

#### JAIPUR ENGINEERING COLLEGE AND RESEARCH CENTRE

Year & Sem – I Year & 1 Sem Subject – Engineering Mathematics Unit – II (SEQUENCES AND SERIES) Presented by – (Dr. Tripati Gupta, Associate Professor)





### VISION AND MISSION OF INSTITUTE

#### **VISION OF INSTITUTE**

To became a renowned centre of outcome based learning and work towards academic professional, cultural and social enrichment of the lives of individuals and communities.

#### **MISSION OF INSTITUTE**

- Focus on evaluation of learning, outcomes and motivate students to research aptitude by project based learning.
- Identify based on informed perception of Indian, regional and global needs, the area of focus and provide platform to gain knowledge and solutions.
- Offer opportunities for interaction between academic and industry .
- Develop human potential to its fullest extent so that intellectually capable and imaginatively gifted leaders may emerge.

#### **Engineering Mathematics: Course Outcomes**

#### **Students will be able to:**

CO1. Understand fundamental concepts of improper integrals, beta and gamma functions and their properties. Evaluation of Multiple Integrals in finding the areas, volume enclosed by several curves after its tracing and its application in proving certain theorems.

CO2. Interpret the concept of a series as the sum of a sequence and use the sequence of partial sums to determine convergence of a series. Understand derivatives of power, trigonometric, exponential, hyperbolic, logarithmic series. **Engineering Mathematics: Course Outcomes** 

CO3. Recognize odd, even and periodic function and express them in Fourier series using Euler's formulae.

CO4. Understand the concept of limits, continuity and differentiability of functions of several variables. Analytical definition of partial derivative. Maxima and minima of functions of several variables Define gradient, divergence and curl of scalar and vector functions.

# CONTENTS (TO BE COVERED)

# SEQUENCES AND SERIES

### A Function whose domain in Sequence:

# the set of natural numbers N and range,

### a subset of real numbers R in called a

Sequence.

A sequence in of the form {(1,x,1),(2,x2),...,(h,x)} where x1, x2,..., x2 are real numbers. The real number on that the sequence associated with the positive integer h is called the image of h under the Sequence.

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Crenerally, it is dended by  $\Sigma x_1, x_2, \ldots, x_n$ . Here x,, x2,..., xc, ere the terms of the Sequence and so xin the nth Jerm of the Sequence A sequence has its ht term denoted by Excly.

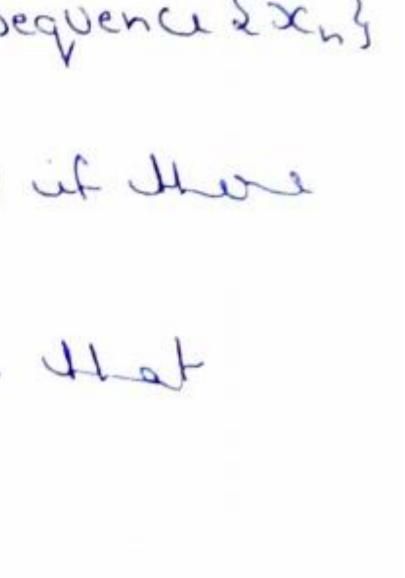
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The Range ! The range set in the set Consisting of all distinct elements of a sequence, without repetition and without regard to the position of a term. Thus, He range may be a finite set or an infinite set.

Bounds of a Sequence! A sequence Sxng cis Bounded - above Sequence: is said to be bounded above if there exists a real number M such that xn EM then.

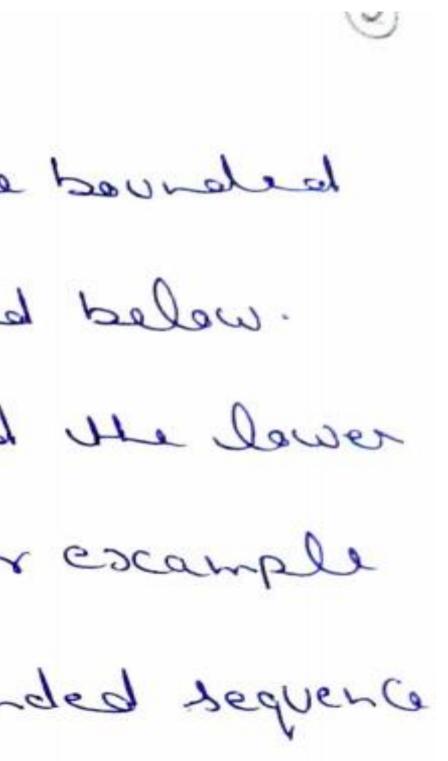
in Bounded-below Sequence : A sequence & xn} eath ti welsed belowed at at the is exists a real number in such that DON ZM. + NEN.

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Cilis Bounded Sequence A sequence pour said to be bounded if it is bounded above and below. Mand in are the upper and the lower bounds of the sequence. For escample Sch = & C-15 ; heng in a bounded sequence



Convergence of a Sequence

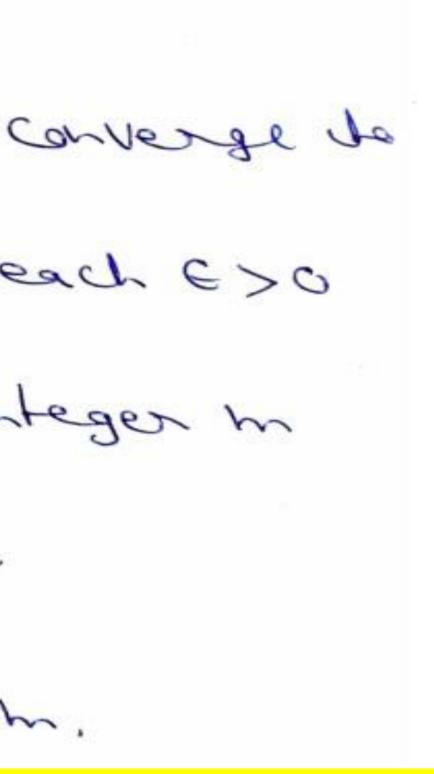
A sequence & xng in said to converge to

a real number " I' if for each E>0

Here exists a positive integer m

( depending on E) such that

1xn-21<E, for all h > m.



Mattenatically, we write

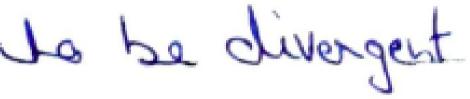
xn->lann->~ or nling xn=l

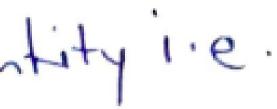
\* A sequence & sch in said to be divergent

it not a finite quantity i.e.

it here xn = + x 0x - x







Examples: cir Let a sequence  $Sx_1 = Sh^2 g$ Then hits he = a c which in hat finite) Hence & x\_2 in divergent. cin Let Sxn3= [th; hen] Then had the = 0 ca finite quantity) Hence, the sequence & scaling Convergent.



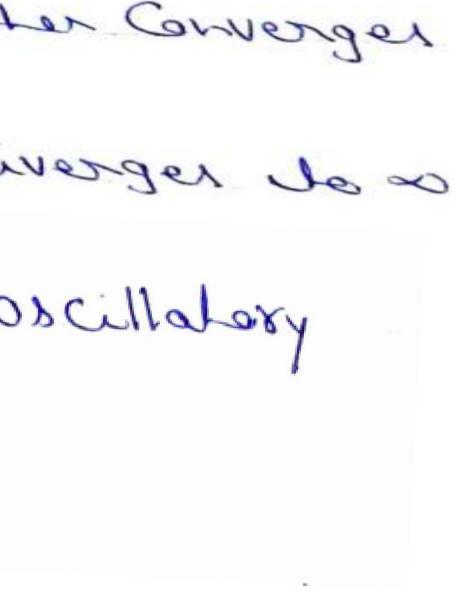
Oscillabory Sequence

A sequence 2 x 2 which neither Converger

Le a finite number nor diverger le 20

08 - ~ in said to be an oscillabory

Sequence.



Escample: 2x,3=2(-1) goscillater finitely

between -1 and 1 and the sequence

2 xn3 ={nc-13 ) Oscillater infinitely between

- a curad as.



Note:

# c's Every convergent sequence has a Unique

limit.

#### Every Convergent sequence in (II)

bounded.

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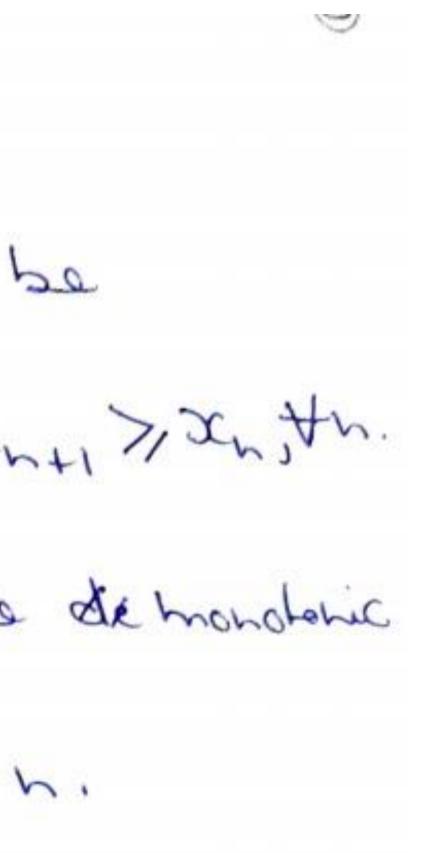
Monotonic Sequence

# A sequence Excide in said to be

monotonic increasing if schild the

A sequence Excit in baid to be demonsteric

decreasing if  $x_{n+1} \leq x_n, \forall h.$ 



Thus, a sequence & xnz in said to be

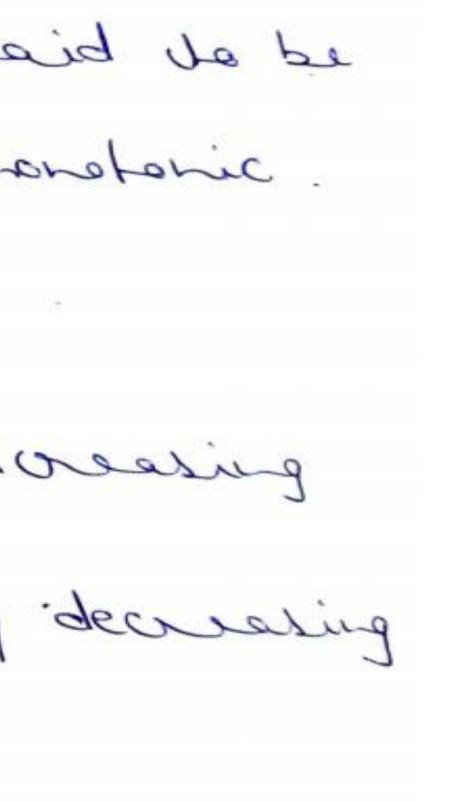
monotonic it it is either monotonic.

increasing or decreasing.

A sequence for fin strictly in creasing

if xny >x, th and strictly decreasing

if xnt, <x, th.

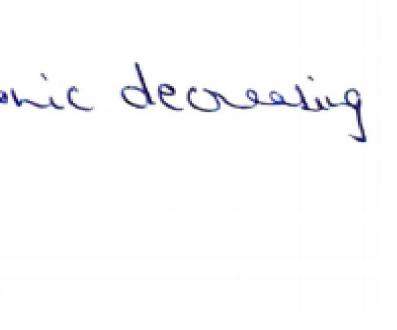


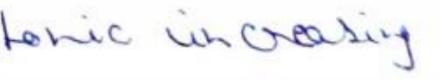
Examples: cisflytyty. ... fin a morehouse decreasing

Sequence.

ciis & sch = { h } in a monotonic in crasing

Sequence.





Infinite Series

IF < Und be a bequence of real numbers

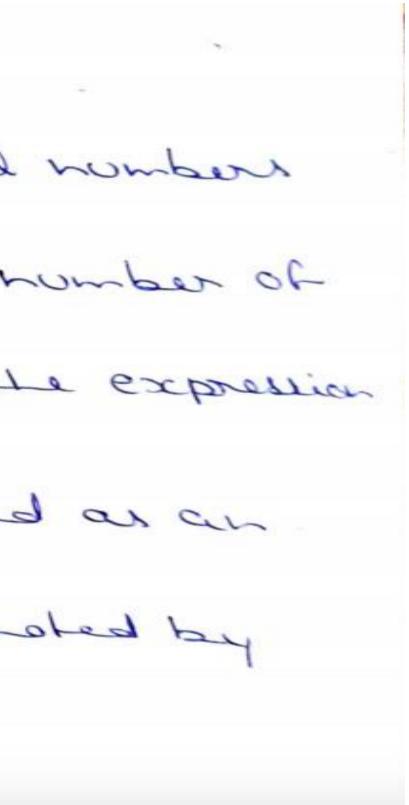
Her the sum of the infinite humber of

deans of this sequence i.e. the expression

Uit U2t. . . tunt. . . in defined as an

infinite series and indended by

Son or Sun.



A sequence < Sh > where Sh denotes the

som of the first ntern of the sociel,

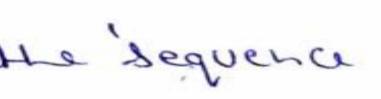
Thus, Sh= Uituzt ton th.

The Sequence < S\_> in called the sequence

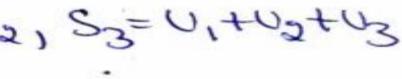
of partial suns of the series and the

partial sums, S,=U, 'S2=U, +U2, S3=U, +U2+U3

and so on.







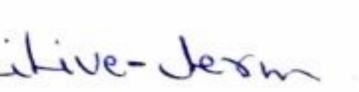
Positive term Series

### The Serier I've in called a positive- Jern

### series if each lexe of this series

in positive i.e.

### $2u_{h} = u_{1} + u_{2} + \cdots + u_{h} + \cdots$





Alternating Series

# A serier whole terms one alternatively

positive and negative i.e.

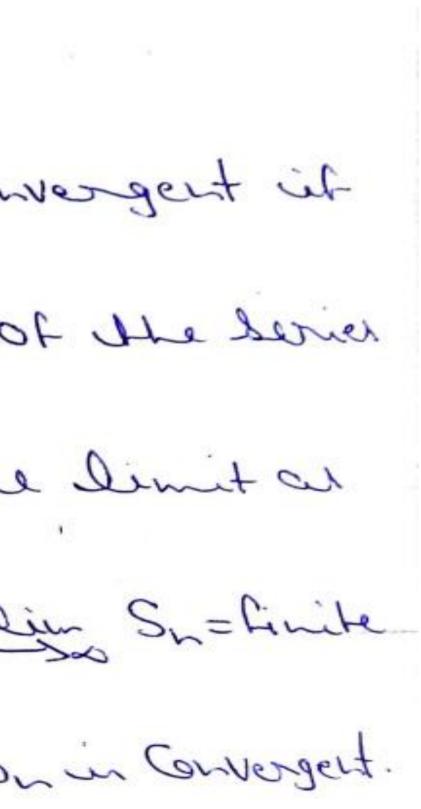
 $\sum v_{n} = v_{1} - v_{2} + v_{3} - v_{4} + \cdots$ 





Convergent Series

A serier Sur in said to be Convergent it the sum of the first n terms of the beries lends to a finite and unique limit a h Jends de infinityjie. if her Sh=finite and unique then the series Ew in Convergent.



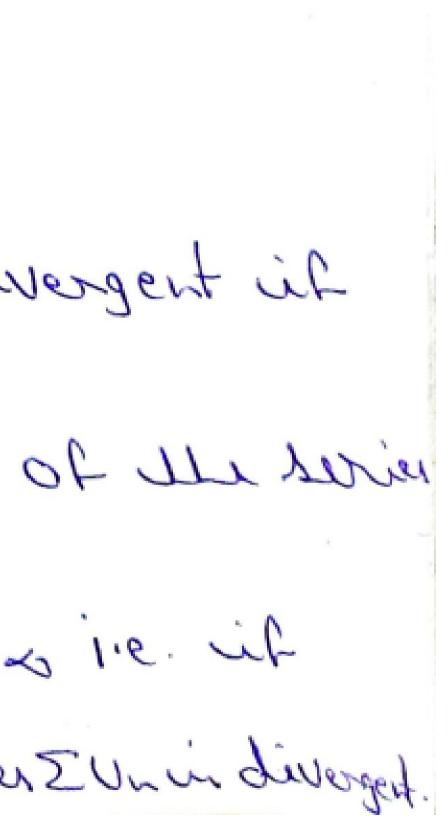
Divergent Series

# Ai serier Ey, in said to be divergent if

the sum of the first in terms of the series

lends to to or - so as h-so it. if

him Shato or-o, then the series ZUnin diverget.



Oscillatory Series

The oscillatory series are two types:

cis oscillate Finitely

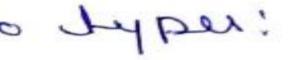
A serier Eur in said to oscillate Finitely

if the sum of its first h terms lends

to a finite but not unique limit as

h lends to infinity, i.e., it has Sn = finite

but not unique the Sun Oscillaler Finitely.

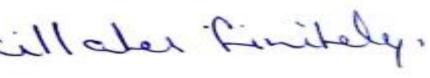












city Oscillabe Infinitely

A series Eq is said to oscillate infinitely if

He sum of its first n term oscillater infinitely,

i.e. if him Sh= too or-oo ball then the series

Zur Oscillaber infinitely.

Note: cis The Convergency or divergency of a

Series is not affected by altering, adding,

Or neglecting a finite number of its terms.

Ciis The convergency or divergency of a

Series in not affected by the multiplication

of all terms of the series by a fixed timber

A Necessary Condition For Convergence

A necessary Condition for a positive - term

serier Sur to Converge in Hat lin Un=0

Note: (i) The above Condition in Lat sufficient.

where Un = 1, i. n Soun = h So h = 0

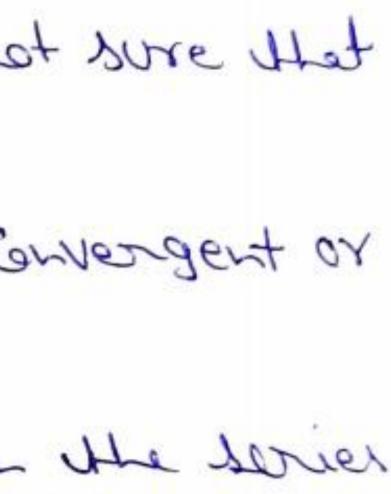


ciis If htmy Un=0, we are not sure that

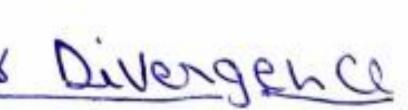
### whether the series I've in Convergent or

# hot but if ling Unito, then the series

Zur in divergent.



Cauchy's Fundamental Test for Divergence IF her Un to, the series Zun in divergent.



Example! Test 110 Convergence of the series Solution! Un = h+1 in hospo Un = heissoh = h= 5 ~ 1 = 1 = 1 = 0 Hence, by Cauchy's test Sun indivergent.



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Examples! Discuss the convergence of the Devies

2= C-15.

Solution: IU = IC-15

 $S_{n=} (-1)^{0} + (-1)^{1} + (-1)^{2} + \cdots + (-1)^{n} + (-1)^{n} + (-1)^{n} + \cdots + (-1)^{n} + (-1)^$ 

= 10x0 accordingly as hoddor even.

i noto Sho I or O i.e. finite but hot unique.

Hence, the series I'm in a finitely oscillating series.

Geometric Series

### The series 1+2c+22+23+ - ain

(i) Convergent if 1x1×1

(ii) Divergent if x >1

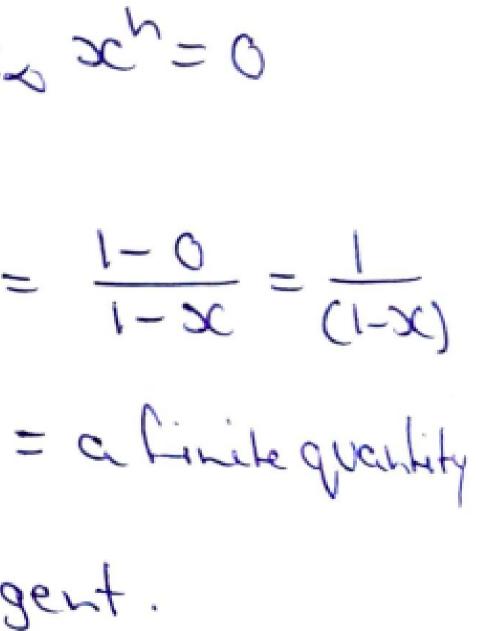
(iii) Oscillabory if x <-1



Proof! Cijuhen Isci<1 then here so sch = 0

 $\frac{1}{1-x} = \frac{1}{1-x} = \frac{1-0}{1-x} = \frac{1}{1-x} = \frac{1}{1-x}$ 

Hence, the series in Convergent.

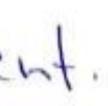


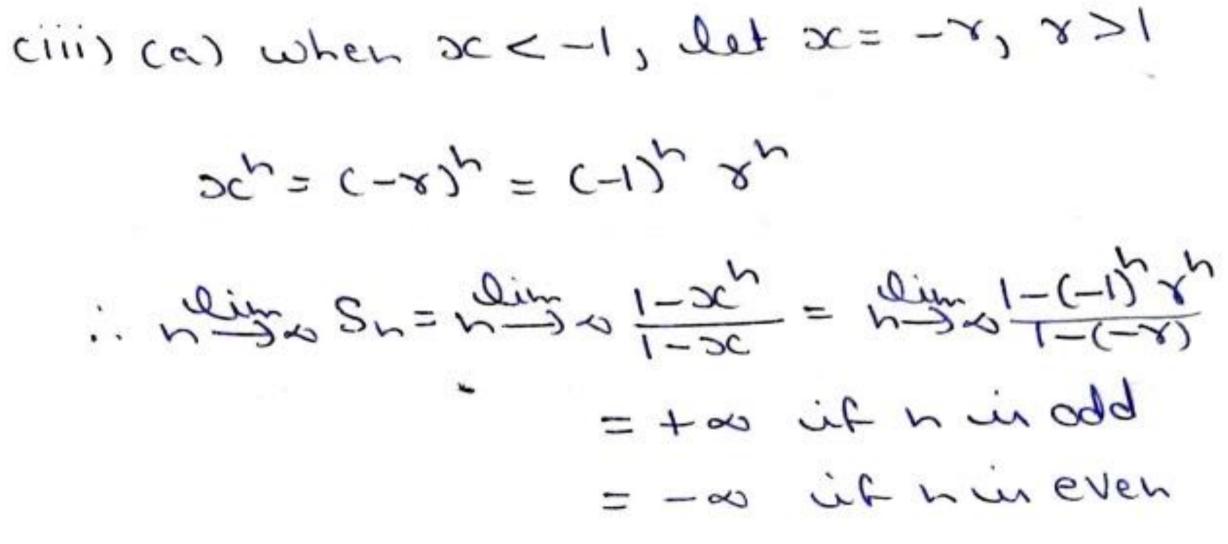
(ii) cas when x>1, lig sc = ~, Hen lim Sn= lim 20-1 = 0

Hence the series in divergent.

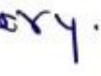
Cb) when x=1, the socies becomes 1+1+1+1+1+.~~ in Sh= 1+1+1+ - . When is the benier indition.







Hence, the series in oscillabory.

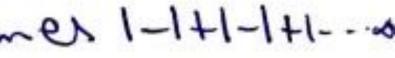


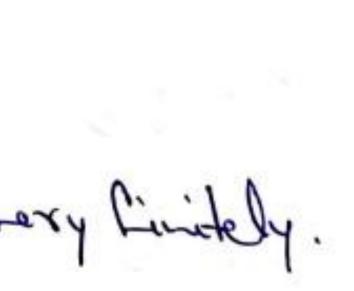
### when x=-1, the series becomes 1-1+1-1+1-1+1-... CP)

.: Sn = 1-1+1-1+ - . . h Hime

noso Sn= 0 if his even = 1 if hinodd

## Hence, the series in Oscillabery Linkly.





Alternating Series

A serier whole terms are alternatively

positive and negative in called an alter-

halling Series.

Example: 1-1+1-4+--0



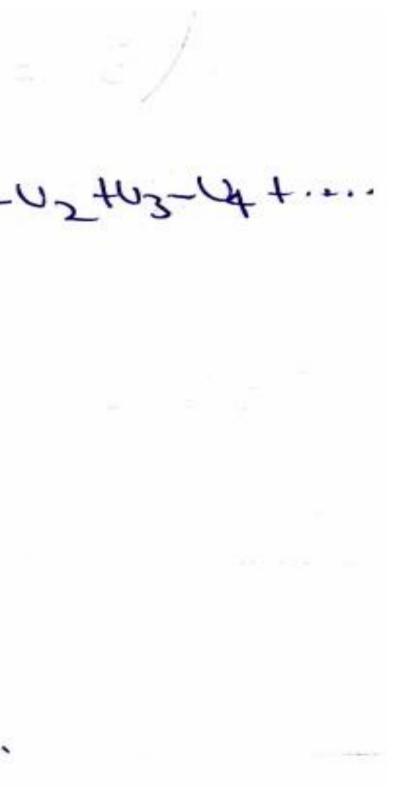
Leibnitz Test

If the alberting series U, -U2 tuz-Uf +...

conso the in such that

cii) Ling UN=0

then the Serier Converger.



Example : Test the series 2 - 3 + 4 - 5 + ... Solution! In the given series, we have cis the terms are alternately, the and -ve ciis the terms are continually decreating ciii) lim Un = lin h+1  $= \lim_{n \to \infty} \frac{1+1}{n^2} = \frac{1+0}{n} = 0$  (finite) Hence, the given alternating series by Leibnitz's lest in Convergent.





Example: Test the series 1-1 + 4 - 5+ 10 - .... Solution ! The given series can be written as  $1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \cdots$ In this series, we find that cis the terms are alternatively + ve and -ve. cii) the levers are Continually decreating citis heimo  $U_n = heimo \frac{1}{2h-1} = 0$ Hence, the given series in Convergent.





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Example: Test the Convergency of the series 1-1=++--++.... Solution:  $U_n = \frac{1}{\sqrt{n}}$ ,  $U_{n-1} = \frac{1}{\sqrt{n-1}}$ In the given series, we have is the Jermi are alternatively the and-re.  $v_{h} < v_{h-1}$ CIII ciii) here un = here = 0 Hence, the alternating series in Convergent.



Escample: Test the Convergency of the teries

 $\frac{\sqrt{2} - \sqrt{1}}{1} = \frac{\sqrt{3} - \sqrt{2}}{2} + \frac{\sqrt{4} - \sqrt{3}}{4} = \frac{\sqrt{5} - \sqrt{4}}{4} + \cdots$ Solution: Here,  $U_n = \left[\frac{\sqrt{1} + \sqrt{1} - \sqrt{n}}{n}\right]$ 

In the given series, we have

cij the terms are continually decreasing as

Un >Unti for all n.

-Ve.



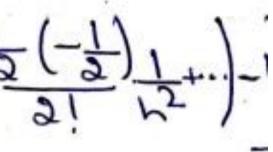
 $\sqrt{h+1} - \sqrt{h} = \lim_{n \to \infty} \frac{1}{n} \left[ \sqrt{h} \frac{1}{n} - \sqrt{h} \right]$ Ciiij

= him 1 [(1+1)/2 -1]

1+ディア+テ(-デ)ブ

 $\frac{1}{8} \left[ \frac{1}{2h} - \frac{1}{8h^2} + \cdots \right] = 0$ 

Lorraling Deries in Convergent. Hence, the all



Example! Test the convergence of the Series 2-3+4-5+--

Solution: Llere Un = <u>h+1</u>.

In the given series, we have (i) the terms are alternately the and we cii) the terms are decreasing order i.e. Un < Un-1 cin)  $\lim_{n\to\infty} u_n = \lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} (1+\frac{1}{n}) = 1 \neq 0$ Hence, the third Condition of the alterating tericrot salisfied, be the series in not Convergent.

However, we can write the given series as



 $(1+1) = (1+\frac{1}{2}) + (1+\frac{1}{2}) - (1+\frac{1}{2}) + ...$ 08 (1-1+1-1+-)+(1-==+=+-) The series in the II'd bracket in Convergent, Since the value of it is log (1+1) i.e. log 2. But, the series in the Ist bracket in an oscillating series whose value in either O or 1 accordingly as his even or odd. i the som of a lever of the given so mode all is (seel+10) ro (seel +0) in zing no accordingly as h in even or odd i.e. leg 201 CI+ legg) if his even or odd. Hence, by the definition, the given series in Oscillating.

Alternating Convergent series

There are show suppor of alloweding Convergent Series CirAbsolutely Convergent series CirConditionally Convergent series

Absolutely Convergent serier: If U, +U2 + --- be such 1U, 1+1U21+1U31+... be Convergent then U; + U2+U3+ --- in Called absolutely Convergent.

Conditionally Convergent Series:

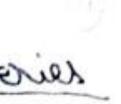
IF IU, I+IU2I+ - . be divergent and U, +U2+U3+ - . be Convergent then U, +U2+U3+ .. in celled Conditionally Convergent.

Example: The serier ZUN=1-1+ + 1= - 1= + 1= -" in absolutely Convergent, because geometric series of positive terms with Common ratio 1 <1. Example: The series 1-1+2-4+... win Conditionally Convergent. because Zlunl=1+ 1++++++++ ... in not Convergent by b-series test. SILL= 2 h= 2 hp so pel. Hence it in divergent.

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Important lest for convergence of Infinite Series The Heavens and rules that have been Considered and discussed in previous section, makes is enable to determine the convergence Of an infilite series in general But, it in alway's not possible do find the sum of n Jerns of Spice (i.e. S.).

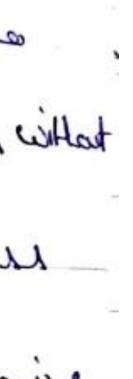




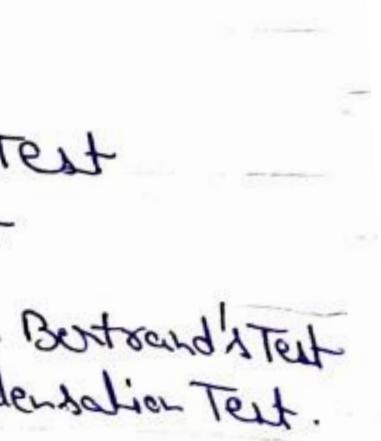




Therefore we require some another way to test the convergence of an infinite. Levier withat using Sn. In this section we will discuss Some c tests' which crable is to determine the Convergence independent of evaluation of Sh. These lests are as follows



1. p-Series Test2. Comparison Test 3. D' Alembert Ratio Test 4. Reabers Test 5. Craws Test G. Cauchy's Root Test 7. Legarithmic Test 8. De Morgan's and Bestrand's Test 9. Cauchy's Integral Test to Cauchy Condensation Test.



Flow chart for Tests the Convergence of positive doon societ Apply Comparison test if it fails, shen apply Alembert radio test if it hails, shen apply Raabe's Test if it fails, Her apply Cauchy's Root Test if it hails, then apply Logarithmic Test if it hails, then apply Crauss Test



5- Series Test The infinite series Eur of the form Zthe = the + the + the + the + ... in known as p-series and it is cis Convergent, if \$>1 CIIS Divergent, if b ≤ 1. Example: The series to the the thet. in divergent as p= 1 < 1. CHerepmean Common Power



Comparison Test

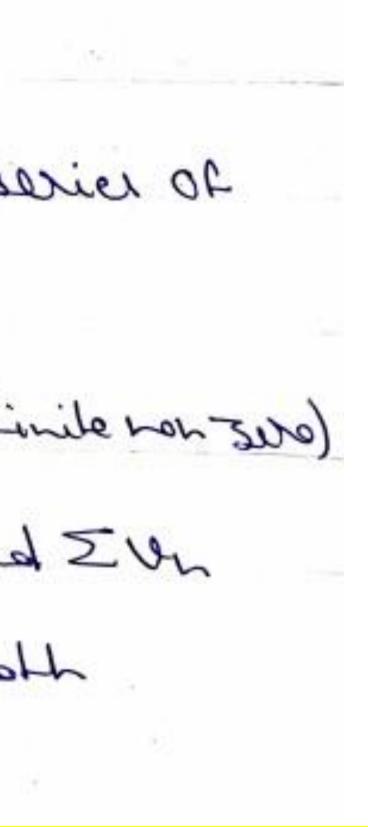
IF I'm and I'v be Juo given series of

positive terms such that here un = 1 = 0 ( I in finile non 300)

No I kno with Sires Du and I'v then

are either both Convergent or both

divergent.

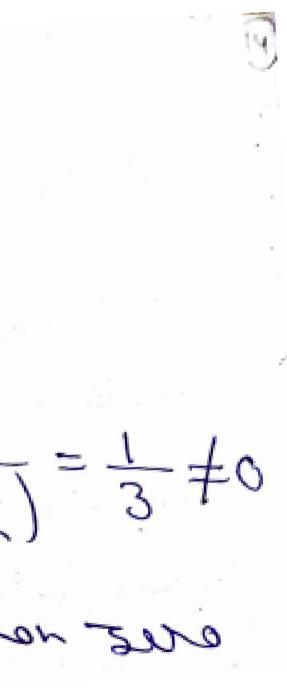


Example: Test the Convergence of the being 
$$\frac{1}{2} + \frac{13}{5} + \frac{13}{8} + \cdots + \frac{1}{3n-1} + \cdots$$
  
Solution! Here, the nth desmost the given in  $\frac{1}{3n-1}$   
i.e.  $U_n = \frac{1}{3n-1} = \frac{1}{3n(1-\frac{1}{3n})}$   
 $= \frac{1}{3\sqrt{n}(1-\frac{1}{3n})}$   
Now, take an auxiliary deriven  $\sum U_n = \frac{1}{3\sqrt{n}} + \frac{1}{\sqrt{n}}$   
Then applying Comparison Test  $\frac{U_n}{U_n} = \frac{1}{3\sqrt{n}(1-\frac{1}{3n})} + \frac{1}{\sqrt{n}}$ 

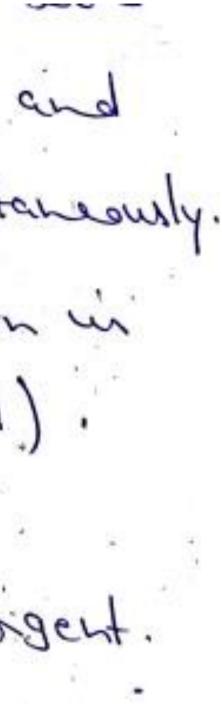
Server

in Series

そう 100 3 = him 1500 -0



Finite quantity: Therefore IUL and Zun Converge and diverge Simultaneously. But I've in divergent (as Zue in of the form 5 1: and b= 1 < 1). Consequently the given series EUn= Zut in also divergent.



Example ! Test the Convergency of the series  $\frac{2}{1p} + \frac{3}{2p} + \frac{4}{3p} + \frac{5}{3p} + -$ Solution! The not seen of the given deriver in  $U_{n} = \frac{h+1}{h} = \frac{h(1+1/h)}{h} = \frac{(1+1/h)}{h}$ Let Un = ---i here Un = here (1+ 1) = 1 (a fink quality



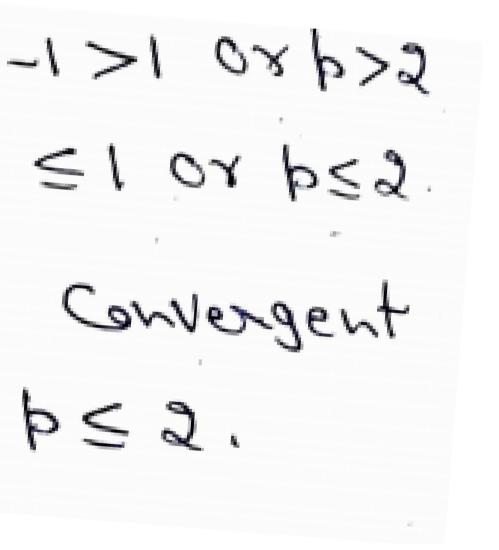








# But Sun in Convergent if p-1>1 0x p>2 and it in divergent if p-1 ≤1 or p≤2. Hence, the given series in Convergent if b>2 and divergent if b < 2.



Example: Test for Convergence of the let  

$$\frac{1}{1\cdot2\cdot3} + \frac{3}{2\cdot3\cdot4} + \frac{5}{3\cdot4\cdot5} + \cdots$$
Solution: Let the given densite be dense  

$$\frac{50n}{n(n+1)(n+2)}$$

$$= \frac{n(2-1/n)}{n(2-1/n)}$$

$$= \frac{(2-1/n)}{n^{2}(1+\frac{1}{n})(1+\frac{2}{n})}$$

$$= \frac{1}{n^{2}}$$
Let  $V_{n} = \frac{1}{n^{2}}$ 

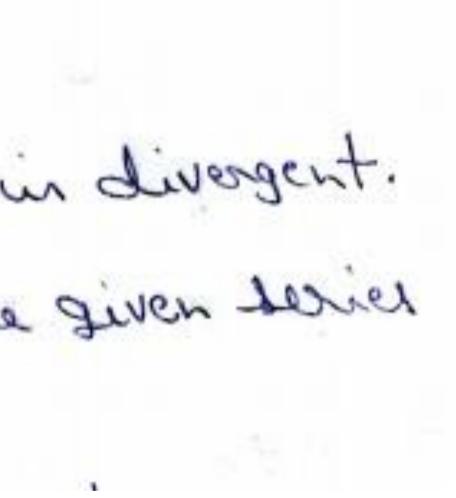
$$\frac{(2-1/n)}{(1+\frac{1}{n})(1+\frac{2}{n})} = 2^{1}$$

vier d by Linte

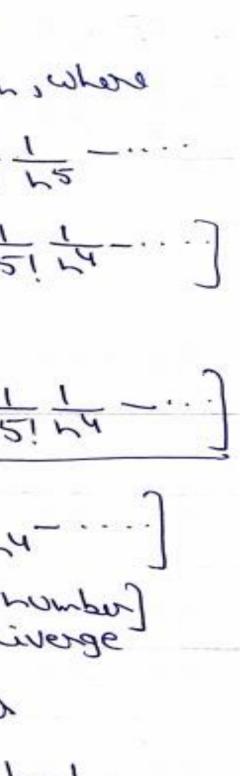
But I un and I've Converge and diverge Limutaneously. Since ZUh = Ztz and it in of the form Z to with b=2>1, therefore Z Up in Convergent. Hence, by Comparison Test Zun in also Convergent.

Example: Test for Convergence of the Series whole general serm in given by Un= V(2+1) - V(2-1) Solution: Un= V(n2+1)-V(h2-1) =  $h(1+\frac{1}{2})^{1/2} - h(1-\frac{1}{2})^{1/2}$  $= h \left[ (1 + \frac{1}{12})^{1/2} - (1 - \frac{1}{12})^{1/2} \right]$  $= h \left[ \left\{ 1 + \frac{1}{2h^2} + \frac{1}{2} \left( -\frac{1}{2} \right) + \frac{1}{h^4} + \frac{1}{2} - \frac{1}{2} \right] \right]$  $= a \left[ \frac{1}{a x^2} + \frac{1}{2} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \frac{1}{b c} + \frac{1}{2} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \frac{1}{b c} + \frac{1}{2} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \frac{1}{b c} + \frac{1}{2} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \frac{1}{b c} + \frac{1}{2} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \frac{1}{b c} + \frac{1}{2} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \frac{1}{b c} + \frac{1}{2} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \frac{1}{b c} + \frac{1}{2} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \frac{1}{b c} + \frac{1}{2} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \frac{1}{b c} + \frac{1}{2} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \frac{1}{b c} + \frac{1}{2} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \frac{1}{b c} + \frac{1}{2} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \frac{1}{b c} + \frac{1}{2} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \frac{1}{b c} + \frac{1}{2} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \frac{1}{b c} + \frac{1}{2} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \frac{1}{b c} + \frac{1}{2} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \frac{1}{b c} + \frac{1}{2} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \frac{1}{b c} + \frac{1}{2} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \frac{1}{b c} + \frac{1}{2} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \frac{1}{b c} + \frac{1}{2} \left( -\frac{1}{2} \right) \frac{1}{b c} + \frac{1}{2} \left( -\frac{1}{2} \right) \frac{1}{b c} \right) \frac{1}{b c} + \frac{1}{2} \left( -\frac{1}{2} \right) \frac{1}{b c} \right) \frac{1}{b c} \frac{1}$  $=\frac{1}{2}+\frac{1}{8}+\frac{1$ Let ZUN= th · h - so Un = h - so h Bh5 -(+---)=1 (finile (1+1-

But IUN = 1 = 5 th Because of p=1, IU = 54 in divergent. Hence, by Comparison best the given besiel I've in divergent.



Example: Test the series 5 Sin (1) Solution ! Let the given series be IUn, where  $U_n = Sin(\frac{1}{n}) = \frac{1}{n} - \frac{1}{31}\frac{1}{n^3} + \frac{1}{51}\frac{1}{15} - \frac{1}{15}$ = - [1-312+514-...] Let SUL= 5t ; where Un = 1 : here Un = here 11-1-1+1+1-... · Mh = h= 50 [1- 1- + 120 h4 - ...] Hence, SUL and SUL Converge or diverge vires proilide all tul. alleget 5 0 = 5 1/2 is divergent as b=1. Hend, the given series in divergent.



D'Alembert & Ratio Test

Let I un be a series of positive learny, such that hen device  $\sum_{n=1}^{\infty} \sum_{n=1}^{\infty} will be$ Cis Convergent if X>1 ciis Divergent if X < 1 and ciii) fails if 1=1



Example : Test the series 
$$\frac{3c_1}{1\cdot 2} + \frac{3c_2}{3\cdot 3} + \frac{3c_3}{3\cdot 4} + \cdots$$
  
Solution:  $U_n = \frac{3c_n}{n(n+1)}$  so  $U_{n+1} = \frac{3c_{n+1}}{(n+1)(n+2)}$   
NOW  $\frac{U_n}{U_{n+1}} = \frac{3c_n}{n(n+1)} \frac{(n+1)(n+2)}{3c_{n+1}}$   
 $= \frac{(n+2)}{n\cdot 3c_{n+1}}$   
 $= \frac{1}{3c_{n+1}} \left(1 + \frac{3}{n}\right)$   
 $\therefore n \rightarrow \infty \frac{U_n}{U_{n+1}} = n \frac{1}{3c_{n+1}} \left(1 + \frac{3}{n}\right) = \frac{1}{3c_{n+1}}$   
From the solic dest, we conclude that

## + - - . (h+2)





the given series EU in and (i) Convergent if 1 >1 or DC<1 cin divergent if 1 <100 x>1 ciii) If x=1 then this test fails and the given series become SUL, where Where Un= 1 = 1 white dean Un= hant = 12(1+1/2)







Let len = 1/2  $h = h = h = h = (\frac{1}{1+1}) = 1$ , which in finite quality. 1202= 512= 5th=>b=2>1, Hence by Comparison Test the given serier. Ethin also Convergent as Ele Convergent as b=2>1=) > 20 in Convergent.

Example: Test the series 
$$\frac{1}{2V_1} + \frac{3c^2}{3V_2} + \frac{3c'_1}{4V_3} + \frac{3c'_1}{5V_1} + \frac{3c'_2}{4V_3} + \frac{3c'_1}{5V_1} + \frac{3c'_1}{4V_3} + \frac{3c'_1}{5V_1} + \frac{3c'_1}{4V_3} + \frac{3c'_1}{5V_1} + \frac{3c'_1}{4V_3} + \frac{3c'_1}{5V_1} + \frac{3c'_1}{4V_3} + \frac{3c'_1}{5V_1} + \frac$$

From the ratio test, we conclude that the given series I've in Convergent or divergent accordingly an 1 >1 or <1 i.e. for is x2<1 series in Govergent. (ii) x2>1 series in divergent. IF sc2=1, then this test fails and He given socies reducer to ZU what Wh Jean Un= ChtDIn = 1312(1+4/2)

the = 1 is then by Comparison Jest him 1 = 1 ( finile quality his Un =. Hence SU and SU Converge or diverge dogetter but SUL = 5-1312 = 5-15 30 /= 3/ Hence, the series in Convergent. Then the given series in Convergent if x2 ≤ 1 and divergent if x2>1





Example: Show that the series  

$$1 + \frac{32}{1!} (\log q) + \frac{32}{2!} (\log q)^2 + \frac{32}{3!} +$$

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Los alls.

Raabe's Test Let I've be an infinite serier of positive tel kno and let  $h = \sum_{n=1}^{\infty} \left[ h \left( \frac{U_n}{U_n + 1} - 1 \right) \right] = \lambda$ Then the series in (1) Convergent if 1>1 and (ii) Divergent if 1<1 ciii) N=1, this test fails



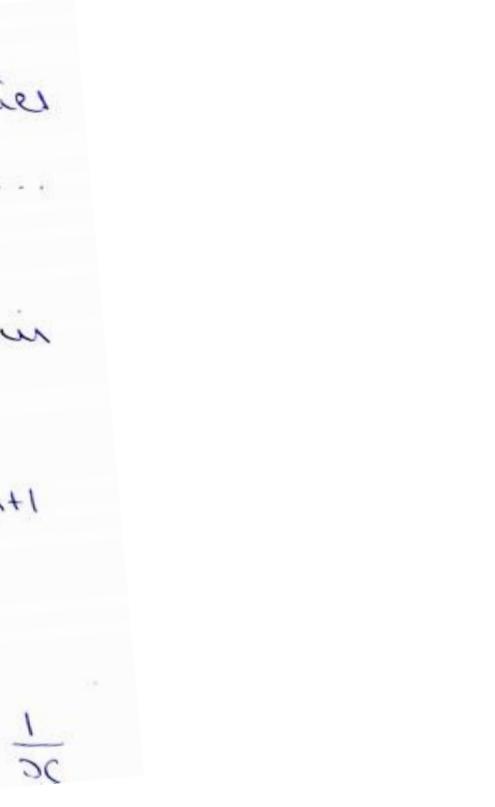
Example! Test the Convergence of the Abried  

$$1 + \frac{3}{7} x + \frac{3 \cdot 6}{7 \cdot 10} x^2 + \frac{3 \cdot 6 \cdot 9}{7 \cdot 10 \cdot 13} x^3 + \dots$$
Solution! The nth Jerm of the given  
Series (heglected first Jerm) in  

$$U_{n} = \frac{3 \cdot 6 \cdot 9 \cdot 12 \cdots (3n)}{7 \cdot 10 \cdot 13 \cdot 16 \cdots (3n + 4)} x^{2n}$$

$$U_{n+1} = \frac{3 \cdot 6 \cdot 9 \cdot 12 \cdots (3n)(3n + 3)}{7 \cdot 10 \cdot 13 \cdot 16 \cdots (3n + 4)} x^{2n+1}$$
Now  $U_{n+1} = (\frac{3n + 7}{3n + 4}) \cdot \frac{1}{3x}$ 

$$\int_{n=1}^{\infty} U_{n+1} = \int_{n=1}^{\infty} \left( \frac{3 + 7 \ln}{3 + 3 \ln} \right) \cdot \frac{1}{3x} = \frac{1}{3}$$



By solio dest, the given series in Galeged  
if 
$$\frac{1}{3} > 1$$
 i.e.  $2c < 1$  and diwergent if  
 $\frac{1}{3} < 1$  or  $2c > 1$ .  
If  $2c = 1$ , this dest fails. Then  
 $\frac{U_n}{U_{n+1}} = \left(\frac{3+7|l_n}{3+3|l_n}\right)$   
 $h\left(\frac{U_n}{U_{n+1}} - 1\right) = h\left[\frac{3+7|l_n}{3+3|l_n} - 1\right]$   
 $= h\left[\frac{4|l_n}{3+3|l_n}\right]$   
 $= h\left[\frac{4|l_n}{3h(1+l_n)}\right]$   
...,  $\lim_{n \to \infty} h\left(\frac{U_n}{U_{n+1}} - 1\right) = \lim_{n \to \infty} \left[\frac{4|3}{1+1|l_n}\right]$   
 $= \frac{4}{3} > 1$ 

Thus, the given series in Convergent if  $x \leq 1$  and divergent if x > 1.

Escanple: Test the Convergence of the A  

$$1 + \alpha + \frac{\alpha(\alpha+1)}{1 \cdot 2} + \frac{\alpha(\alpha+1)(\alpha+2)}{1 \cdot 2 \cdot 3} + \frac{\alpha(\alpha+1)(\alpha+2)}{1 \cdot 2 \cdot 3} + \frac{\alpha(\alpha+1)(\alpha+2)}{1 \cdot 2 \cdot 3} + \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)}{1 \cdot 2 \cdot 3\cdots n}$$

$$0_{n+1} = \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)(\alpha+1)}{1 \cdot 2 \cdot 3\cdots n(n+1)}$$

$$\frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)(\alpha+1)}{1 \cdot 2 \cdot 3\cdots n(n+1)}$$

$$\frac{\alpha(\alpha+1)}{\alpha(n+1)} = \frac{\alpha(\alpha+1)}{\alpha(n+1)} + \frac{\alpha(\alpha+1)}{\alpha(n+1)} + \frac{\alpha(\alpha+1)}{\alpha(n+1)} = \frac{\alpha(\alpha+1)}{\alpha(n+1)} = \frac{1}{\alpha(n+1)}$$
Hence, the solio dest is fails.  
Now,  $n\left(\frac{(0_{n+1}-1)}{(0_{n+1}-1)}\right) = h\left(\frac{(n+1)}{\alpha+1}\right)$ 

$$= \frac{1-\alpha}{1+\alpha(n+1)}$$

$$\frac{\alpha(\alpha+1)}{\alpha(n+1)} = \frac{1-\alpha}{n+1}$$



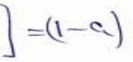






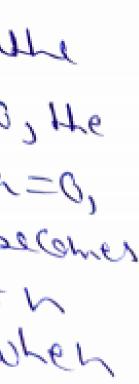


~ (1+1/m)

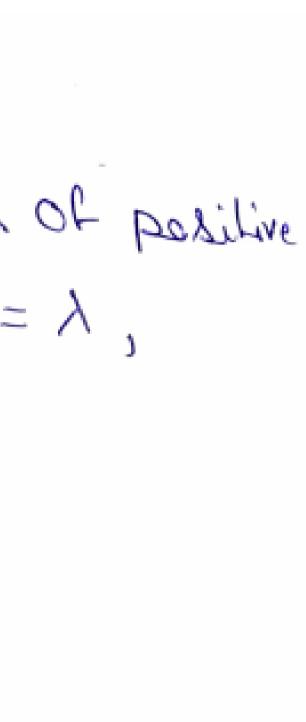


From Baabe's Jestif (1-a)>100 aco, the Series in Convergent. if (1-a)<10ra>0, He series in divergent and if 1-2=1 or 2=0, this last fails and the given series becomes 1 tototot ... the sum of where first h Jernin always equal Jol. Hence when a=0, the series in Convergent. Thur, the given series in Convergent if a <0 and the series in divergent

if aso.

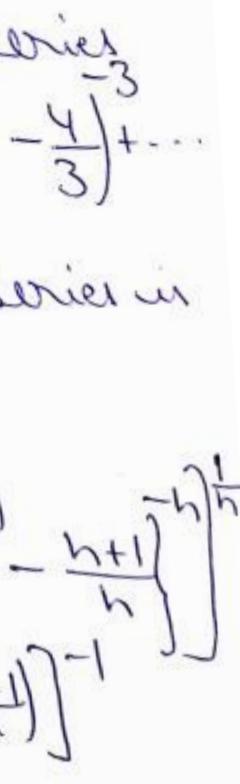


Cauchy's Root Test Let I've be an infinite serier of positive tern and let lin [un] h= 1. Then the series in (i) Convergent if 1<1, and (ii) Divergent if 1>1 (iii) IF  $\lambda = 1$ , this lest fails.

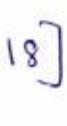


Escample: Test the Convergence of the series  $\left(\frac{2^{2}}{1^{2}}-\frac{2}{1}\right)^{-1}+\left(\frac{3^{3}}{3^{3}}-\frac{3}{2}\right)^{-2}+\left(\frac{4^{4}}{3^{4}}-\frac{4}{3}\right)^{-3}+\cdots$ 

Solution: The not dean of the given series in  $U_{h} = \left[ \frac{C_{h} + I_{h}}{h} + \frac{h}{h} - \frac{h}{h} \right]^{-h}$ · n ->>> [Un] = lim [2 (n+1) (n+1) - h+1] = him ( ( h+1) h+1 ( h+1) -1



= heits [( h+1)x { ( h+1) - 1 ] = him (1+1) (1+1) -1 }  $= \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^{-1} \left[ \left( 1 + \frac{1}{n} \right)^{-1} \right]^{-1}$  $= 1.(e_{-1})^{-1}$ = <u>1</u> <1 [: 2<e<2 and e-1 <1 [: 2<e<2 and we take e=2.718] Hence, by Carly's root test, the given Acrier in Convergent.



(15

Example! Test the Convergence of the series  $\sum \frac{hh^2}{(h+1)h^2}$ Solution: Here UL= hh Ch+11/2  $\frac{dim}{dm} \left[ \frac{dm}{dm} \right] = \frac{dim}{dm} \left[ \frac{h^2}{(h+1)h^2} \right]$ = his [hh (h+1)h]  $= \lim_{n \to \infty} \left[ \frac{1}{(1+\frac{1}{n})} \right] = \frac{1}{e} < 1.$ Hence, by Cauchy's test, the given stried in Chlogert.



Example: Text the Convergence of the there  

$$\frac{1}{2} + \frac{3}{3}x + \left(\frac{3}{4}\right)^2 x_c^2 + \left(\frac{4}{5}\right)^3 x_c^3 + \dots$$
Solution: Neglecting the first derm, we obta  
the n<sup>th</sup> term of the given derive in  

$$U_n = \left(\frac{h+1}{h+2}\right)^h x_c^h$$

$$U_n = \left(\frac{h+1}{h+2}\right)^h x_c^h \left[\left(\frac{h+1}{h+2}\right)^n x_c^h\right]^{th}$$

$$= n - 3x_0 \left[\left(\frac{h+1}{h+2}\right)^n x_c^h\right]$$

$$= n - 3x_0 \left[\left(\frac{h+1}{h+2}\right)^n x_c^h\right] = 3(1 - 3x_0) \left[\frac{h+1}{h+2} + 3x_0\right]$$

## 

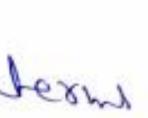


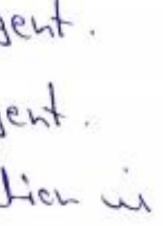
- Ultah From Cauchy's root test, the given series in Convergent if sec1, and divergent if x shad that will be first fi bus fails. Now, putting x=1 in the given strier, we ge line (httph httph now Un = line (httph httph = h-Jo (1+1)  $= n - 3 \circ \frac{(1 + 1/n)^{h}}{(1 + 2/n)^{h}} = \frac{e}{e^{2}} = \frac{1}{e} = \frac{1}{e$ Hence, He series in diverger Thur, the give beier Convergent itscal and divegent if DC 21.



Logarithmic Test

IF ZULLE à serier of positive terms Juch that h= [mu] gel n] ac then (is if 1>1, EUn in Grougent. ciis if 1<1, ZUL in divergent. ciii) if N=1, further investigation is required.





Example: Test for the Convergence of the boien  

$$\begin{array}{r} 2c+\frac{2^{2}\cdotxc^{2}}{2!}+\frac{3^{3}x^{3}}{3!}+\frac{4^{4}\cdotx^{4}}{4!}+\cdots\\ \text{Selution:} \quad \text{Hene } U_{n}=\frac{h^{n}xc^{h}}{h!}\\ \text{As } U_{n+1}=\frac{(h+1)^{n+1}}{(h+1)!}\frac{x^{h+1}}{k!}\\ \xrightarrow{} 0 \frac{U_{n}}{U_{n+1}}=\frac{h^{h}}{(h+1)^{h+1}}\cdot\frac{1}{3c}\\ =\frac{1}{(1+\frac{1}{h})^{h}}\cdot\frac{1}{3c}\\ \text{Air } \frac{U_{n}}{U_{n+1}}=\frac{h^{in}}{h-3c}\frac{1}{(1+\frac{1}{h})^{h}}\frac{1-\frac{1}{2}\cdot\frac{1}{c}}{c}\\ \end{array}$$

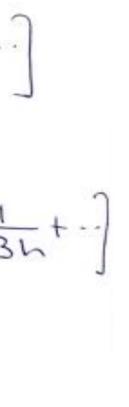
N



Therefore the given series Convergent if the >1 or sc< the cby salenber Ratio Lest) and socies in divergent if I <1 or oc >1 . when x = 1 then fuller investigation

in required. Now

$$\begin{split} & \text{log}\left(\frac{U_{n+1}}{U_{n+1}}\right) = \text{log}\left(\frac{e}{(1+\frac{1}{n})^{h}}\right) \\ &= \text{loge} - \text{log}\left(1+\frac{1}{n}\right)^{h} \\ &= 1 - n \text{ log}\left(1+\frac{1}{n}\right) \\ &= 1 - n \left[\frac{1}{n} - \frac{1}{2n^{2}} + \frac{1}{3n^{3}} - \cdots\right] \\ &= \frac{1}{2n} - \frac{1}{3n^{2}} + \cdots \\ &\text{lime } n \text{ log}\left(\frac{U_{n}}{U_{n+1}}\right) = \frac{\text{lime}}{n - \frac{1}{2n^{2}}} \left[\frac{1}{2} - \frac{1}{3n^{3}}\right] \\ &= \frac{1}{2} < 1 \\ &\text{Hence } \text{ He line } \text{ liver } \text{ series in diversed} \\ & \text{at } x = \frac{1}{e} \end{split}$$



gent

Crows Test  
If 
$$\Sigma \cup_n in a$$
 series of positive der  
tuch that  
 $\frac{\bigcup_n}{\bigcup_{n+1}} = \alpha + \frac{\beta}{n} + \text{terms of higher}$   
 $\alpha > 0$   
ci) if  $\alpha > 1$ , then the series  $\Sigma \cup_n$  Conver  
and if  $\alpha < 1$  then the series  $\Sigma \cup_n$  divo  
whatever  $\beta$  may be  
cii) if  $\alpha = 1$  then series  $\Sigma \cup_n$  Converger in  
and diverger if  $\beta \leq 1$ .





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F B>1

Example! Test the Convergence of the series  

$$1 + \frac{2^{2}}{3^{2}} + \frac{2^{2} \cdot 4^{2}}{3^{2} \cdot 5^{2}} + \frac{3^{2} \cdot 4^{2} \cdot c^{2}}{3^{2} \cdot 5^{2} \cdot 7^{2}} + \cdots$$
Solution: Here the general term of the given  
Series in (hegleding first term)  

$$U_{n} = \frac{2^{2} \cdot 4^{2} \cdot c^{2} \cdots (2n-2)^{2}}{3^{2} \cdot 5^{2} \cdot 7^{2} \cdots (2n-2)^{2}}$$

$$U_{n+1} = \frac{2^{2} \cdot 4^{2} \cdot c^{2} \cdots (2n-2)^{2} (2n)^{2}}{3^{2} \cdot 5^{2} \cdot 7^{2} \cdots (2n-2)^{2} (2n+1)^{2}}$$

$$= 1 + \frac{1}{n} + \frac{1}{4n^{2}}$$
Now applying Craws test; here i, B=1

Le Me given serier in divergent.

Example! Test for convergence of the Series  $1 + \frac{a \cdot b}{1 \cdot c} x + \frac{a(a+1)b(b+1)}{1 \cdot 2 \cdot c(c+1)} z^2$  $+ \alpha(\alpha + i)(\alpha + 2) b(b+i)(b+2)x_{1}^{3} + \cdots + \alpha_{j}b_{j}call$ 1.2.3.C (C+1)(C+2) being positive.

Solution! Criven series in an infinite series The nature of series will not be changed if we neglect first Jerm, then the general Jern of the series can be expressed at





 $\sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\dots(\alpha+h-1)b(b+1)\dots(b+h-1)x^{h}}{\alpha(\alpha+1)\dots(\alpha+h-1)}$  $\chi_{h+1} = \alpha (\alpha + 1) \cdots (\alpha + h) b(b+1) \cdots (b+h) b(b+1) \cdots (b+h) b(b+1) (b+1) b(b+1)$  $\frac{2ch}{2cht} = \frac{(h+1)(cth)}{(ath)(bth)} \frac{1}{2c}$  $\lim_{h \to \infty} \frac{\chi_{h}}{\chi_{h+1}} = \lim_{h \to \infty} \frac{(1+\frac{1}{h})(1+\frac{c}{h})}{(1+\frac{c}{h})(1+\frac{b}{h})\chi}$  $=\frac{1}{2c}$ He test is a set of



Then according the Alembert Datic test the given series in convergent if I >1 crxxl and divergent if x>1. Test of Convergence at sc=1  $\frac{x_{n+1}}{x_{n+1}} = \frac{(1+\frac{1}{2})(1+\frac{1}{2})}{(1+\frac{1}{2})(1+\frac{1}{2})}$  $= (1+\frac{1}{2})(1+\frac{1}{2})(1+\frac{1}{2})(1+\frac{1}{2})(1+\frac{1}{2})$  $= (1+\frac{1}{2})(1+\frac{2}{2})(1-\frac{2}{2}+\frac{2^{2}}{2}+...)(1-\frac{2}{2}+\frac{2^{2}}{2}+...)$  $= (1+\frac{1}{2}+\frac{c^{2}}{2})(1-\frac{c^{2}}{2}+\frac{c^{2}}{2}-\cdots)(1-\frac{b}{2}+\frac{b^{2}}{2}-\cdots)$ = 1+ 1-C-a-b+all the terms Cartaining higher pavers of L



Appleing Causs Jest Here 2=1, B= (1+C-Q-b) So given spier in Convergent if (1+c-q-5)>1 and divergent if (1+C-9-6)<1 at x=1.

De Morgan's and Bestrand's Test A serier 5 Un of positive term such that  $h = \left[ \lambda = \left[ \lambda = \left( \frac{\omega_{n+1}}{\omega_{n+1}} - 1 \right) - 1 \right] = \lambda$ then cit I Un in Convergent if 1>1 (11) I'm divergent if 1<1.









Example: Test for Convergence of the series  

$$I^{b} + \left(\frac{1}{2}\right)^{b} + \left(\frac{1\times3}{2\times4}\right)^{b} + \left(\frac{1\times3\times5}{2\times4\times6}\right)^{b} + \cdots$$
Solution: The given deriver un  

$$I^{b} + \left(\frac{1}{2}\right)^{b} + \left(\frac{1\times3}{2\times4}\right)^{b} + \left(\frac{1\times3\times5}{2\times4\times6}\right)^{b} + \cdots$$
Neglecting first tom of the deriver, weget  
the general derm of deriver as  

$$U_{n} = \left(\frac{1\times3\times5\times-\times(2n-3)}{2\times4\times6\times\times\times(2n-3)}\right)^{b}$$

$$U_{n+1} = \left(\frac{1\times3\times5\times\cdots\times(2n-3)}{2\times4\times6\times\times\times(2n-3)(2n)}\right)^{b}$$

$$\frac{U_{n}}{U_{n+1}} = \frac{1}{(2n-1)}b^{-1}\frac{1}{(1-\frac{1}{2n})b} = 1$$

$$\sum_{n=1}^{1} \frac{U_{n}}{U_{n+1}} = 1$$

$$\sum_{n=1}^{1} O' Alendust'' Test fails.$$

Now, we apply Baabe's Test  

$$h\left(\frac{U_{n}}{U_{n+1}}-1\right) = h\left[\left(1-\frac{1}{2n}\right)^{-b}-1\right]$$

$$= h\left[1+\frac{b}{2n}+\frac{b(b+1)}{8h^{2}}+\cdots-1\right]$$

$$= \frac{b}{2}+\frac{b(b+1)}{8h}+\cdots$$

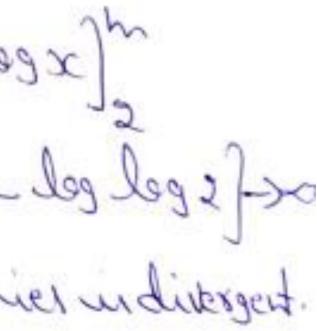
$$\lim_{n\to\infty}h\left(\frac{U_{n}}{U_{n+1}}-1\right) = \frac{b}{2}$$
If  $\frac{b}{2} > 1$  i.e.  $b > 2$ , the strict in Goldergen  
and divergent if  $\frac{b}{2} < 1$  i.e.  $b < 2$ .  
This test fails if  $\frac{b}{2} = 1$  i.e.  $b = 2$ .





· Cauchy & Integral lest A positive term sovies  $F(1) + F(2) + F(3) + \cdots + F(n) + \cdots$ where foundecreases are increases, Coverger or diverger according to the integral J Fcx 1 doc in finite or infinite.

Example: Examine the Convergence of Z hlogh Solution: Here fix1= - 1 Jugoc fxeel eel ] actim = xb xeel x cl of seel cel - Meel cel Join = By Cauchy's Integral test, the series indikegent.



Example! Examine the convergence of She he トニト Solution: Hore  $L(x) = xe^{-xc^2}$ Now  $\int_{1}^{\infty} x e^{-x^2} dx = \lim_{n \to \infty} \int_{1}^{\infty} \left[ \frac{e^{-x^2}}{-x} \right]_{n=1}^{n}$ =  $\lim_{n \to \infty} \int e^{-h^2} + e^{-1}$ = e = 1 - 2 which infinite. Hence, the given Spier in Convergent.

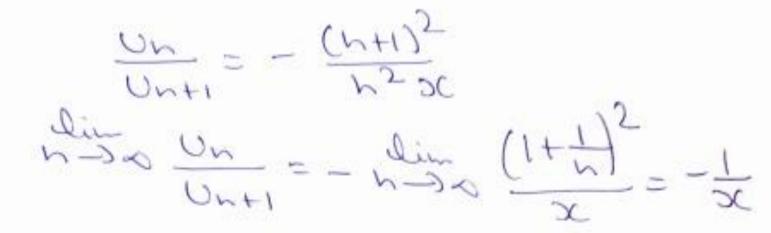


Power series insc

A sociel of the form Zanx or actaix tag 2 + - + 2 at a top of is sale a power series in x, where a 's are independent of x.

Example! Find the values of x for which the series  $x - x^2 + x^3 - \frac{3c'}{4^2} + \dots < Carloger$ 

Solution!  $U_{h} = (-1)^{h-1} \frac{2c}{h^2}; U_{h} = \frac{(-1)^{h} 2c^{h+1}}{(h+1)^2}$ 

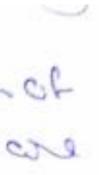


By D' Alembert's Test the given deriel in Convergent for IxI < I and divergent it At x=+1, the series becomes 1-1-+++-++-This is an alterably convergent series At 2c= -1, the given beier becomes This is also convergent series; b=2. Here, the interval of Convergence in -15 X 51.



## Taylor's series

A Taylor's series in a representation of Cordian as an infinite terms that are Calculated at a single point ZUN= Z F(a) (x-a) where fical denotes the nth order derivative of f evaluated at a. Taylor's series are het Convergent series in general. Some Convergent Taylor's Series are (i) Exponential Series ciis Legiraltuic Jeries (III) Trigonometric Leviel





Exponential Seriel  $e^{x} = 1 + x + \frac{x^{2}}{21} + \frac{x^{3}}{21} + \dots + \frac{x^{h}}{2h} + \frac{x^{h}}{21} + \dots$ Convergent for all values of sc Proof:  $U_n = \frac{3ch-1}{ch-1}$   $3U_{n+1} = \frac{3ch}{h!}$ him Uh = h = x Hence by D'Alembert's Test the exponential series in Convergent for all values of x. D'Alembert Dalio Test L hoso (Un) = + 0, then SUL in Convergent].



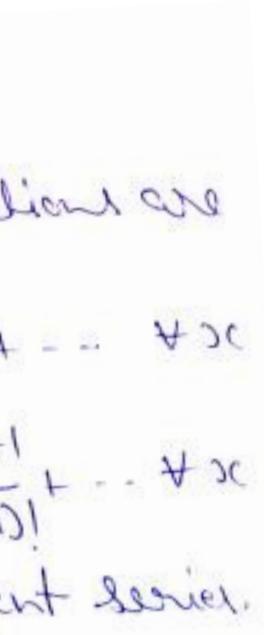
Logarithmic Series

Series of the form  $log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{2} - \frac{x}{2} + \dots + (-1)^{-1} \frac{x^{-1}}{2}$ in Convergent for -1< x ≤1. <u>Proof</u>:  $U_n = (-1)^{h-1} \frac{3c}{3c}, U_{n+1} = (-1)^{h} \frac{3c^{h+1}}{3c}$  $\lim_{h \to \infty} \bigcup_{u \to 1} = \lim_{h \to \infty} \left( \frac{h+1}{h} \right) \frac{1}{2c} = \lim_{h \to \infty} \left( \frac{1}{1+\frac{1}{h}} \right) \frac{1}{2c}$ = -1 Thus, the series in convergent if IxI<I and divergent Isc1>1. At x=1, the series becomes 1- 1+ 1 - 1+ - which in Convergent. At x=-1, the series becomes -1- to -to + - - which in divergent.

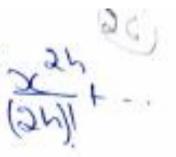
> Dr. Tripati Gupta(Associate Professor, Deptt. of Mathematics), JECRC, JAIPUR

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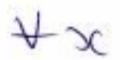
Trigonometric Function The power series of circular functions are given be  $C_{9SDC=1-\frac{x^2}{21}+\frac{x^4}{41}+\dots+(-1)\frac{x^{2h}}{(2h)}+\dots$  $Sinsc = x - \frac{x^3}{31} + \frac{x^5}{51} + - + (-1)\frac{x^{-1}}{2n+1} + - + x$ These region are absolutely Convergent Series.



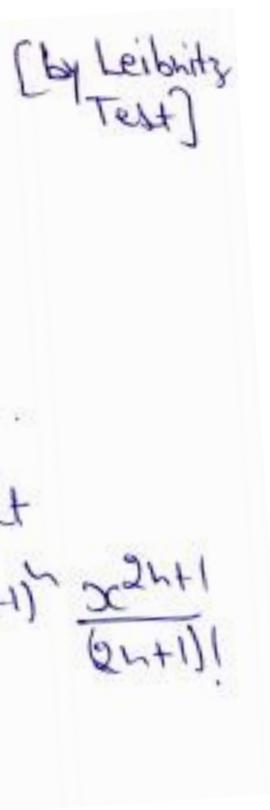
Proof: 
$$(asx = 1 - \frac{3c^2}{2!} + \frac{3c'}{4!} + - +(-1)^{n}$$
  
 $(bn = (-1)^{n} \frac{x^{2n}}{(2n)!}$  of  
Since all Jerns in the terricored  
Lign to it in an allowating terricored  
 $(bn+1 = (-1)^{n+1} \frac{3c^{2n+2}}{(2n+2)!}$   
 $(bn > 0n+1 : \frac{3c^{2n}}{2n!} > \frac{3c^{2n+2}}{(2n+2)!}$   
 $(bn > 0n = \frac{1}{n-3} - \frac{1}{(2n+2)!} = 0$ 



allerhabive



Criven albending series Guvergent. [by Leibidz Test]  $|U_n| = \frac{x^2}{21} + \frac{x^4}{41} + \frac{x^6}{61} + \dots$ him 10-1=0 => Cossc in absolutely Convergent. Similarly it can be shown that  $Sinx = x - \frac{x^3}{31} + \frac{x^5}{51} + - \cdot + (-1)^{-\frac{x^{2h+1}}{2n+1}}$ in absolutely Convergent series.



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