EXAMPLE :- Find $Z^{-1}\left[\frac{1}{(z-3)(z-2)}\right]$ If ROC is given as

(a) $|z| < 2$ (b) $2 < |z| < 3$ (c) $|z| > 3$

Sol:→ since $\bar{u}(z) = \frac{1}{(z-2)(z-3)}$

$$= \frac{1}{z-3} - \frac{1}{z-2} \quad \text{[Using Partial Fraction]}$$

(a) $|z| < 2 \Rightarrow |z| < 3$

$\therefore \left|\frac{z}{2}\right| < 1$ and $\left|\frac{z}{3}\right| < 1$

so $\bar{u}(z) = \frac{1}{z-3} - \frac{1}{z-2}$

$$= -\frac{1}{3(1-\frac{z}{3})} + \frac{1}{2(1-\frac{z}{2})}$$

$$= -\frac{1}{3}\left(1-\frac{z}{3}\right)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= -\frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{3^2}+\cdots\right) + \frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\cdots\right)$$

$$= -\frac{1}{3}\sum_{n=0}^{\infty}\frac{z^n}{3^n} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n}$$

$$= -\sum_{n=0}^{\infty} 3^{-n-1} z^n + \sum_{n=0}^{\infty} 2^{-n-1} z^n$$

$$= -\sum_{n=0}^{\infty} 3^{-n-1} z^n + \sum_{n=0}^{\infty} 2^{-n-1} z^n$$

$$= \sum_{n=0}^{\infty}\left(2^{-n-1} z^n - 3^{-n-1} z^n\right)$$

$$= \sum_{n=0}^{\infty}\left(2^{-n-1} - 3^{-n-1}\right) z^n$$

Coefficient of $z^{n} = \bar{2}^{n-1} - \bar{3}^{n-1}$; $n \geq 0$

coefficient of $\bar{z}^{n} = 2^{n-1} - 3^{n-1}$; $n \leq 0$

$\therefore \; Z^{-1}[\bar{u}(z)] = u_n = 2^{n-1} - 3^{n-1}$ Ans

(b) If $2 < |z| < 3$ i.e $2 < |z|, |z| < 3$

or $\left|\frac{2}{z}\right| < 1$, $\left|\frac{z}{3}\right| < 1$

$$\therefore \quad \bar{u}(z) = \frac{1}{z-3} - \frac{1}{z-2}$$

$$= -\frac{1}{3}\left(1 - \frac{z}{3}\right)^{-1} - \frac{1}{z}\left(1 - \frac{2}{z}\right)^{-1}$$

$$= -\frac{1}{3}\left(1 + \frac{z}{3} + \frac{z^2}{3^2} + \cdots + \frac{z^n}{3^n} + \cdots\right)$$

$$- \frac{1}{z}\left(1 + \frac{2}{z} + \frac{2^2}{z^2} + \cdots + \frac{2^{n-1}}{z^{n-1}} + \cdots\right)$$

$$= -\left(3^{-1} + 3^{-2}z + 3^{-3}z^2 + \cdots + 3^{-n-1}z^n + \cdots\right)$$

$$- \left(z^{-1} + 2z^{-2} + 2^2 z^{-3} + \cdots + 2^{n-1}z^{-n} + \cdots\right)$$

Coefficient of z^n in I series $= -3^{-n-1}$; $n \geq 0$

$\therefore$ Coeff. of $z^{-n} = -3^{n-1}$; $n \leq 0$ $(n \to -n)$

and Coeff. of z^{-n} in 2nd series

$= -2^{n-1}$; $n > 0$

$$\therefore \; Z^{-1}[\bar{u}(z)] = \begin{cases} -2^{n-1} & ; \; n > 0 \\ -3^{n-1} & ; \; n \leq 0 \end{cases} \quad \text{Ans}$$

(c) If $|z| > 3 \Rightarrow \left|\frac{3}{z}\right| < 1$

$$\therefore \bar{u}(z) = \frac{1}{z-3} - \frac{1}{z-2}$$

$$= \frac{1}{z}\cdot\left(1-\frac{3}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= \frac{1}{z}\cdot\left(1+\frac{3}{z}+\frac{3^2}{z^2}+\cdots\right) - \frac{1}{z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\cdots\right)$$

$$= \left(\frac{1}{z}+\frac{3}{z^2}+\frac{3^2}{z^3}+\cdots\right) - \left(\frac{1}{z}+\frac{2}{z^2}+\frac{2^2}{z^3}+\cdots\right)$$

$$= \sum_{n=1}^{\infty} 3^{n-1} z^{-n} - \sum_{n=1}^{\infty} 2^{n-1} z^{-n}$$

$$= \sum_{n=1}^{\infty} \left(3^{n-1} - 2^{n-1}\right) z^{-n}.$$

$\therefore Z^{-1}[\bar{u}(z)] = u_n = 3^{n-1} - 2^{n-1}$; $n \geq 1$ Ans

EXAMPLE. Find $Z^{-1}\left[\frac{1}{(z-a)^2}\right]$, when

(a) $|z| < a$ (b) $|z| > a$

Sol:→ (a) $|z| < |a| \Rightarrow \left|\frac{z}{a}\right| < 1$

$$\therefore \bar{u}(z) = \frac{1}{(z-a)^2}$$

$$= \frac{1}{a^2}\left(1-\frac{z}{a}\right)^{-2}$$

$$= \frac{1}{a^2}\left[1+2\cdot\frac{z}{a}+3\cdot\left(\frac{z}{a}\right)^2+\cdots\right]$$

$$= \frac{1}{a^2} + 2\cdot\frac{z}{a^3} + 3\cdot\frac{z^2}{a^4} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(n+1)}{a^{n+2}} z^n$$

$\therefore$ Coefficient of $z^n = \dfrac{n+1}{a^{n+2}}$; $n \geq 0$

and coeff of $z^n = \dfrac{-n+1}{a^{-n+2}}$; $n \leq 0$

Hence $Z^{-1}\left[\dfrac{1}{(z-a)^2}\right] = \dfrac{-n+1}{a^{-n+2}}$, $n \leq 0, |z| < a$

(b) when $|z| > a \Rightarrow \left|\frac{a}{z}\right| < 1$

$$\therefore \bar{u}(z) = \frac{1}{(z-a)^2}$$

$$= \frac{1}{z^2}\left(1 - \frac{a}{z}\right)^{-2}$$

$$= \frac{1}{z^2}\left(1 + 2\cdot\frac{a}{z} + 3\cdot\frac{a^2}{z^2} + \cdots\right)$$

$$= \frac{1}{z^2} + \frac{2a}{z^3} + \frac{3a^2}{z^4} + \cdots$$

$$= \sum_{n=2}^{\infty} (n-1)\, a^{n-2} z^{-n}$$

Hence $Z^{-1}\left[\dfrac{1}{(z-a)^2}\right] = (n-1)\, a^{n-2}$; $n \geq 2$, $|z| < a$.

EXAMPLE. Find $Z^{-1}\left[\dfrac{z}{(z+1)^2}\right]$ by division Method.

Sol:- For $|z| > 1 \Rightarrow \left|\frac{1}{z}\right| < 1$

$$\frac{1}{z} - \frac{2}{z^2} + \frac{3}{z^3}$$

$z^2+2z+1 \;)\; z$

$z + 2 + \frac{1}{z}$

$- \quad - \quad -$

$-2 - \frac{1}{z}$

$-2 - \frac{4}{z} - \frac{2}{z^2}$

$+ \quad + \quad +$

$\frac{3}{z} + \frac{2}{z^2}$

$\frac{3}{z} + \frac{6}{z^2} + \frac{3}{z^3}$

$- \quad - \quad -$

$-\frac{4}{z^2} - \frac{3}{z^3}$

$$\therefore \quad \frac{z}{(z+1)^2} = \sum_{n=1}^{\infty} (-1)^{n-1} n z^{-n}$$

$$\Rightarrow \quad Z^{-1}\left[\frac{z}{(z+1)^2}\right] = Z^{-1}\left[\sum_{n=1}^{\infty} (-1)^{n-1} \cdot n \cdot z^{-n}\right]$$

$$= (-1)^{n-1} \cdot n \quad \text{if } n > 0.$$

Example 7. *Find the sequence* $\{u_n\}$ *if*

$$\bar{u}(z) = \frac{15z}{(4-z)\,(4z-1)}; \qquad \frac{1}{4} < |z| < 4$$

Sol. Since the region of convergence is $\frac{1}{4} < |z| < 4 \Rightarrow \left|\frac{1}{4z}\right| < 1$ *and* $\left|\frac{z}{4}\right| < 1$.

Since $\quad \bar{u}(z) = \dfrac{15z}{(4-z)(4z-1)} = 15z\left[\dfrac{1}{15(4-z)} + \dfrac{4}{15(4z-1)}\right]$

$$\Rightarrow \bar{u}(z) = \frac{15z}{(4-z)(4z-1)} = \frac{z}{4-z} + \frac{4z}{4z-1}$$

$$= \frac{z}{4\left(1-\frac{z}{4}\right)} + \frac{1}{\left(1-\frac{1}{4z}\right)}$$

$$= \frac{z}{4}\left[1-\frac{z}{4}\right]^{-1} + \left[1-\frac{1}{4z}\right]^{-1}$$

$$= \frac{z}{4}\left[1+\frac{z}{4}+\left(\frac{z}{4}\right)^2+\left(\frac{z}{4}\right)^3+\ldots\right] + \left[1+\frac{1}{4z}+\left(\frac{1}{4z}\right)^2+\left(\frac{1}{4z}\right)^3+\ldots\right]$$

$$= \left[\frac{z}{4}+\left(\frac{z}{4}\right)^2+\left(\frac{z}{4}\right)^3+\ldots\right] + \left[1+\frac{1}{4z}+\left(\frac{1}{4z}\right)^2+\left(\frac{1}{4z}\right)^3+\ldots\right]$$

$$= \sum_{n=1}^{\infty}\left(\frac{z}{4}\right)^n + \sum_{n=0}^{\infty}\left(\frac{1}{4z}\right)^n$$

$$= \sum_{n=1}^{\infty} 4^{-n} z^n + \sum_{n=0}^{\infty} 4^{-n} z^{-n}$$

Putting $n = -n'$ in the first term

$$= \sum_{n'=-1}^{-\infty} 4^{n'} z^{-n'} + \sum_{n=0}^{\infty} 4^{-n} z^{-n}$$

$$\therefore \quad Z^{-1}\left[\frac{15z}{(4-z)(4z-1)}\right] = \begin{cases} 4^n; & n<0 \\ 4^{-n}; & n \ge 0 \end{cases}.$$