JAIPUR ENGINEERING COLLEGE AND RESEARCH CENTRE

Year & Sem – I Year & II Sem

Subject –Engineering Mathematics-II

Unit – I

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VISION AND MISSION OF INSTITUTE

VISION OF INSTITUTE

To became a renowned centre of outcome based learning and work towards academic professional, cultural and social enrichment of the lives of individuals and communities .

MISSION OF INSTITUTE

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- Offer opportunities for interaction between academic and industry .
- Develop human potential to its fullest extent so that intellectually capable and imaginatively gifted leaders may emerge.

CONTENTS (TO BE COVERED)

Cayley-Hamilton Theorem

Cayley - Hamilton Theorem

Every square matrix satisfies its own characteristic equation.

If A is a square matrix of order n, then

$$|A - \lambda I_n| = 0$$

Q.1 Find the characteristic eq^n of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

and verify that it is satisfied by A. Hence find A^{-1}.

Sol^n

The characteristic eq^n is

$$A = \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

Now

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A \cdot A^2$$

$$A^3 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$A^3 - 6A^2 + 9A - 4I_3$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$-4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0$$

A satisfies its own characteristic eq^n.

To find A^{-1}

$$(A^3 - 6A^2 + 9A - 4I_3)A^{-1} = 0$$

$$A^{-1} = \frac{1}{4}\left[A^2 - 6A + 9I\right]$$

$$= \frac{1}{4}\left\{\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Q.2 If $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$

Find the characteristic eq^n of A. Prove that A satisfies this equation and hence find A^{-1}.

Sol^n The characteristic eq^n is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 3 & 4-\lambda & 5 \\ 0 & -6 & -7-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(4-\lambda)(-7-\lambda)+30\right] - 1(-18) = 0$$

$$\lambda^3 + 2\lambda^2 - \lambda - 20 = 0 \qquad -(1)$$

Now

$$A^2 = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6 & 6 \\ 15 & -14 & -18 \\ -18 & 18 & 19 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 6 & 6 \\ 15 & -14 & -18 \\ -18 & 18 & 19 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix} = \begin{bmatrix} 19 & -12 & -13 \\ -27 & 52 & 41 \\ 36 & -42 & -25 \end{bmatrix}$$

These all values satisfy characteristic eq^n ①.

Now to find A^{-1} we multiply A^{-1} by $A^3 + 2A^2 - A - 20I_3 = 0$

$$A^{-1}[A^3 + 2A^2 - A - 20I_3] = 0$$

$$A^2 + 2A - I_3 - 20A^{-1} = 0$$

$$A^{-1} = \frac{1}{20}\left[A^2 + 2A - I_3\right]$$

$$A^{-1} = \frac{1}{20}\left[A^2 + 2A - I_3\right]$$

$$= \frac{1}{20}\left\{\begin{bmatrix} 1 & 6 & 6 \\ 15 & -14 & -18 \\ -18 & 18 & 19 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{20}\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

Q. 3 Find the characteristic eq^n of the matrix

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

Show that eq^n is satisfied by A and hence obtain the inverse of A.

Sol^n The characteristic eq^n is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(2-\lambda)(1-\lambda)-6]-3[4(1-\lambda)-3]+7[8-(2-\lambda)]=0$$

$$\lambda^3-4\lambda^2-20\lambda-35=0 \qquad -(1)$$

which is the characteristic eq^n of A.

Now

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

Now

$$A^3 - 4A^2 - 20A - 35I = 0$$

$$\begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$-35\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

Hence A satisfies its characteristic eqⁿ

Now multiplying A^{-1} by $A^3-4A^2-20A-35I=0$

$$A^{-1}[A^3-4A^2-20A-35I]=0$$

$$A^2-4A-20I-35A^{-1}=0$$

$$A^{-1}=\frac{1}{35}[A^2-4A-20I]$$

$$= \frac{1}{35}\left\{\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 4\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 20\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{35}\begin{bmatrix} -4 & +11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$$

Q 4 State Cayley Hamilton theorem. Verify for the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}.$$ Hence find A^{-1}.

Sol^n The characteristic eq^n is $|A - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & 1 & 2 \\ 1 & 2-\lambda & 3 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$-\lambda[(2-\lambda)(1-\lambda)-3]-1[(1-\lambda)-9]+2[1-3(2-\lambda)]=0$$

$$\lambda^3-3\lambda^2-8\lambda+2=0 \qquad — (1)$$

which is the characteristic eqⁿ of A.

$$A^2=\begin{bmatrix}0&1&2\\1&2&3\\3&1&1\end{bmatrix}\begin{bmatrix}0&1&2\\1&2&3\\3&1&1\end{bmatrix}=\begin{bmatrix}7&4&5\\11&8&11\\4&6&10\end{bmatrix}$$

$$A^3=\begin{bmatrix}7&4&5\\11&8&11\\4&6&10\end{bmatrix}\begin{bmatrix}0&1&2\\1&2&3\\3&1&1\end{bmatrix}=\begin{bmatrix}19&20&31\\41&38&57\\36&26&36\end{bmatrix}$$

Now $A^3 - 3A^2 - 8A + 2I_3$

$$= \begin{bmatrix} 19 & 30 & 31 \\ 41 & 38 & 57 \\ 36 & 26 & 36 \end{bmatrix} - \begin{bmatrix} 21 & 12 & 15 \\ 33 & 24 & 33 \\ 12 & 18 & 30 \end{bmatrix} - \begin{bmatrix} 0 & 8 & 16 \\ 8 & 16 & 24 \\ 24 & 8 & 8 \end{bmatrix}$$

$$+ \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 0$$

Hence A satisfies its characteristic eqn.

Now $A^{-1} = \frac{-1}{2}[A^2 - 3A - 8I_3]$

$$= -\frac{1}{2}\left\{\begin{bmatrix} 7 & 4 & 5 \\ 11 & 8 & 11 \\ 4 & 6 & 10 \end{bmatrix} - \begin{bmatrix} 0 & 3 & 6 \\ 3 & 6 & 9 \\ 9 & 3 & 3 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$$

Q.5 Find the characteristic eq^n of matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$ Hence find the value of

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I_3$$

Sol^n The characteristic eq^n is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\left[(1-\lambda)(2-\lambda)\right]+1\left[-(1-\lambda)\right]=0$$

$$\lambda^3-5\lambda^2+7\lambda-3=0 \qquad -(1)$$

which is characteristic eq^n of A.

Now

$$A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I_3$$

$$=A^5(A^3-5A^2+7A-3I_3)+A(A^3-5A^2+7A-3I_3)+A^2+A+I_3$$

Since A satisfies its characteristic eq^n so

$$= 0 + 0 + A^2 + A + I_3$$

Now

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A^2 + A + I_3$$

$$\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Refrences

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Thank you!