JAIPUR ENGINEERING COLLEGE AND RESEARCH CENTRE

Year & Sem. – B. Tech I year, Sem.-I

Subject –Engineering Mathematics

Unit – I

Presented by – Dr. Ruchi Mathur

Designation - Associate Professor

Department - Mathematics

VISION OF INSTITUTE

To become a renowned centre of outcome based learning, and work towards academic, professional, cultural and social enrichment of the lives of individuals and communities.

MISSION OF INSTITUTE

❖**Focus on evaluation of learning outcomes and motivate students to inculcate research aptitude by project based learning.**

❖**Identify, based on informed perception of Indian, regional and global needs, the areas of focus and provide platform to gain knowledge and solutions.**

❖**Offer opportunities for interaction between academia and industry.**

❖**Develop human potential to its fullest extent so that intellectually capable and imaginatively gifted leaders may emerge in a range of professions.**

Engineering Mathematics: Course Outcomes

Students will be able to:

CO1. **Understand fundamental concepts of improper integrals, beta and gamma functions and their properties. Evaluation of Multiple Integrals in finding the areas, volume enclosed by several curves after its tracing and its application in proving certain theorems**.

CO2. Interpret the concept of a series as the sum of a sequence, and use the sequence of partial sums to determine convergence of a series. Understand derivatives of power, trigonometric, exponential, hyperbolic, logarithmic series.

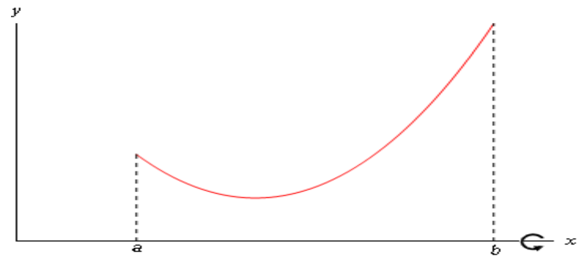
CO3. Recognize odd, even and periodic function and express them in Fourier series using Euler's formulae.

CO4. Understand the concept of limits, continuity and differentiability of functions of several variables. Analytical definition of partial derivative. Maxima and minima of functions of several variables Define gradient, divergence and curl of scalar and vector functions.

Volume of the Solid of Revolution (Cartesian Equations)

(i) Revolution about $x - axis$

The volume of the solid generated by the revolution about the $x - axis$ of the area bounded by the curve $y = f(x)$, the $x - axis$, and the ordinates = $x = a, x = b$ is



$\int_a^b \pi\, y^2\, dx$

(ii) Revolution about $y - axis$

The volume of the solid generated by the revolution about the $y - axis$ of the area bounded by the curve $x = f(y)$, the $y - axis$, and the abscissae $y = c, y = d$ is

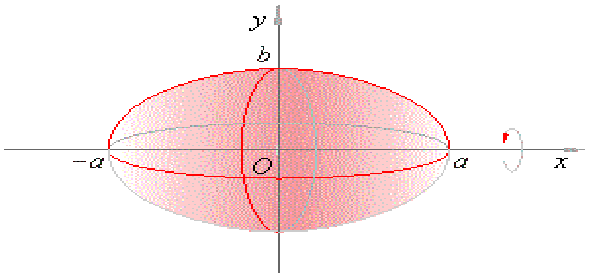
$$\int_c^d \pi\, x^2\, dy$$

(iii) Revolution about a Line

If the generated curve $f(x)$ revolves about any line (other than $x - axis$ and $y - axis$), say x=a or y=c

$$\int \pi(y - c)^2 dx \; or \; \int \pi(x - a)^2 dy$$

Example **Find the volume of the solid obtained by the revolution of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about its minor axis.**



Solution: The equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or

$$x^2 = \frac{a^2}{b^2}(b^2 - y^2)$$

Volume of the ellipse = twice the volume generated by the revolution of the arc BA about the minor axis, i.e., $y = axis$. For

the arc BA, y varies from $y = 0$ to $y = b$.

$\therefore$ the required volume = $2 \int_{y=0}^{b} \pi \ x^2 \ dy$

$$= 2\pi \int_0^b \frac{a^2}{b^2}(b^2 - y^2) \ dy = \frac{2\pi a^2}{b^2} \int_0^b (b^2 - y^2) \ dy$$

$$= \frac{2\pi a^2}{b^2}\left[b^2 y - \frac{y^3}{3}\right] = \frac{2\pi a^2}{b^2}\left[b^3 - \frac{b^3}{3}\right] = \frac{4\pi a^2 b}{3}$$

Volume of Revolution for Parametric Equations

Let $x = f(t), y = g(t)$ be the equations of the parametric form, where 't' is a parameter. There are two cases arise.

Case I

The volume generated by the revolution of the area bounded by the given equations of curve, the $x - axis$, and the ordinates $x = a,\ x = b$ about the $x - axis$ is

$$\int_a^b \pi\, y^2\, dx = \pi \int_{t_2}^{t_1} [g(t)]^2 \frac{dx}{dt} dt$$

where $t = t_1$ when $x = a$, and $t = t_2$ when $x = b$.

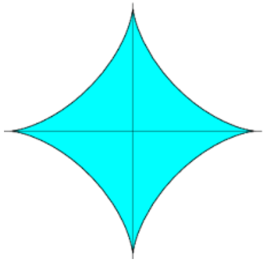
Case II

The volume generated by the revolution of the area bounded by the given equations of the curve, the $y - axis$, and the abscissae $y = c,\ y = d$ about the $y - axis$is

$$\int_c^d \pi\, x^2\, dy = \pi \int_{t_3}^{t_4} [f(t)]^2 \frac{dy}{dt} dt$$

Where $t = t_3$ when $y = c,$ and $t = t_4$when $y = d$.

Q. Find the volume of the solid generated by the revolution of the curve $x = a\,cos^3 t, y = a\,sin^3 t,$



about the $x - axis$.

Solution: The parametric equations of the astroid are $x = a\,cos^3 t,$ and $y = a\,sin^3 t,$

The curve is symmetrical about both the axes. It cuts the axis at $t = 0$ and $= \frac{\pi}{2}$. For the portion of the curve in the first quadrant, t varies from 0 to $\frac{\pi}{2}$.

The required volume

$$= 2\int_0^{\pi/2} \pi\, y^2\, \frac{dx}{dt}\, dt$$

$$= 2\pi \int_0^{\pi/2} (a\,sin^3 t)^2 . (-3a\,cos^2 t\sin t)\, dt$$

$$= -6\pi\, a^3 \int_0^{\pi/2} sin^7 . cos^2 t\ \ dt \quad = -6\pi\, a^3 . \frac{\Gamma 4\, .\, \Gamma\frac{3}{2}}{2\,\Gamma\frac{11}{2}}$$

$$= -6\pi\, a^3 . \frac{3.2\,.1.\frac{1}{2}\sqrt{\pi}}{2.\frac{9}{2}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}}$$

$$= -\frac{32\,\pi\, a^2}{105}$$

Volume $= \frac{32\,\pi\, a^2}{105}$ (in magnitude)

Volume of the Solid of Revolution for Polar Coordinates

(i) The volume of the solid generated by the revolution of the area bounded by the curve $r = \int(\theta)$ and the radii vectors $\theta = \alpha$ and $\theta = \beta$ about the initial line $\theta = 0$ is given by

$$\frac{2\pi}{3}\int_{\alpha}^{\beta} r^3 \sin\theta \, d\theta$$

(ii) The volume of the solid generated by the revolution of the area about the line $\theta = \pi/2$ of the area bounded by the curve
$r = \int(\theta)$ and the radii vectors $\theta = \alpha,\ \theta = \beta$ is given by

$$\frac{2\pi}{3}\int_{\alpha}^{\beta} r^3 \cos\theta \, d\theta$$

(iii) If the curve $r = \int(\theta)$ revolves about the initial line $(i.e., x - axis)$ then the volume generated is

$$\int_{a}^{b} \pi\ y^2\ dx = \int_{\alpha}^{\beta} \pi\, y^2 \frac{dx}{d\theta}\ d\theta$$

Where $\theta = \alpha$ when $x = \alpha$, and $\theta = \beta$, when $x = b$.
But $x = r\cos\theta, y = r\sin\theta, \frac{dx}{d\theta} = \frac{d}{d\theta}(r\cos\theta)$

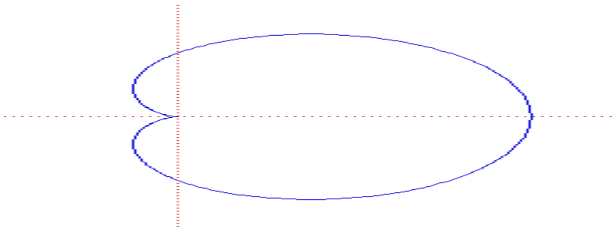
$\therefore$ Volume = $\int_{\alpha}^{\beta} \pi\,(r\sin\theta)^2 \frac{d}{d\theta}.(r\cos\theta)d\theta$

Similarly, if $r = \int(\theta)$ revolves about the $y = axis$ then

Volume = $\int_{\alpha'}^{\beta'} \pi\,(r\cos\theta)^2 \frac{d}{d\theta}.(r\sin\theta)d\theta$

where $\theta = \alpha'$ when $y = c$, and $\theta = \beta'$ when $y = d$.

Example: If the cardioid $r = a\,(1 + cos\theta)$ revolves about the initial line, find the volume generated.



Solution: The equation of the curve $r = a\,(1 + cos\theta)$

The curve (1) is symmetrical about the initial line and for the

Upper half, θ varies from 0 to π.

$\therefore$ the required volume $= \frac{2\pi}{3}\int_0^{\pi} r^3 \sin\theta \, d\theta$

$$= \frac{2\pi}{3}\int_0^{\pi} a^3\,(1+\cos\theta)^3 . \sin\theta\, d\theta$$

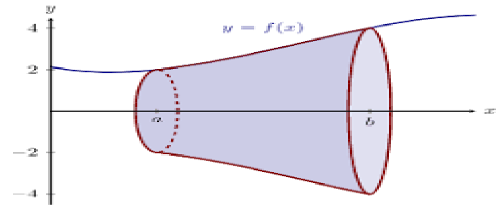
$$= \frac{2\pi a^3}{3}\int_0^{\pi}(1 + cos\theta) . \sin\theta\, d\theta$$

$$= \frac{2\pi a^3}{3}\left[\frac{(1+cos\theta)}{4}\right]_0^{\pi} = -\frac{1}{6}\pi a^3(0-16)$$

$$= \frac{8}{3}\,\pi a^3$$

Surface of the Solid of Revolution in Cartesian Coordinates

The curved surface of a solid generated by revolution about the $x = axis$, of the area bounded by the curve $y = \int (x)$, the $x = axis$, and the ordinates $x = a, x = b$ is



$$\int_{x=a}^{b} 2\pi \; y \, dS$$

where S is the length of the arc of the curve measured from a fixed point on it to any point (x, y).

Note 1 We know that $\frac{dS}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$. Therefore, the above formula may be rewritten as the required

$$\text{curved surface} = \frac{2}{\pi} \int_{x=a}^{b} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx.$$

Note 2 Similarly, the curved surface of the solid generated by the revolution about the $y - axis$ of the area

Bounded by the curve $x = f(y)$, the $y - axis$, and the abscissa $y = c$ and $y = d$ is

$$\int_{y=e}^{d} 2\pi \, dS = \int_{y=e}^{d} 2\pi \, x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

Or

$$= 2\,\pi \int_{y=c}^{d} x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

Surface of the Solid of Revolution in Parametric Equations

The curved surface of the solid generated by the revolution about the $x - axis$ of the area bounded by the curve

$x = \emptyset(t), y = \psi(t)$, the $x - axis$, and the ordinates $t = t_1, t = t_2$ is

$$\int 2\pi\, y\, dS = \int_{t=t_1}^{t_2} 2\pi\, y\, \frac{dS}{dt}\, dt$$

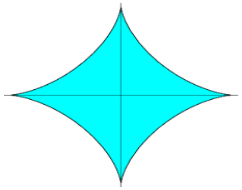
where $\frac{dS}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

Similarly, the revolution about the $y - axis$ gives

$$\int 2\pi\ x\, dS = \int_{t=t_1}^{t_2} 2\pi\, y\, \frac{dS}{dt}\, dt$$

where $\frac{dS}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

Q. Find the surface of the solid generated by the revolution of the curve $x = a\cos^3 t,\ y = a\sin^3 t$ about the



$x - axis$.

Solution The parametric equations are

$$x = a\cos^3 t,\ y = a\sin^3 t$$

$$\frac{dx}{dt} = -3a\cos^2 t\,.\sin t,\ \frac{dy}{dt} = 3a\sin^2 t\cos t$$

$$\frac{dS}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]} = \sqrt{(-\,3a\cos^2 t\sin)^2 + (3a\sin^2 t\cos t)^2}$$

$$= 3a\sqrt{\cos^4 t\,.\sin^2 t + \sin^4 t\cos^2 t} = \frac{dS}{dt} = 3a\,.\sin t\cos t$$

The required surface = 2 x Area of the surface generated by revolution about the $x - axis$

$$= 2\int_{t=0}^{\pi/2} 2\pi\, y\, dS = 2\int_{t=0}^{\pi/2} 2\pi\, y\, \frac{dS}{dt}.dt$$

$$= 4\pi \int_{t=0}^{\pi/2} a\sin^3 t(3a\sin t\cos t)\, dt$$

$$= 4\pi a^2 \int_{t=0}^{\pi/2} 3\sin^4 t\cos t\, dt = 12\,\pi\, a^2 \int_0^{\pi/2} \sin^4 t\,.cost\, dt$$

$$= 12\,\pi\, a^2 \left[\frac{\sin^5 t}{5}\right]_0^{\pi/2} = 12\,\pi\, a^2\left[\frac{1}{5} - 0\right] = \frac{12}{5}\,\pi\, a^2$$

Surface of the Solid of Revolution in Polar Form

The curved surface of the solid generated by the revolution about the $x-axis$ of the area bounded by the curve

$r = \int(\theta)$, and the radii vectors $\theta = \alpha,\ \theta = \beta$ is

$$\int_{\theta=a}^{\beta} 2\,\pi\, y\, dS = 2\pi \int_{a}^{\beta} y \frac{dS}{d\theta}\, d\theta$$

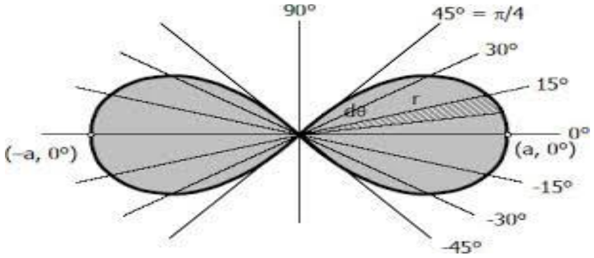
where $\quad \frac{dS}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}\ and\ y = r\sin\theta$

Similarly, revolution about the $y-axis$of the area bounded by the curve $\theta = \int(r)$ is

$$\int_{\theta=a}^{\beta} 2\,\pi\, x\, dS = 2\pi \int_{a}^{\beta} x \frac{dS}{d\theta}\, d\theta$$

where $\quad \frac{dS}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}\ and\ x = r\cos\theta.$

Example: Find the surface area generated by the revolution of the loops of the lemniscate $r^2 = a^2 cos\, 2\theta$ about the initial line.



Solution: The given curve is symmetrical about the initial line and the line $\theta = \frac{\Pi}{2}$. The curve consist of two loops and in the first quadrant, for half the loop, θ varies from 0 to $\frac{\Pi}{4}$.

$$\therefore\ r^2 = a^2 cos\, 2\theta \ or\ r = a\sqrt{cos\, 2\theta}$$

$$2r\frac{dr}{d\theta} = a^2(-2sin\, 2\theta)$$

$$\frac{dr}{d\theta} = -\frac{a^2 sin\, 2\theta}{r}$$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2 cos\, 2\theta + \frac{a^4 sin^2\, 2\theta}{r^2}}$$

$$= \sqrt{a^2 cos\, 2\theta + \frac{a^4 sin^2\, 2\theta}{r^2}}$$

$$\sqrt{a^2\cos 2\theta + \frac{a^4\sin^2 2\theta}{a^2\cos 2\theta}} = \sqrt{\frac{a^2}{\cos 2\theta}} = \frac{a}{\sqrt{\cos 2\theta}}$$

The required surface

$=$

$2\int_0^{\pi/4} 2\pi y \frac{ds}{d\theta}\, d\theta =$

$4\pi \int_0^{\pi/4} r\sin\theta \frac{ds}{d\theta}\, d\theta = 4\pi \int_0^{\pi/4} a\sqrt{\cos 2\theta}\,\sin\theta \frac{a}{\sqrt{\cos 2\theta}}\, d\theta$

$$4\pi a^2 \int_0^{\pi/4} \sin\theta\, d\theta = 4\pi a^2[-\cos\theta]_0^{\pi/4}$$

$$= 4\pi a^2\left(1 - \frac{1}{\sqrt{2}}\right)$$

Suggested links from NPTEL & other Platforms:

- Advanced Engineering Mathematics: Erwin Kreyszig, Wiley plus publication
- Engineering Mathematics : CB Gupta, SR Singh, Mukesh Kumar, Mc Graw Hill
- https://nptel.ac.in/courses/111/105/111105122/

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Thank you!