Example . *Using Fourier transform, solve* $\frac{\partial^4\theta}{\partial x^4}+\frac{\partial^2\theta}{\partial t^2}=0 \quad -\infty<x<\infty, t\geq 0$

satisfying the conditions,

(i) $\theta = f(x)$ *when* $t=0, x>0$

(ii) $\frac{\partial\theta}{\partial t}=0,$ *when* $t=0, x>0$

(iii) $\theta(x,t)$ *and its first three partial derivatives with respect to x tend to zero as* $x\to\pm\infty$.

Sol. Taking the Fourier transform of both the sides, we have

$$F\left[\frac{\partial^4\theta}{\partial x^4}\right]+F\left[\frac{\partial^2\theta}{\partial t^2}\right]=0$$

$$\Rightarrow \quad (-is)^4 F(\theta)+\frac{d^2\overline{\theta}}{dt^2}=0$$

$$\Rightarrow \quad \frac{d^2\overline{\theta}}{dt^2}+s^4\,\overline{\theta}=0$$

$$\Rightarrow \quad (D^2+s^4)\overline{\theta}=0$$

A.E. is $m^2+s^4=0 \Rightarrow m=\pm is^2$

$\therefore\ \overline{\theta}=A\cos s^2 t+B\sin s^2 t$ .....(1)

But we are given that $\theta=f(x)$ when $t=0$

$\overline{\theta}=\overline{f}(s)$ *when* $t=0$

Applying this condition in (1) we obtain

$\overline{f}(s)=A+0 \Rightarrow A=\overline{f}(s)$ .....(2)

From the second boundary condition, we have

$\frac{\partial\theta}{\partial t}=0$ when $t=0$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial \theta}{\partial t} e^{isx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 0 \cdot e^{isx}\, dx = 0$$

$$\Rightarrow \frac{\partial \bar{\theta}}{\partial t} = 0 \text{ when } t = 0$$

Now Differentiating (1) with respect to t, we have

$$\frac{d\bar{\theta}}{dt} = -\mathrm{A}s^2 \sin(s^2 t) + \mathrm{B}s^2 \cos(s^2 t)$$

$\therefore\ 0 = 0 + \mathrm{B}s^2 1 \qquad \therefore \mathrm{B} = 0 \qquad (as\, s^2 \neq 0)$ .....(3)

Putting the values of A and B from (2) and (3) in the equation (1), we have

$\bar{\theta} = \bar{f}(s) \cos s^2 t$

Again applying the inversion formula, we get

$$\theta(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{\theta} e^{-isx}\, ds$$

$$\theta(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(s) \cos(s^2 t) e^{-isx}\, ds$$

Its solution is $\bar{u}_s = Ae^{-2s^2 t}$ ...(3)

But given that $u(x, 0) = e^{-x}$, $\therefore$ when $t = 0$

$$\bar{u}(s, 0) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x, 0) \sin sx\, dx$$

$$\bar{u}(s, 0) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin sx\, dx$$

$$\bar{u}(s,0) = \sqrt{\frac{2}{\pi}} \frac{s}{1+s^2} \qquad ...(4)$$

using (4) in (3) when $t = 0$, we get

$$A = \sqrt{\frac{2}{\pi}} \frac{s}{1+s^2}$$

$\therefore$ from (3) $\bar{u}_s = \sqrt{\frac{2}{\pi}} \frac{s}{1+s^2} e^{-2s^2 t}$

Now, applying the inverse Fourier sine transform, we have

$$u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{s}{1+s^2} e^{-2s^2 t} \sin sx\, ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{se^{-2s^2 t}}{(1+s^2)} \sin sx\, ds \qquad \textit{Ans.}$$

2. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, if $u_x(0, t) = 0$, $u(x, 0) = \begin{cases} x, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$ and $u(x, t)$ is bounded where $x > 0, t > 0$.

Sol. Given equation is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

Taking the Fourier cosine transform of both the sides, we have

$$F_c\left\{\frac{\partial u}{\partial t}\right\} = F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\Rightarrow \quad \frac{d\bar{u}_c}{dt} = -\sqrt{\frac{2}{\pi}}\left(\frac{\partial u}{\partial x}\right)_{x=0} - s^2\bar{u}_c(s, t)$$

$$\Rightarrow \quad \frac{d\bar{u}_c}{dt} + s^2\bar{u}_c = -\sqrt{\frac{2}{\pi}}\left(\frac{\partial u}{\partial x}\right)_{x=0}$$

$$\Rightarrow \quad \frac{d\bar{u}_c}{dt} + s^2\bar{u}_c = 0$$

Its solution is $\bar{u}_c = Ae^{-s^2 t}$ ...(2)

But $u(x, 0) = \begin{cases} x, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$; when $t = 0$

$$\Rightarrow \quad \bar{u}_c(s, 0) = f_c\{u(x, 0)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty u(x, 0)\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^1 x\cos sx\, dx\right] = \sqrt{\frac{2}{\pi}}\left[\frac{x\sin sx}{s} + \frac{\cos sx}{s^2}\right]_0^1$$

$$\bar{u}_c(s, 0) = \sqrt{\frac{2}{\pi}}\left[\frac{\sin s}{s} + \frac{\cos s - 1}{s^2}\right] \quad ...(3)$$

$\therefore$ using the above condition (3) in (2), we have

$$A = \sqrt{\frac{2}{\pi}}\left[\frac{\sin s}{s} + \frac{\cos s - 1}{s^2}\right] \quad ...(4)$$

putting the value of A in (2), we get

$$\bar{u}_c(s, t) = \sqrt{\frac{2}{\pi}}\left[\frac{\sin s}{s} + \frac{\cos s - 1}{s^2}\right]e^{-s^2 t}$$

Now applying the inverse Fourier cosine transform, we have

$$u(x, t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{u}_c(s, t)\cos sx\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \left(\frac{\sin s}{s} + \frac{\cos s - 1}{s^2}\right)e^{-s^2 t}\cos sx\, ds$$

Ans.

4. Use the method of Fourier transform to determine the displacement $y(x, t)$ of an infinite string, given that the string is initially at rest and that the initial displacement is $f(x)$, $-\infty < x < \infty$. Show that the solution can also be put in the form

$$y(x, t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$$

Sol. Displacement of the string is governed by one dimensional wave

equation $\quad \dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2} \quad$ ...(1)

where $y(x, t)$ is the displacement of string at any time t,

$-\infty < x < \infty, t > 0$ and $c^2 = \dfrac{T}{\rho}$

Taking the Fourier transform of both the sides of equation (1), we have

$$F\left[\frac{\partial^2 y}{\partial t^2}\right]=c^2F\left[\frac{\partial^2 y}{\partial x^2}\right]$$

$$\Rightarrow \quad \frac{d^2\bar{y}}{dt^2}=c^2(-is)^2\bar{y}(s,t)=-c^2s^2\bar{y}(s,t)$$

$$\Rightarrow \quad \frac{d^2\bar{y}}{dt^2}(s,t)+c^2s^2\bar{y}(s,t)=0$$

whose solution is

$$\bar{y}(s,t)=A\cos cst+B\sin cst \qquad \text{...(2)}$$

Initially the string is at rest *i.e.*. $\frac{\partial y}{\partial t}=0$ at $t=0$

$$\therefore \quad \left(\frac{d\bar{y}}{dt}\right)_{t=0}=F\left[\left(\frac{\partial y}{\partial t}\right)_{t=0}\right]=0 \qquad \text{...(3)}$$

differentiating (3) w.r.to t, we have

$$\frac{d\bar{y}}{dt}=-Acs\sin cst+Bcs\cos cst \qquad \text{...(4)}$$

using (3) in (4), we get $Bcs=0$ or $B=0$

$\therefore$ (2) can be written as $\bar{y}(s,t)=A\cos cst$ ...(5)

Also at $t=0, y=f(x)$

$$\therefore \quad \bar{y}(s,0)=F[y(x,0)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{isu}\,du=\bar{f}(s)$$

$$\therefore \quad \bar{y}(s,0)=\bar{f}(s) \qquad \text{...(6)}$$

at $t=0$ from (5) and (6), we get $A=\bar{f}(s)$

$$\therefore \quad \bar{y}(s,t)=\bar{f}(s)\cos cst$$

Taking the inverse Fourier transform, we have

$$y(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{f}(s)\cos cst\, e^{-isx}\,ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(s) \left\{ \frac{e^{icst} + e^{-icst}}{2} \right\} e^{-isx} ds$$

$$= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(s) \left\{ e^{-is(x-ct)} + e^{-is(x+ct)} \right\} ds$$

$$= \frac{1}{2} \{ f(x-ct) + f(x+ct) \}$$

$$\therefore \quad y(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)]$$

Ans.

5. If the initial temperature of an infinite bar is given by

$$u(x) = \begin{cases} u_0 & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases}$$

Determine the temperature at any point x at any instant t.

Sol. We know that the heat diffusion equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{...(1)}$$

where $u(x, t)$ is the temperature at any point x and at any instant t.
Taking the Fourier transform of both the sides of (1), we get

$$F\left\{ \frac{\partial u}{\partial t} \right\} = c^2 F\left[\frac{\partial^2 u}{\partial x^2} \right]$$

$$\Rightarrow \quad \frac{d\bar{u}}{dt}(s, t) = c^2(-is)^2 \bar{u}(s, t)$$

$$\Rightarrow \quad \frac{d\bar{u}}{dt} = -c^2 s^2 \bar{u}$$

whose solution is $\bar{u} = Ae^{-c^2 s^2 t}$ ...(2)

But given that when $t = 0$, $u(x, 0) = f(x) = \begin{cases} u_0, & |x| < a \\ 0, & |x| > a \end{cases}$

$$\therefore \quad \bar{u}(s, 0) = F[u(x, 0)]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, 0) e^{isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} u_0 e^{isx} dx$$

$$\bar{u}(s,0)=\frac{u_0}{\sqrt{2\pi}}\frac{1}{is}\left[e^{isx}\right]_{-a}^{a}=\frac{u_0}{is\sqrt{2\pi}}\left(e^{isa}-e^{-isa}\right)$$

$$=\frac{u_0}{s\sqrt{2\pi}}2\sin as=\sqrt{\frac{2}{\pi}}u_0\frac{\sin as}{s}$$

$$\therefore\quad \bar{u}(s,0)=\sqrt{\frac{2}{\pi}}u_0\frac{\sin as}{s} \qquad ...(3)$$

Now at $t = 0$, using (3) in (2) we get $A=\sqrt{\frac{2}{\pi}}u_0\frac{\sin as}{s}$ ...(4)

$$\therefore\quad \bar{u}=\sqrt{\frac{2}{\pi}}\frac{u_0\sin as}{s}e^{-c^2s^2t}$$

now taking inverse Fourier transform, we get

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{u}(s,t)e^{-isx}ds$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}u_0\frac{\sin as}{s}e^{-c^2s^2t}e^{-isx}ds$$

$$=\frac{u_0}{\pi}\int_{-\infty}^{\infty}\frac{\sin as}{s}e^{-c^2s^2t}(\cos sx-i\sin sx)ds$$

$$=\frac{2u_0}{\pi}\int_{0}^{\infty}\frac{\sin as}{s}e^{-c^2s^2t}\cos sx\,ds$$

(by definition of odd/even function)

$$\Rightarrow\quad u(x,t)=\frac{u_0}{\pi}\int_{0}^{\infty}\frac{e^{-c^2s^2t}}{s}[\sin(a+x)s+\sin(a-x)s]ds \qquad \textit{Ans.}$$