JECRC Foundation

JAIPUR ENGINEERING COLLEGE AND RESEARCH CENTRE

JAIPUR ENGINEERING COLLEGE AND RESEARCH CENTRE

Year & Sem – I Year & II Sem

Subject –Engineering Mathematics-II

Unit – 5

Presented by – (Dr. Sunil K Srivastava, Associate Professor)

VISION AND MISSION OF INSTITUTE

VISION OF INSTITUTE

To became a renowned centre of outcome based learning and work towards academic professional, cultural and social enrichment of the lives of individuals and communities .

MISSION OF INSTITUTE

- Focus on evaluation of learning, outcomes and motivate students to research aptitude by project based learning.
- Identify based on informed perception of Indian, regional and global needs, the area of focus and provide platform to gain knowledge and solutions.
- Offer opportunities for interaction between academic and industry .
- Develop human potential to its fullest extent so that intellectually capable and imaginatively gifted leaders may emerge.

METHOD OF SEPERATION OF VARIABLE:

In this method , we assume the dependent variable is the product of two function, each of which involves only one of the independent variable. The following example explains this method widely.

Example: Using the method of separation of variable,

Solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial y} + u$, where $u(x,0) = 6e^{-3x}$.

Solution: The given partial differential equation is

- $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial y} + u$ (1)
- Let the required solution of equation (1) is of the form
- $u(x,y) = X(x).Y(y)$ (2)
- where X is the function of x only and Y is the function of y only.
- Differentiating both side of (2) w.r.t. x and y , we get
- $\frac{\partial u}{\partial x} = YX'$, $\frac{\partial u}{\partial y} = XY'$

- Putting the value of u , $\frac{\partial u}{\partial x}$, and $\frac{\partial u}{\partial y}$ in equation (1), we get
- $YX' = \ 2XY' + XY$
- Or $$(X' - X)Y = \ 2XY'$$
- Separating the variable ,
- $\frac{(X\prime - X)}{2X} \ = \ \frac{Y\prime}{Y} \ = k,$ where k is constant.
- Or $$\frac{1}{2X}(X' - X) \ = \frac{Y\prime}{Y} = k$$
- Or $$\frac{1}{2X}(X' - X) = k \qquad (3)$$
- And $$\frac{Y\prime}{Y} = k \qquad (4)$$

- From (3) , we have $X'/X = 2k + 1$
- On integration , $\log X = (1 + 2k)x + \log c_1$
- Or $X(x) = c_1 e^{(1+2k)x}$
- And on integration (4), we have
- $Y(y) = c_2 e^{ky}$
- Putting the value of $X(x)$ and $Y(y)$ in (1), we have
- $u(x, y) = c_1 e^{(1+2k)x} c_2 e^{ky} = c_1 c_2 e^{(1+2k)x+ky}$
- $u(x, y) = A e^{(1+2k)x+ky}$(5)
- where $A = c_1 c_2$
- Using $u(x, 0) = 6e^{-3x}$ in (5) , we have
- $A = 6$, $1 + 2k = -3 \implies k = -2$
- Therefore , the required solution is
- $u(x, y) = 6e^{-(3x+2y)}$

One Dimensional wave Equation :

- This is the partial differential equation giving transverse vibrations of the string.

- One Dimensional Heat flow equation is given by
- $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ (1)
- where $c^2 = \frac{T}{m}$,where T denote the tension and be the mass per unit length of string.
- Solution of waveEquation:
- Let $\mathrm{y} = XT$ (2)
- be a solution of the given wave equation where X is function of x only and T is function of t only.
- Then $\frac{\partial^2 y}{\partial t^2} = XT''$ and $\frac{\partial^2 y}{\partial x^2} = TX''$

- Substitute in the given equation, we have
- $XT'' = c^2\, TX''$
- Or $\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = k(say)$(3)
- $\Longrightarrow$ Left hand side of (2) is a function of x only and right hand side is a function of t alone. Since x and t are independent variable so above equation hold good if each side is equal to a constant k. So

$X'' - kX = 0$ and $T'' - c^2kT = 0$

- Solving above we have

- CASE I: If k is positive i.e. $k = p^2$
- From equation (3), $X'' - p^2 X = 0$
- A. E is given by, $m^2 - p^2 = 0 \Longrightarrow m = \pm p$
- $\Longrightarrow X = c_1 e^{px} + c_2 e^{-px}$
- And from equation (3), $T'' - c^2 p^2 T = 0$
- A. E is given by, $m^2 - p^2 c^2 = 0 \Longrightarrow m = \pm pc$
- $\Longrightarrow T = c_3 e^{cpt} + c_4 e^{-cpt}$

$so \quad y = XT = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt})$ (a)

- Case 2: if k is zero i.e. $k = 0$
- Let From equation (2) , $X'' = 0$
- on integration $\Longrightarrow X = (b_1 + b_2 x)$
- From equation (3) , $T'' = 0$
- on integration $\Longrightarrow T = (b_3 + b_3 t)$
- $y = XT = (b_1 + b_2 x)(b_3 + b_3 t)$ (b)

- Case 3: if k is negative i.e. $k = -p^2$
- From equation (2) , we have
- $X'' + p^2 X = 0$
- A. E is given by, $m^2 + p^2 = 0 \Longrightarrow m = \pm pi$

$\Longrightarrow \quad X = c_5 cospx + c_6 sinpx$

- And $\qquad T'' + c^2 p^2 T = 0$
- A. E is given by, $m^2 + p^2 c^2 = 0 \Longrightarrow m = \pm pc$i

- $\Longrightarrow \qquad T = c_7 \cos cpt + c_8 \sin cpt$
- $\Longrightarrow$ y $= XT = (c_5 cospx + c_6 sinpx)(c_7 \cos cpt + c_8 \sin cpt)$ ©

So all possible Solution are

` $$y = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_2 e^{-cpt})$$

$$y = (b_1 + b_2 x)(b_3 + b_3 t)$$

$$y = (c_5 cospx + c_6 sinpx)(c_7 \cos cpt + c_8 \sin cpt)$$

Out of these three solution we have to choose that solution which is consistent with physical nature of the problem. As we are dealing with problem on vibration, y must be periodic function of x and t. Hence there solution must involve trigonometry terms.

Therefore solution of wave equation is given by

$$y = (c_5 cospx + c_6 sinpx)(c_7 \cos cpt + c_8 \sin cpt)$$

is the only suitable solution of the wave equation.

Example: A tightly stretched string with fixed ends points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 sin^3\left(\frac{\pi x}{l}\right)$. if is released from rest find the displacement $y(x,t)$.

Solution: The equation of string is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad (1)$$

The boundary conditions are

$$y(o,t) = 0 \qquad (2)$$

$$y(l,t) = 0 \qquad (3)$$

The initial conditions are

$$y(x,o) = y_0 sin^3\left(\frac{\pi x}{l}\right) \qquad (4)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \qquad (5)$$

- Since vibration of the string is periodic. Therefore solution of equation (1) is

$$y = (c_1 cospx + c_2 sinpx)(c_3 \cos cpt + c_4 \sin cpt)$$

Now using the condition (2) , we have

$$y(o, t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$$

for this to be true all time, $c_1 = 0$

So $$y = c_2 sinpx(c_3 \cos cpt + c_4 \sin cpt)$$

Now using condition (3), we have

$$y(l, t) = c_2 sinpl(c_3 \cos cpt + c_4 \sin cpt = 0$$

This gives $pl = n\pi$ or $p = \frac{n\pi}{l}$

- Thus
- $y = c_2 \sin\frac{n\pi x}{l}(c_3 \cos\frac{cn\pi t}{l} + c_4 \sin\frac{cn\pi t}{l})$
- $\Longrightarrow \frac{\partial y}{\partial t} = c_2 \sin\frac{n\pi x}{l}(-c_3 \sin\frac{cn\pi t}{l} + c_4 \cos\frac{cn\pi t}{l})\frac{n\pi c}{l}$
- using condition (5) in above , we have
- $\left(\frac{\partial y}{\partial t}\right)_{t=0} = c_2 \sin\frac{n\pi x}{l}\left(c_4 \frac{n\pi c}{l}\right) = 0$
- $\Longrightarrow c_4 = 0$
- So $y = c_2 c_3 \sin\frac{n\pi x}{l}\cos\frac{cn\pi t}{l} = b_n \sin\frac{n\pi x}{l}\cos\frac{cn\pi t}{l}$

- Adding all such solution , the general solution of given wave equation is
- $y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cos\frac{cn\pi t}{l}$
- Now again using condition (4), we have
- $y_0 sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$
- Now calculating the values of $b_1, b_2, \ldots..$
- We have the solution of given equation.

One Dimensional Heat Equation :

- One Dimensional Heat flow equation is given by
- $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ (1)
- where $c^2 = \frac{k}{\rho s}$, is called diffusivity of the substance.
- Solution of Heat Equation:
- Let $u = XT$………… (2)
- be a solution of the given heat equation where X is function of x only and T is function of t only.
- Then $\frac{\partial u}{\partial t} = XT'$ and $\frac{\partial^2 u}{\partial x^2} = TX''$

- Substitute in the given equation, we have
- $XT' = c^2\, TX''$
- Or $\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = k(say)$ (3)
- $\Longrightarrow$ Left hand side of (2) is a function of x only and right hand side is a function of t alone. Since x and t are independent variable so above equation hold good if each side is equal to a constant k. So

$X'' - kX = 0$ and $T' - c^2kT = 0$

- Solving above we have

- CASE I: If k is positive
- Let $k = p^2$
- From equation (3), $X'' - p^2 X = 0$
- $X = c_1 e^{px} + c_2 e^{-px}$
- And from equation (3), $T' - c^2 p^2 T = 0$
- $T = c_3 e^{c^2 p^2 t}$
- $\Longrightarrow \quad u = XT = c_1 e^{px} + c_2 e^{-px}) c_3 e^{c^2 p^2 t}$ (a)

- Case 2: if k is zero
- Let k=0
- From equation (2) $X'' = 0$
- $\Longrightarrow X = (b_1 + b_2 x)$
- From equation (3) $T' = 0 \Longrightarrow T = b_3$
- $u = XT = (b_1 + b_2 x) b_3 \qquad (b)$

- Case 3: if k is negative
- Let $k = -p^2$
- From equation (2) , we have
- $X'' + p^2 X = 0$
- $X \ = c_1 cospx + c_2 sinpx$
- And $\qquad T' + c^2 p^2 T = 0$
- $T = c_3 e^{-c^2 k^2 t}$
- $\Longrightarrow u = XT = (c_1 cospx + c_2 sinpx) c_3 e^{-c^2 p^2 t}$

- So all possible Solution are
- \` $u = XT \quad = (c_1 cospx + c_2 sinpx)c_3 e^{-c^2p^2t}$
- $u \ = \ XT \ \ = \ (b_1 + b_2 x)b_3$

$$u = XT \quad = c_1 e^{px} + c_2 e^{-px})c_3 e^{c^2p^2t}$$

- Out of these three solution we have to choose that solution which is consistent with physical nature of the problem. As we are dealing with problem on heat conduction, it must be a transient solution . i.e. u is to decrease with increase of time t.
- Therefore solution of heat equation is given by
- $u(x,t) = (c_1 cospx + c_2 sinpx)c_3 e^{-c^2p^2t}$
- $u(x,t) = (Acospx + Bsinpx)e^{-c^2p^2t}$
- where A=$c_1 c_2$ and B=$c_2 c_3$

Two Dimensional Heat Equation :

- This is the partial differential equation giving temperature distribution of the plane in transient state.
- Two Dimensional Heat flow equation is given by
- $\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$ (1)
- where $c^2 = \frac{k}{\rho s}$, is called diffusivity of the substance.
- In steady state, u is independent of time , so that $\frac{\partial u}{\partial t} = 0$ and above gives the Laplace equation in two dimensions i.e. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Laplace Equation in two dimensional is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Solution of Laplace Equation:

- Let $\mathrm{u} = XY$(1)
- be a solution of the given Laplace equation where X is function of x only and Y is function of y only.
- Then $\frac{\partial^2 u}{\partial x^2} = YX''$ and $\frac{\partial^2 u}{\partial Y^2} = XY''$
- Substitute in the given equation, we have
- $YX'' = -XY''$

- Or $$\frac{X''}{X} = -\frac{Y''}{Y} = k(say) \qquad (2)$$
- $\Longrightarrow$ Left hand side of (2) is a function of x only and right hand side is a function of y alone. Since x and y are independent variable so above equation hold good if each side is equal to a constant k. So

$X'' - kX = 0$ and $Y'' + kT = 0$

- Solving above we have

- CASE I: If k is positive i.e. $k = p^2$
- From equation (3), $X'' - p^2 X = 0$
- A. E is given by, $m^2 - p^2 = 0 \Longrightarrow m = \pm p$
- $\Longrightarrow X = c_1 e^{px} + c_2 e^{-px}$
- And from equation (3), $Y'' + p^2 Y = 0$
- A. E is given by, $m^2 + p^2 = 0 \Longrightarrow m = \pm pi$
- $\Longrightarrow$ Y$= c_3 \cos py + c_4 \sin py$

$so \quad u = XY = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$

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- Case 2: if k is negative i.e. $k = -p^2$
- From equation (2) , we have
- $X'' + p^2X = 0$
- A. E is given by, $m^2 + p^2 = 0 \Longrightarrow m = \pm pi$

$\Longrightarrow \quad X = c_5 cospx + c_6 sinpx$

- And $Y'' + p^2Y = 0$
- A. E is given by, $m^2 - p^2 = 0 \Longrightarrow m = \pm p$

- $\Longrightarrow \quad Y = c_7 e^{py} + c_8 e^{-py}$
- $\Longrightarrow$ u $= XY = (c_5 cospx + c_6 sinpx)(c_7 e^{py} + c_8 e^{-py})$

- Case 3: if k is zero i.e. $k = 0$
- Let From equation (2) $X'' = 0$
- $\Longrightarrow \; X = c_9 + c_{10}x$
- From equation (3) $Y'' = \; 0$
- $\Longrightarrow \; Y = c_{11} + c_{12}y$

- $u = XY = (c_9 + c_{10}x)(c_{11} + c_{12}y)$

So all possible Solution are

$$u = XY = (c_1e^{px} + c_2e^{-px})(c_3cos\,py + c_4\,sin\,py)$$

$$u = XY = (c_5cospx + c_6sinpx)(c_7e^{py} + c_8e^{-py})$$

$$u = XY = (c_9+c_{10}x)(c_{11}+c_{12}y)$$

Out of these three solution we have to choose that solution which is consistent with given boundary condition of the problem.

Example: Solve the Laplace equation subjected to the condition $u(0, y) = 0$, $u(l, y) = 0$, $u(x, 0) = 0$ and $u(x, \propto) = \sin\frac{n\pi x}{l}$

Solution.The Laplace equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (1)$$

The boundary conditions are

$$u(0, y) = 0 \qquad (2)$$

$$u(l, y) = 0 \qquad (3)$$

$$u(x, 0) = 0 \qquad (4)$$

$$u(x, \propto) = \sin\frac{n\pi x}{l} \qquad (5)$$

Now all three possible Solution are

$$u = XY = (c_1e^{px} + c_2e^{-px})(c_3cos\,py + c_4\,sin\,py)$$

$$u = XY = (c_5cospx + c_6sinpx)(c_7e^{py} + c_8e^{-py})$$

$$u = \ XY \ = (c_9+c_{10}x)(c_{11}+c_{12}y)$$

Now we have to solve Laplace equation satisfying the boundary conditions.

Now using the boundary condition (2) and (3) we have

$c_1 + c_2 = 0$ and $c_1e^{pt} + c_2e^{-pt}$

On solving these , we get $c_1 = c_2 = 0$ which lead to trivial solution.

Similarly using condition (2) and (3) last solution of Laplace Equation gives trival solution.

Therefore suitable solution for the present problem is

$$u = (c_5 cospx + c_6 sinpx)(c_7 e^{py} + c_8 e^{-py}) \quad (6)$$

Now using the condition (2) in (6) , we have

$$c_5(c_7 e^{py} + c_8 e^{-py}) = 0$$

for this to be true all time, $c_5 = 0$

So u= $c_6 sinpx(c_7 e^{py} + c_8 e^{-py})$

Now using condition (3), we have

$$c_6 sinpl(c_7 e^{py} + c_8 e^{-py}) = 0$$

This gives $pl = n\pi$ or $p = \frac{n\pi}{l}$

where n=0,1,2..

- so
- $u = c_6 \sin\frac{n\pi x}{l}\left(c_7 e^{\sin\frac{n\pi y}{l}} + c_8 e^{-\sin\frac{n\pi y}{l}}\right)$

Using condition (4) in above , we have

$c_7 + c_8 = 0 \quad \Rightarrow \quad c_7 = -c_8$

- So $\mathrm{u} = c_6 c_7 \sin\frac{n\pi x}{l}\left(e^{\sin\frac{n\pi y}{l}} - e^{-\sin\frac{n\pi y}{l}}\right)$
- $= b_n \sin\frac{n\pi x}{l}\left(e^{\sin\frac{n\pi y}{l}} - e^{-\sin\frac{n\pi y}{l}}\right)$

- Now again using condition (5), we have
- $sin\left(\frac{\pi x}{l}\right) = b_n \sin\frac{n\pi x}{l}(e^{\sin\frac{n\pi\propto}{l}} - e^{-\sin\frac{n\pi\propto}{l}})$
- so $b_n = 1/(e^{\sin\frac{n\pi\propto}{l}} - e^{-\sin\frac{n\pi\propto}{l}})$

Putting the value in above , we have required solution.

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Thank you!