7. Solve $\frac{\partial u}{\partial t} = c^2 \cdot \frac{\partial^2 u}{\partial x^2}, t > 0$ subject to initial conditions

$u(x, 0) = f(x), -\infty < x < \infty$.

Sol. Taking the Fourier transform of both the sides of the equation.

$$F\left\{\frac{\partial u}{\partial t}\right\} = c^2 F\left\{\frac{\partial^2 u}{\partial x^2}\right\} \quad ...(1)$$

$$\frac{d\bar{u}}{dt} = c^2(-is)^2\bar{u} = -c^2 s^2 \bar{u}$$

Its solution is $\bar{u} = Ae^{-c^2 s^2 t}$ ...(2)

But. at $t = 0, \quad v = f(x)$

$$\therefore \quad \bar{u}(s, 0) = F\{u(x, 0)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx = \bar{F}(s)$$

$\therefore$ at $t = 0 \quad \bar{u} = \bar{F}(s)$ ...(3)

using (3) in (2) we have

$\therefore \quad A = \bar{F}(s)$

$\therefore \quad \bar{u} = \bar{F}(s) e^{-c^2 s^2 t}$

On taking inverse Fourier transform, we get

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{u}(s, t) e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} \bar{F}(s) e^{-c^2 s^2 t} ds \quad \textit{Ans.}$$