JAIPUR ENGINEERING COLLEGE AND RESEARCH CENTRE

Year & Sem – I Year & II Sem
Subject –Engineering Mathematics-II
Unit – 4
Presented by – (Dr. Sunil K Srivastava, Associate Professor)

VISION AND MISSION OF INSTITUTE

VISION OF INSTITUTE

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MISSION OF INSTITUTE

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- Offer opportunities for interaction between academic and industry .
- Develop human potential to its fullest extent so that intellectually capable and imaginatively gifted leaders may emerge.

Partial differential equation:

A differential equation involving partial derivatives with respect to more than one independent variable is called partial differential equation.

We generally take x and y as independent variable and z is taken as dependent variable , the we shall employ following notations:

$$p = \frac{\partial z}{\partial x}\ , q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}\ , s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$$

For example, $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = kz,\ \ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ are form of the partial differential equation.

Order and Degree of a PDE

- The order of a partial differential equation is the order of the highest order partial derivative involve in the equation and degree is the degree of highest order derivative occurring in the equation. The above equation have order 1 ,2 and degree 1, 1 respectively.

Formation of partial differential equation:

- The partial differential equation are formed in two way namely:
- **By elimination of arbitrary constant:**
- **By elimination of arbitrary functions:**

(a) :By elimination of arbitrary constant

consider the relation

- $f(x, y, z, a, b) = 0$(1)
- where x ,y z are variable and a , b are arbitrary constant.
- In order to eliminate the two constant a and b we differentiate (1) partially w.r.t. x and y respectively,
- $$\left.\begin{aligned}\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = 0\\ \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} = 0\end{aligned}\right\} \qquad (2)$$
- Eliminating a and b from (1) and (2) we shall obtain the equation of the form
- $F(x, y, z, a, b) = 0$(3)
- which is partial differential equation of first order.

Example:Form the partial differential equation by eliminating arbitrary constant $z = ax + by + ab$

- **Solution:**Given$z = ax + by + ab$ (1)
- Differentiating (1) partially with respect x and y, we get
- $\frac{\partial z}{\partial x} = p = a$ and $\frac{\partial z}{\partial y} = q = b$ (2)
- Putting the value of a and b from (2) in(1), we get
- $z = px + qy + pq$,
- which is required partial differential equation.

Example: Form the partial differential equation by eliminating arbitrary constant $z = (x + a)(y + b)$

Solution: Given $z = (x + a)(y + b)$ (1)

- Differentiating (1) partially with respect x and y, we get
- $\frac{\partial z}{\partial x} = p = (y + b)$ and $\frac{\partial z}{\partial y} = q = (x + a)$ (2)
- Putting the value of a and b from (2) in(1), we get
- $z = pq$,
- which is required partial differential equation.

(b): Formation of p d .e. by elimination of arbitrary function

Let $f(u, v) = 0$ (1)

Differentiating (1) partially with respect x and y, we get

- $\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$ (2)
- And
- $\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0$ (3)
- Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from 2 and 3 we get required p.d.e.

Example: Form the p.d.e. by elimination of arbitrary function $z = f(x^2 + y^2)$

- Solution: we have here
- $z = f(x^2 + y^2)$ (1)
- Differentiating (1) partially with respect x and y, we get
- $\frac{\partial z}{\partial x} = p = f'(x^2 + y^2)2x$ (2)
- and
- $\frac{\partial z}{\partial y} = q = f'(x^2 + y^2)2y$ (3)
- Division of (2) and (3) , gives
- $yp + xq = 0$
- which is required equation.

Example: Form the p.d.e. by elimination of arbitrary

$$f(x^2 - y^2, z - xy) = 0$$

- Solution: Given $f(x^2 - y^2, z - xy) = 0$
- Let $u = x^2 - y^2$ and $v = z - xy$, the above equation can be written as
- $f(u, v) = 0$ (1)

Differentiating (1) partially with respect x and y, we get

- $\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$
- $\frac{\partial f}{\partial u}(2x) + \frac{\partial f}{\partial v}(-y + q) = 0$ (2)

Solution Continue….

- And
- $\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + q\,\frac{\partial u}{\partial z}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + q\,\frac{\partial v}{\partial z}\right) = 0$
- $\frac{\partial f}{\partial u}(2y) + \frac{\partial f}{\partial v}(-x + q) = 0$ (3)
- Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (2) and (3), we have
- $\begin{vmatrix} 2x & -y + p \\ 2y & -x + q \end{vmatrix} = 0$
- Or $xq - yp = x^2 - y^2$
- which is required p.d.e.

Linear differentiation equation of first order:

- A linear partial differential equation of the form
- $Pp + Qq = R$ (1)
- Where P, Q and R are function of x, y, z. this equation is also called **Lagrange's linear** equation or **Quasi – linear** equation. The equation (1) is linear if P, Q, R are independent of z.
- **Method to solve:**
- **Write down the** given partial differential equation in standard form $Pp + Qq = R$
- Form auxiliary equation $\frac{dx}{\mathrm{P}} = \frac{dy}{\mathrm{Q}} = \frac{dz}{\mathrm{Q}}$
- Solve the simultaneous equation find in (ii) and find its solution $u(x, y, z) = a$ and $v(x, y, z) = b$.
- Finally write the complete solution as $f(u, v) = 0$ or $u = f(v)$ where f is an arbitrary function.

Example: Solve $\frac{y^2 z}{x} p + xzq = \mathrm{y}^2$

- Solution: The given equation can be written as
- $y^2 zp + x^2 zq = \mathrm{xy}^2$
- The Subsidiary equation is given by
- $\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{\mathrm{xy}^2}$ (1)
- Taking first and second fraction , we have
- $\frac{dx}{y^2} = \frac{dy}{x^2}$
- Or $x^2 dx = y^2 dy$

- On integration, we have
- $x^3 - y^3 = c_1$ (2)
- Taking second and third fraction , we have
- $\frac{dx}{z} = \frac{dz}{\mathrm{x}}$
- Or $xdx = ydy$
- On integration, we have
- $x^2 - y^2 = c_2$ (3)
- Therefore in view of (2) and (3) complete solution is given by
- $f(x^3 - y^3, x^2 - y^2) = 0$, where f is arbitrary function.

Example: Solve $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$.

- **Solution: The** given equation is $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$
- Now comparing the given equation with standard form $Pp + Qq = R$, we have,
- $P = x^2(y-z), \quad Q = y^2(z-x) \quad$ and $\quad R = z^2(x-y)$
- The Subsidiary equation is given by
- $\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$ (1)
- Taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, each fraction of (1) is , we have
- each fraction $= \frac{\frac{dz}{x}+\frac{dz}{y}+\frac{dz}{z}}{0}$
- i.e. $\frac{dz}{x} + \frac{dz}{y} + \frac{dz}{z} = 0$

- On integration, we have
- $\log x + \log y + \log z = \log c_1 \Rightarrow xyz = c_1$
- Again using $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ as multipliers, each fraction in (1) is
- $= \frac{\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2}}{0}$
- i.e. $\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0$
- on integrating , we have
- $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_2$
- Therefore solution is given by
- $f\left(xyz, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0$ where f is an arbitrary function.

Type-1: $f(p,q) = 0$, the solution of this type of equation is given by $z = ax + by + c$ where a and b are connected by relation $f(a,b) = 0$

- **Example: Solve $\sqrt{\boldsymbol{p}} + \sqrt{\boldsymbol{q}} = \mathbf{1}.$**
- **Solution: the** given equation is of the type-1 i.e. $f(p,q) = 0.$
- Therefore the complete solution is given by
- $z = ax + by + c$
- Where $\sqrt{\boldsymbol{a}} + \sqrt{\boldsymbol{b}} = \mathbf{1} \Longrightarrow \boldsymbol{b} = (\mathbf{1} - \sqrt{\boldsymbol{a}})^{\mathbf{1/2}}$
- **Thus** $z = ax + (\mathbf{1} - \sqrt{\boldsymbol{a}})^{\mathbf{1/2}} y + c$ is required solution.
-
- **Example: Solve $\boldsymbol{p}^{\mathbf{2}} - \boldsymbol{q}^{\mathbf{2}} = \mathbf{1}.$**
- **Solution: The** given equation is of the type-1 i.e. $f(p,q) = 0.$
- Therefore the complete solution is given by
- $z = ax + by + c$
- Where $\boldsymbol{a}^{\mathbf{2}} - \boldsymbol{b}^{\mathbf{2}} = \mathbf{1} \Longrightarrow \boldsymbol{b} = (\boldsymbol{a}^{\mathbf{2}} - \mathbf{1})^{\mathbf{1/2}}$
- **Thus** $z = ax + (\boldsymbol{a}^{\mathbf{2}} - \mathbf{1})^{\mathbf{1/2}} y + c$ is required solution.

Reducible form:

Example: solve $x^2p^2 + y^2q^2 = z^2$

- **Solution: the given equation can be written as**
- $x^2\left(\frac{\partial z}{\partial x}\right)^2 + y^2\left(\frac{\partial z}{\partial y}\right)^2 = z^2$
- Or $\left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1$ **(1)**
- **Put** $X = \log x$, $Y = \log y$, $Z = \log z$ **then**
- $dX = \frac{1}{x}dx$, $dY = \frac{1}{y}dy$, $dz = \frac{1}{z}dz$
- **Then Equation (1) take the form**
- $\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1$ **which is of the type-1 .i.e.** $f(P, Q) = 0$
- where
- $P = \frac{\partial Z}{\partial X}$, $Q = \frac{\partial Z}{\partial Y}$

- Therefore the complete solution is given by
- $Z = aX + bY + C$
- Where $a^2 + b^2 = 1 \Longrightarrow b = (1 - a^2)^{1/2}$
- Thus $log\, z = a\, log\, x + b\, log\, y + log\, c$
- Or $log\, z = cos\, \alpha\, log\, x + sin\, \alpha\, log\, y + log\, c$
- Where $a = cos\, \alpha\,,\ b = (1 - a^2)^{1/2} = (1 - cos^2\alpha)^{1/2} = sin\, \alpha$
- Or $log\, z = log\, x^{cos\, \alpha} + log\, y^{sin\, \alpha} + log\, c$
- Or $z = x^{cos\, \alpha}\, y^{sin\, \alpha} c$ is required solution.

Type-2: Equations of the form $(z, p, q) = 0$, i.e. the equations containing z, p and q only and x and y do not occur in equation.

The equation of the form

$$f(p, q, z) = 0 \qquad (1)$$

Charpit auxiliary equation is

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial x}} = \frac{dp}{-\frac{\partial f}{\partial y}}$$

Which implies

$$\frac{dp}{p\frac{\partial f}{\partial z}} = \frac{dq}{q\frac{\partial f}{\partial z}} = \dots\dots\dots$$

Taking (1) and (2) term we get

$$\log q = \log p + \log a$$

Which implies

$$q = pa \text{ or } p = qa$$

$$\therefore \quad dz = pdx + qdy$$

$$= pdx + pa\,dy = p(dx + ady$$

$$dz = pd(x + ay) = \; pdu \qquad \text{where } u = \; x+ay$$

$$\Rightarrow p = \frac{dz}{dx} \quad \text{and} \quad p = \; a\frac{dz}{dx}$$

Putting the value of p and q in (1) we get

$$f\left(\frac{dz}{dx}, a\frac{dz}{dx}, z\right) = 0$$

which is O.D.E of first order , solving above we get z as function of u.

$Z = f(u)$ be the solution where $u = x + ay$.

Example: Find a complete integral of $9(p^2 z + q^2) = 4$

Solution: The given equation is of the form $f(p, q, z) = 0$

Let u = x+ay where a being arbitrary constant.

Now replacing p and q by $p = a\frac{dz}{du}$, $q = a\frac{dz}{du}$ the given equation becomes

$$9\left[z\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2\right] = 4 \text{ or}$$

$$\left(\frac{dz}{du}\right)^2 = \frac{4}{9(z+a^2)} \text{ or}$$

$$du = \frac{3}{2}(z + a^2)^{1/2}dz$$

On integration it gives

$$u + b = (z + a^2)^{3/2}$$

$$(u + b)^2 = (z + a^2)^3$$

$$(x + ay + b)^2 = (z + a^2)^3$$

Type-3: The equation of the form $f_1(x,p) = f_2(y,p)$

It is the equation in which the variable z does not appear and the terms containing p and x can be separate

To solve such equations we put each expression equal to an arbitrary constant a. Thus

$f_1(x,p) = a$ and $f_2(y,q) = a$

For solving p and q we get

$\varphi_1(x,a) = p$ and $\varphi_2(y,b) = q$

Now we know

$dz = pdx + qdy$

On integration we get

$\int dz = \int pdx + \int qdy$

$\int dz = \int \varphi_1(x,a)dx + \int \varphi_2(y,b)dy + b$

Example : Find the complete integral of $p^2 + q^2 = x + y$

Solution: The given partial differential equation can be written as

$p^2 - x = -q^2 + y$

To solve we assume

$p^2 - x = -q^2 + y = a$

Therefore

$p = \sqrt{a+x}$ and $q = \sqrt{y-a}$

Now

$dz = pdx + q\,dy$

$dz = \sqrt{a+x}\,dx + \sqrt{y-a}\,dy$

Integrating both sides we get

$$z = \frac{3}{2}(a+x)^{\frac{3}{2}} + \frac{3}{2}(y-a)^{\frac{3}{2}} + b$$

Which is the required complete integral

Example : Solve $pq = xy$.

Solution: The given P.D.E can be written as

$$\frac{p}{x} = \frac{y}{q}$$

Which is the standard form III i.e $f_1(p, x) = f_2(q, y)$

To solve we assume

$$f_1(p, x) = f_2(q, y) = a$$

i.e. $\frac{p}{x} = \frac{y}{q} = a$

Now $p = ax$ and $q = \frac{y}{a}$

$dz = p\,dx + q\,dy$ gives

$$dz = ax\,dx + \frac{y}{a}\,dy$$

Integrating both sides we get

$$z = \frac{ax^2}{2} + \frac{y^2}{2a} + b$$

Which is the require complete integral

Type-4: if the differential equation is of the form$z = px + qy + f(p, q)$ This type of partial differential equation is known as **Clairaut'sform.**

A first order P.D.E is called a Clairaut's Form

$$\boldsymbol{z = px + qy + f(p, q)}$$

The complete solution of this type of equation is obtained by replacing p by a and q by b i.e.

$$\boldsymbol{z = ax + by + f(a, b)}$$

Find the complete integral of

$$z = px + qy + p^2 + q^2.$$

Given $z = px + qy + p^2 + q^2$.

This equation is of the form z =px +qy f+(p, q) .

By Clairaut's type,put p=a , q =b.

Therefore the complete integral is

Therefore the complete integral is $z = ax + by + a^2 + b^2$.

Charpit's method:

- **This is general** method of solving equation of first order but of any degree, linear or non-linear with two independent variable.
- Let the give equation be
- $f(x, y, z, p, q) = 0$(1)
- Since z is function of two independent variable x and y, therefore
- $dz = pdx + qdy$(2)
- If we can find another relation
- $F(x, y, z, p, q) = 0$(3)
- other than (1) in $x, y, z, p\ and\ q$ such that when the values of p and q could be obtained by solving (1) and (3) and after putting the value of p and q in (2) it becomes integrable. The integration of (2) will give complete solution of (1).

- In order to find the relation (3), we differentiate partially (1) and (3) w.r.t. x and y, we have
- $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0$ (4)
- $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0$ (5)
- And $\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = 0$ (6)
- $\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} p + \frac{\partial F}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial y} = 0$ (7)
- Eliminating $\frac{\partial p}{\partial x}$ from (4) and (5), we get
- $\left(\frac{\partial f}{\partial x}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial x}\frac{\partial f}{\partial p}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial p}\right) p + \left(\frac{\partial f}{\partial q}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial q}{\partial x} = 0$ (8)

- And Eliminating $\frac{\partial q}{\partial y}$ from (6) and (7), we get
- $\left(\frac{\partial f}{\partial y}\frac{\partial F}{\partial p}-\frac{\partial F}{\partial y}\frac{\partial f}{\partial q}\right)+\left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial z}\frac{\partial f}{\partial q}\right)p+\left(\frac{\partial f}{\partial p}\frac{\partial F}{\partial q}-\frac{\partial F}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y}=0$(9)
- Adding (8) and (9) and rearranging the term, we get
- $\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial p}+\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial q}+\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial z}+\left(-\frac{\partial f}{\partial p}\right)\frac{\partial F}{\partial x}+\left(-\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial y}=0$ (10)
- { the last term cancel out as $\frac{\partial q}{\partial x}=\frac{\partial p}{\partial y}=\frac{\partial^2 z}{\partial x\partial y}$ }
- This is a linear partial differential equation of first order with F as dependent variable and $x, y, z, p, and\ q$ as independent variable. Therefore auxiliary equations are given by
- $\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dF}{0}$ (11)
- Find the values of p and q from above equation (11) and put it in $dz=pdx+qdy$ which on integration gives the required solution of differential equation(1).

Example : Solve $z = pq$, using Chapit's method.

- **Solution:** Here $f(x, y, z, p, q) = z - pq = 0$
- $\therefore \frac{\partial f}{\partial x} = 0, \ \frac{\partial f}{\partial y} = \ 0, \ \frac{\partial f}{\partial z} = 1, \frac{\partial f}{\partial p} = -q \ , \frac{\partial f}{\partial q} = -p$
- Now Charpit's subsidiary(auxiliary) equation are
- $\frac{dp}{p} = \frac{dq}{q} = \frac{dx}{q} = \frac{dy}{p} = \frac{dz}{2pq}$
- or $\frac{dp}{2z-2qy} = \frac{dq}{0} = \frac{dx}{x^2-q} = \frac{dy}{2xy-p} = \frac{dz}{px^2-2pq+2qxy}$
- from above we have $\frac{dp}{p} = \frac{dq}{q}$ so that $\log p = \log q + \log c$
- or $p = qc$
-

- putting $p = qc$ in given equation we have $q = \sqrt{\frac{z}{c}} \Longrightarrow p = \sqrt{cz}$
- therefore putting the values of p and q in $dz = pdx + qdy$, we have
- $dz = \sqrt{cz}dx + \sqrt{\frac{z}{c}}dy$
- or $\sqrt{\frac{c}{z}}dz = cdx + dy$
- on integration, $2\sqrt{cz} = cx + y + d$
- which is required solution

Example : Solve $pxy + pq + qy = yz$, using Chapit's method

- **Solution:** Here $f(x, y, z, p, q) = pxy + pq + qy - yz = 0$
- $\therefore \frac{\partial f}{\partial x} = py,\ \frac{\partial f}{\partial y} = px + q,\ \frac{\partial f}{\partial z} = -y \frac{\partial f}{\partial p} = xy + q\ , \frac{\partial f}{\partial q} = y + p$
- Now Charpit's subsidiary(auxiliary) equation are
- $\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}}$
- or $\frac{dp}{0} = \frac{dq}{px+q+qy} = \frac{dx}{-(xy+q)} = \frac{dy}{-(p+y)} = \frac{dz}{-p(y+q)-q(p+y)}$
- from above we have $dp = 0$ or $p = a$

- putting $p = a$ in given equation we have $q = \frac{y(z-ax)}{a+y}$
- therefore putting the values of p and q in $dz = pdx + qdy$, we have
- $dz = adx + \frac{y(z-ax)}{a+y}dy$
- or $\frac{dz-adx}{z-ax} = (1 - \frac{a}{a+y})dy$
- on integration, $\log(z - ax) = y - \log(a + y) + \log b$
- or $y = \log(z - ax) + \log(a + y)^a - \log b$
- $\Longrightarrow$ $(z - ax)(a + y)^a = be^y$
- which is required solution

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Thank you!