JECRC Foundation

JAIPUR ENGINEERING COLLEGE AND RESEARCH CENTRE

JAIPUR ENGINEERING COLLEGE AND RESEARCH CENTRE

Year & Sem. – B. Tech I year, Sem.-I

Subject –Engineering Mathematics-1

Unit – 4 (**Multivariable Calculus (Differentiation)**

Presented by – Dr. Sunil Kumar Srivastava

Designation - Associate Professor

Department - Mathematics

VISSION OF INSTITUTE

To become a renowned centre of outcome based learning, and work towards academic, professional, cultural and social enrichment of the lives of individuals and communities.

MISSION OF INSTITUTE

❖**Focus on evaluation of learning outcomes and motivate students to inculcate research aptitude by project based learning.**

❖**Identify, based on informed perception of Indian, regional and global needs, the areas of focus and provide platform to gain knowledge and solutions.**

❖**Offer opportunities for interaction between academia and industry.**

❖**Develop human potential to its fullest extent so that intellectually capable and imaginatively gifted leaders may emerge.**

Engineering Mathematics-1: Course Outcomes

Students will be able to:

On completion of this course students will be expected to:

CO1. Understand fundamental concepts of improper integrals, beta and gamma functions and their properties. Evaluation of Multiple Integrals in finding the areas, volume enclosed by several curves after its tracing and its application in proving certain theorems.

CO2. Interpret the concept of a series as the sum of a sequence, and use the sequence of partial sums to determine convergence of a series. Understand derivatives of power, trigonometric, exponential, hyperbolic, logarithmic series.

CO3. Recognize odd, even and periodic function and express them in Fourier series using Euler's formulae.

CO4. Understand the concept of limits, continuity and differentiability of functions of several variables. Analytical definition of partial derivative. Maxima and minima of functions of several variables Define gradient, divergence and curl of scalar and vector functions.

Limit:

- Limit: Let f be a function of two variable defined in some region then limit of f is equal to l if for every $\in > 0, \exists\ \delta > 0$ st
- $|f(x, y) - l| < \in$ where $0 < \sqrt{(x - x0)^2 + (y - y0)^2} < \delta$
- Then $\lim_{(x,y)\to(x0,y0)} f(x, y) = l$
- Method:
- If $x \to a, y \to b$ **then** evaluate limit by normal procedure.
- If $x \to o, y \to o$ then evaluate limit along y=mx and $y = mx^n$. If in both case limit is same then limit exits.

Limit:

- Q: Find the limit $\lim\limits_{(x,y)\to(0,0)}\left[\frac{x^2y}{x^4+y^2}\right]$
- Solution: along the path y=mx
- $\lim\limits_{(x,y)\to(0,mx)}\left[\frac{x^2mx}{x^4+m^2x^2}\right]=0$
- And along the path $y=mx^n$.
- $\lim\limits_{(x,y)\to(0,mx^n)}\left[\frac{x^2mx^n}{x^4+m^2x^4}\right]=\lim\limits_{x\to0}\left[\frac{m}{1+m^2}\right]$ which depend on m and different for different values of m so limit does not exit.

Continuity:

- Continuity: a f be a function of two variable is said to be continuous at the point (x_0, y_0) if
- $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$
- Test of continuity:
- Find $f(x, y)$ at (a,b)
- $\lim_{(x,y)\to(a,b)} f(x, y)$must be exit and is equal to $f(a, b)$.

Q: find the limit and test the for continuity of the function

$$f(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2} & if\ x,y \neq 0 \\ 0 & if\ x,y = 0 \end{cases}$$ at the point (0,0).

Solution: by definition $f(0,0) = 0$

- Now limit along path y=mx
- $$\lim_{(x,y)\to(0,mx)} \left[\frac{x^3-m^3x^3}{x^2+m^2x^2}\right] = 0$$
- And along the path $y = mx^2$.
- $$\lim_{(x,y)\to(0,mx^2)} \left[\frac{x^3-m^3x^6}{x^2+m^2x^4}\right] = 0$$
- Since limit is same and is equal to value of the function at (0,0). So function is continuous at (0,0).

Partial Derivative:

if $z = f(x, y)$ be a function of two independent variable x and y the derivative of z , with respect to x, keeping y as constant , is called partial derivative of z with respect to x and is denoted as $\frac{\partial z}{\partial x}$ and is defined as

- $$\frac{\partial z}{\partial x} = \lim_{h \to o} \frac{f(x+h,y)-f(x,y)}{h}$$

And similarly the derivative of z , with respect to y, keeping x as constant , is called partial derivative of z with respect to x and is denoted as $\frac{\partial z}{\partial y}$ and is defined as

- $$\frac{\partial z}{\partial y} = \lim_{k \to o} \frac{f(x,y+k)-f(x,y)}{k}$$

Similarly $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$ denote the second order partial derivative and so on.

Q: if $u = e^{xyz}$ then find the value of $\frac{\partial^3 y}{\partial x \partial y \partial z}$

Solution: Here given $u = e^{xyz}$ (1)

- Differentiating (1) partially with respect to z taking x and y as constant, we have
- $$\frac{\partial u}{\partial z} = e^{xyz}(xy)$$
- Also $\frac{\partial^2 u}{\partial y \partial z} = e^{xyz}.x + e^{xyz}x^2yz$
- Finally $\frac{\partial^3 y}{\partial x \partial y \partial z} = e^{xyz}(1 + 2xyz) + e^{xyz}yz(x + x^2yz)$
- $= e^{xyz}\ (1 + 2xyz + xyz + x^2y^2z^2$

Total Derivative :

- In partial derivative of a function two or more variable, only one variable varies but in total derivative increments are given in all the variables.
- Let $z = f(x, y)$ then total derivative of z is
- $$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial x} dy$$
- **Note: (1)** if $z = f(x, y)$ and $x = \emptyset(t), y = \emptyset(t)$ then total derivative of z is
- $$\frac{dz}{dt} = \frac{\partial f}{\partial x} dx/dt + \frac{\partial f}{\partial x} dy/dt$$
- **(2)** if $z = f(x, y)$ and $x = \emptyset(u, v), y = \emptyset(u, v)$ then total derivative of z is
- $$\frac{dz}{du} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial x}\ \frac{\partial y}{\partial u} \text{ and } \frac{dz}{dv} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial x}\ \frac{\partial y}{\partial v}$$

Tangent plane and Normal to a surface:

- The equation of tangent plane to the point $P(x, y, z)$ to the surface $f(x, y, z)$ is
- $$\frac{\partial f}{\partial x}(X - x) + \frac{\partial f}{\partial y}(Y - y) + \frac{\partial f}{\partial z}(Z - z) = 0$$
- Where (X,Y,Z) are the current coordinate of the any point on this tangent plane.
- The equation of normal plane to the point $P(x, y, z)$ to the surface $f(x, y, z)$ is
- $$\frac{X-x}{\frac{\partial f}{\partial x}} = \frac{Y-y}{\frac{\partial f}{\partial y}} = \frac{Z-z}{\frac{\partial f}{\partial z}}$$

Maxima And Minima of Function of two variable:

- Def. A Function $f(x, y)$ is said to have a maximum or minimum at $x = a, y = b$, according as
- $f(a + h, b + k) <$ or $f(a, b),$ for all positive or negative values of h and k.

- In other words if $f(a + h, b + k) - f(a, b)$ is of the same sign for all small values of h, k an then $f(a, b)$ is a maximum d if this sign is negative , then $f(a, b)$ is a maximum . If this sign is then $f(a, b)$ is a minimum.

Working Rule to find the maximum and minimum values of $\boldsymbol{f(x, y)}$.

1. find the $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and each equate tom zero. Solve these simultaneous equations in x and y and find the pair of values of the form (a, b).....
2. Calculate the values of $r = \frac{\partial^2 f}{\partial x^2}$, $\mathrm{s} = \frac{\partial^2 f}{\partial x \partial y}$, t $= \frac{\partial^2 f}{\partial y^2}$ for each pair of values.
3. (i) if $rt - s^2 > 0$ and $r < 0$ at (a, b) then $f(x, y)$ is maximum and have maximum values $f(a, b)$.

   (ii) if $rt - s^2 > 0$ and $r > 0$ at (a, b) then $f(x, y)$ is minimum and have minimum values $f(a, b)$.

   (iii) if $rt - s^2 < 0$ at (a, b) then $f(x, y)$ is not have extreme values i.e $f(a, b)$ is a saddle point.

   (iv) if $rt - s^2 = 0$ at (a, b) then case is doubtful and needs further investigation.

Q. Discuss the maxima and minima of the function

$$f(x,y) = x^3y^2(1-x-y)$$

Solution: Here we ha

$$f(x,y) = x^3y^2(1-x-y)$$

Now $\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$ and $\frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2$

And
$$r = \frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 6x^2y - 8x^3y - 9x^2y^2$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y$$

for stationary values we have $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

So we have $x^2y^2(3-4x-3y) = 0$ and $x^2y\,(2-2x-3y) = 0$

Solving these we have $(0,0)$ and $(\frac{1}{2},\frac{1}{3})$ as stationary points.

- Now $rt - s^2 = \frac{1}{14} > 0$ at $(1/2, 1/3)$
- And also $r = -\frac{1}{9} < 0$ at $(1/2, 1/3)$
- So function is maximum at the point at $(1/2, 1/3)$ and maximum value is 1/432.
- At (0,0) , $rt - s^2 = 0$ and hence further investigation is required.

- Practice problem:

- **Q:** in a plane triangle , find the maximum value of $CosACosBCosC$.

Lagrange's method of undetermined multipliers:

- Let $u = f(x, y, z)$ (1)
- Be a function of three variables x, y, z related by the relation
- $$\emptyset(x, y, z) = 0 \qquad (2)$$
- For stationary points we have
- $$\frac{\partial f}{\partial x} = 0\ , \quad \frac{\partial f}{\partial y} = 0\ , \quad \frac{\partial f}{\partial z} = 0$$
- So $du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$ (3)
- Differentiating (2) gives
- $$\frac{\partial \emptyset}{\partial x}dx + \frac{\partial \emptyset}{\partial y}dy + \frac{\partial \emptyset}{\partial z}dz = 0 \qquad (4)$$
- Now multiplying (4) by λ and then adding in (3)
- $$du = \left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \emptyset}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \emptyset}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \emptyset}{\partial z}\right)dz$$
- $$\Longrightarrow \quad \frac{\partial f}{\partial x} + \lambda\frac{\partial \emptyset}{\partial x} = 0, \quad \frac{\partial f}{\partial y} + \lambda\frac{\partial \emptyset}{\partial y} = 0\ , \quad \frac{\partial f}{\partial z} + \lambda\frac{\partial \emptyset}{\partial z} = 0$$
- Solving above equation with (2) we can find the values of x, y, z and λ for which function f have stationary values.

Q: a rectangular box open at the top is to have volume of 32 cubic ft. find the dimension of the box requiring least material for its construction.

- **Solution:** Let x, y , z be dimension of the box and S be its surface.
- The $S = xy + 2yz + 2zx = f(x, y, z)$ (1)
- And $xyz = 32 = \emptyset(x, y, z)$ (2)
- Now we have Lagrange's function
- $F = f(x, y, z) + \lambda = \emptyset(x, y, z)$
- For stationary $dF = 0$ so we have
- $\left(\frac{\partial f}{\partial x} + \lambda \frac{\partial \emptyset}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y} + \lambda \frac{\partial \emptyset}{\partial y}\right) dy + \left(\frac{\partial f}{\partial z} + \lambda \frac{\partial \emptyset}{\partial z}\right) dz = 0$
- Or $(y + 2z + \lambda yz)dx + (x + 2z + \lambda zx)dy + (2y + 2x + \lambda xy)dz = 0$
- Or $y + 2z + \lambda yz = 0$, $x + 2z + \lambda zx = 0$, $2y + 2x + \lambda xy = 0$
- On solving above equation we have $x = y$ and $y = 2z$
- So we have $x = y = 2z$
- There fore dimension of the box are 4,4,2

The Vector Differential Operator Del (∇):

- The Vector Differential Operator Del ∇ (read as nabla) is defined as
- $$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$
- Gradient of Scalar point function:
- The gradient of a scalar point function $\emptyset(x, y, z) = c$, where c is constant, is denoted as grad∅ or $\nabla\emptyset$ and is defined as
- $$\text{grad}\emptyset \text{ or } \nabla\emptyset = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\emptyset$$
- $$= \ i\frac{\partial\emptyset}{\partial x} + j\frac{\partial\emptyset}{\partial y} + k\frac{\partial\emptyset}{\partial z}$$

Gradient:

- **Important to Note:**
- The gradient of a scalar point function $\emptyset(x, y, z)$ is a vector point function.
- The grad$\emptyset$ is vector normal to the surface.
- **The Directional Derivative:**
- The directional derivative of a scalar field f at the point $P(x, y, z)$ in the direction of a vector $\vec{a}$ is given by
- $\frac{\partial \mathrm{f}}{\partial \mathrm{x}} = \nabla f.\acute{a}$ where $a = \frac{\vec{a}}{|\vec{a}|}$

Q: find the directional derivative of $\emptyset = x^2yz - 4xyz^2$ at the point $P(1,3,1)$ in the direction of the vector $2i - j - 2k$.

- **Solution:** Here given surface $\emptyset = x^2yz - 4xyz^2$
- So grad∅ or $\nabla\emptyset$ $= \mathrm{i}\frac{\partial\emptyset}{\partial \mathrm{x}} + j\frac{\partial\emptyset}{\partial \mathrm{y}} + k\frac{\partial\emptyset}{\partial \mathrm{z}}$
- $= (2xyz - 4yz^2)i + (x^2z - 4xz^2)j + (x^2y - 8xyz)k$
- Now $\nabla\emptyset$ at the point $P(1,3,1) = -6i - 3j - 21k$
- Let a be unit vector in the direction of the vector $2i - j - 2k$. Then
- $$a = \frac{\vec{a}}{|\vec{a}|} = \frac{-6i-3j-21k}{|-6i-3j-21k|} = -\frac{2}{3}i - \frac{1}{3}j - \frac{2}{3}k$$
- Now we know that The directional derivative of a scalar field f at the point $P(x, y, z)$ in the direction of a vector $\vec{a}$ is given by
- $$\frac{\partial \mathrm{f}}{\partial \mathrm{x}} = \nabla f.\acute{a} = (-6i - 3j - 21k).(-\frac{2}{3}i - \frac{1}{3}j - \frac{2}{3}k) = 11$$

Divergence:

- **Divergence of a vector point function:**
- The divergence of a vector point function $\vec{V} = v_1 i + v_2 j + v_3 k$ is a scalar point function is denoted as div $\vec{V}$ or $\nabla.\vec{V}$ and is defined as
- div $\vec{V}$ = $\nabla.\vec{V} = \left(\mathrm{i}\frac{\partial}{\partial \mathrm{x}} + j\frac{\partial}{\partial \mathrm{y}} + k\frac{\partial}{\partial \mathrm{z}}\right).(v_1 i + v_2 j + v_3 k)$
- $= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$
- Which shows that divergence of a vector point function is a scalar function.
- Divergence represents the rate of flow out per unit volume per unit time at a point of the fluid.
- If div $\vec{V} = 0$ then vector field is said to be solenoidal.

Curl:

- **Curl of a vector point function:**
- The Curl of a vector point function $\vec{V} = v_1 i + v_2 j + v_3 k$ is a vector point function is denoted as curl $\vec{V}$ or $\nabla \times \vec{V}$ and is defined as
- curl $\vec{V}$ or $\nabla \times \vec{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial x} \\ v_1 & v_2 & v_3 \end{vmatrix}$
- Curl measure the angular velocity at any point of the vector field .
- A vector field is called irrotational if curl $\vec{V} = 0$.
- If vector field represents the flow velocity of a moving fluid then curl is circulation density of the fluid.

Q: a fluid motion is given by

$$\vec{V} = (y+z)i + (z+x)j + (x+y)k$$

Is this motion irrotational. If so find scalar potential

- **Solution:** Motion will be irrotational if curl $\vec{V} = 0$ so we have to show curl $\vec{V} = 0$.
- Now $\nabla \times \vec{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial x} \\ v_1 & v_2 & v_3 \end{vmatrix} = (1-1)i + (1-1)j + (1-1)k = 0$
-
- Therefore given vector is irrotational.
- Since fluid is irrotational, therefore $\vec{V} = \text{grad}\emptyset = \nabla\emptyset$
- $(y+z)i + (z+x)j + (x+y)k = \mathrm{i}\frac{\partial\emptyset}{\partial \mathrm{x}} + j\frac{\partial\emptyset}{\partial \mathrm{y}} + k\frac{\partial\emptyset}{\partial \mathrm{z}}$
- $\Longrightarrow \frac{\partial\emptyset}{\partial \mathrm{x}} = y+z \; , \; \frac{\partial\emptyset}{\partial \mathrm{y}} = x+z \; , \; \frac{\partial\emptyset}{\partial \mathrm{x}} = x+y$
- Now $d\emptyset = \frac{\partial\emptyset}{\partial \mathrm{x}}dx + \frac{\partial\emptyset}{\partial \mathrm{y}}dy + \frac{\partial\emptyset}{\partial \mathrm{z}}dz$
- $= (y+z)dx + (z+x)dy + (x+y)dz$
- $d\emptyset = \mathrm{d(yx)} + \mathrm{d(zx)} + \mathrm{d(zy)}$
- On integration , we get
- $\emptyset = xy + xz + yz + c$

- https://nptel.ac.in/courses/111/107/111107108
- https://www.youtube.com/watch?v=XzaeYnZdK5o&fea
- https://www.youtube.com/watch?v=HyWagR_7x-o&feature=emb_logoture=emb_logo

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Thank you!

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