now Using G.P

$a+ar+ar^2+\cdots+ar^{n-1}$. Then

$$S_n=\frac{a(1-r^n)}{1-r}\ (r<1);\ \frac{a(r^n-1)}{r-1}\ (r>1)$$

$$=\left[\beta^n\left\{\frac{\left(\frac{\alpha}{\beta}\right)^{n+1}-1}{\left(\frac{\alpha}{\beta}\right)-1}\right\}\right]$$

$$=\left[\beta^n\left\{\frac{\alpha^{n+1}-\beta^{n+1}}{\beta^n(\alpha-\beta)}\right\}\right]$$

$$\therefore Z^{-1}\left[\frac{z^2}{(z-\alpha)(z-\beta)}\right]=\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}$$

② Use Convolution Theorem, evaluate

$$Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]$$

Sol:- since $Z^{-1}\left(\frac{z}{z-1}\right) = (1)^n = u_n$ and

$$Z^{-1}\left(\frac{z}{z-3}\right) = 3^n = v_n$$

$\therefore$ By convolution Theorem

$$Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = \sum_{m=0}^{n} 1^m \, 3^{n-m}$$

$$= 3^n + 3^{n-1} + 3^{n-2} + \cdots + 1$$

$$= 3^n\left[1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots + \left(\frac{1}{3}\right)^n\right]$$

$$= 3^n\left[\frac{1-\left(\frac{1}{3}\right)^{n+1}}{\left(1-\frac{1}{3}\right)}\right]$$

$$= \frac{3^n\left[3^{n+1}-1\right]}{2\cdot 3^n} = \frac{(3^{n+1}-1)}{2}.$$

<u>EXAMPLE</u> ③ show that $\frac{1}{\lfloor n} * \frac{1}{\lfloor n} = \frac{2^n}{\lfloor n}$.

Sol:- since $\frac{1}{\lfloor n} * \frac{1}{\lfloor n} = \sum_{m=0}^{n} \frac{1}{\lfloor m} \cdot \frac{1}{\lfloor n-m}$

$$= \frac{1}{\lfloor n} + \frac{1}{\lfloor 1} \cdot \frac{1}{\lfloor n-1} + \frac{1}{\lfloor 2} \cdot \frac{1}{\lfloor n-2} + \cdots + \frac{1}{\lfloor n}.$$

$$= \frac{1}{\lfloor n}\left[1 + n + \frac{n(n-1)}{\lfloor 2} + \cdots + 1\right]$$

$$\left\{\because \frac{\lfloor n}{\lfloor n-1} = \frac{n\cancel{\lfloor n-1}}{\cancel{\lfloor n-1}} = n;\ \frac{\lfloor n}{\lfloor n-2} = \frac{n(n-1)\cancel{\lfloor n-2}}{\cancel{\lfloor n-2}}\right\}$$

$$= \frac{1}{\lfloor n}(1+1)^n = \frac{2^n}{\lfloor n}$$ proved.

EXAMPLE 4 $\quad Z^{-1}\left[\frac{z^2}{(z-a)^2}\right]$

Sol:→ since $Z^{-1}\left[\frac{z}{z-a}\right] = a^n;\ n \geq 0$

By Convolution Theorem

$$Z^{-1}\left[\left(\frac{z}{z-a}\right)\cdot\left(\frac{z}{z-a}\right)\right] = \sum_{m=0}^{n} a^m \cdot a^{n-m}$$

$$= a^n \sum_{m=0}^{n} a^m \cdot a^{-m}$$

$$= a^n \sum_{m=0}^{n} a^0 = a^n \sum_{m=0}^{n} 1$$

$$= a^n [1 + 1 + \cdots + 1] \quad (n+1) \text{ times}$$

$$\therefore \quad Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = \{(n+1)a^n\} \quad \underline{\text{Ans}}$$